2008 Final Exam Solution

노트 제목 2008-05-29

1. $L(\underline{A}, \mu) = \frac{2}{\sqrt{3}} A_1 + A_2 + \mu\left(F\left(\frac{8}{\sqrt{3}A_1} + \frac{3}{A_2}\right) - \sigma_0\right)$

i) stationary condition

$$\frac{\partial L}{\partial A_1} = \frac{2}{\sqrt{3}} - \mu F \frac{8}{\sqrt{3}} \frac{1}{A_1^2} = 0 \Rightarrow A_1 = \sqrt{4\mu F}$$

$$\frac{\partial L}{\partial A_2} = 1 - \mu F\, 3 \frac{1}{A_2^2} = 0 \Rightarrow A_2 = \sqrt{3\mu F}$$

ii) feasibility condition

$$\frac{\partial L}{\partial \mu} = F\left(\frac{8}{\sqrt{3}A_1} + \frac{3}{A_2}\right) - \sigma_0 \le 0$$

iii) orthogonality condition

$$\mu\left(F\left(\frac{8}{\sqrt{3}A_1} + \frac{3}{A_2}\right) - \sigma_0\right) = 0$$

iv) non-negativity condition

$$\mu \ge 0$$

5pt

$\mu \ne 0$

$$F\left(\frac{8}{\sqrt{3}A_1} + \frac{3}{A_2}\right) - \sigma_0 = 0$$

$$F\left(\frac{4}{\sqrt{3}} \frac{1}{\sqrt{\mu F}} + \frac{3}{\sqrt{3}} \frac{1}{\sqrt{\mu F}}\right) - \sigma_0 = 0 \Rightarrow \mu = \frac{49}{3} \frac{F}{\sigma_0^2}$$

$\mu = 0 \Rightarrow$ stationary condition을 만족하지 않음.

$$\Rightarrow \mu_1 = \frac{49}{3}\frac{F}{\sigma_0^2}, \quad A_1 = \frac{14}{\sqrt{3}}\frac{F}{\sigma_0} \quad A_2 = 7\frac{F}{\sigma_0}$$ 5pt

- Sufficient condition

$$\underline{H}_L = \begin{bmatrix} 16\sqrt{3}\mu F A_1^{-3} & 0 \\ 0 & 6\mu F A_2^{-3} \end{bmatrix} \Rightarrow P-D$$

$F > 0,\ \sigma_0 > 0$

$A_1 > 0 \quad A_2 > 0 \quad \mu > 0$

$$\Rightarrow \underline{d}^T \underline{H}_L \underline{d} > 0$$ 5pt

$$\therefore f_{min} = \frac{49}{3}\frac{F}{\sigma_0} \quad (A_1, A_2) = \left(\frac{14}{\sqrt{3}}\frac{F}{\sigma_0},\ 7\frac{F}{\sigma_0}\right)$$ 5pt

2. $L = (x_1-3)^2 + (x_2+1)^2 + \mu_1(x_1+x_2-1.5) + \mu_2(x_1-1)$

$\triangleq L_1(x_1;\underset{\sim}{\mu}) + L_2(x_2;\underset{\sim}{\mu}) + L_0$

$= (x_1-3)^2 + (\mu_1+\mu_2)x_1 + (x_2+1)^2 + \mu_1 x_2 - 1.5\mu_1 - \mu_2$ ✓ 5pt

$\phi = \min_{x_1, x_2} L(x_1, x_2, \underset{\sim}{\mu})$

$$\begin{cases} \frac{\partial L}{\partial x_1} = 2(x_1-3) + (\mu_1+\mu_2) = 0 \\ \frac{\partial L}{\partial x_2} = 2(x_2+1) + \mu_1 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1^* = 3 - \frac{1}{2}(\mu_1+\mu_2) \\ x_2^* = -1 - \frac{1}{2}\mu_1 \end{cases}$$ ✓ 5pt

$\phi(\underset{\sim}{\mu}) = -\frac{1}{2}\mu_1^2 - \frac{1}{4}\mu_2^2 - \frac{1}{2}\mu_1\mu_2 + \frac{1}{2}\mu_1 + 2\mu_2$ ✓ 10pt

i) $\mu_1 > 0, \mu_2 > 0$

$$\left.\begin{aligned} \frac{\partial \phi}{\partial \mu_1} &= -\mu_1 - \frac{1}{2}\mu_2 + \frac{1}{2} = 0 \\ \frac{\partial \phi}{\partial \mu_2} &= -\frac{1}{2}\mu_2 - \frac{1}{2}\mu_1 + 2 = 0 \end{aligned}\right) \Rightarrow \mu_1 = \underset{0}{\underset{\wedge}{-3}}, \mu_2 = 7 \Rightarrow \text{모순}$$ ✓ 6pt

ii) $\mu_2 > 0, \mu_1 = 0$ (i.e. $g_1 \le 0$)

$\mu_2 = 4$, $x_1^* = 1$, $x_2^* = -1$ $\quad g_1 = -1.5 \le 0$ ✓ 6pt

iii) $\mu_1 > 0$, $\mu_2 = 0$ (i.e $g_2 \le 0$)

$\mu_1 = \frac{1}{2}$, $x_1^* = \frac{11}{4}$ $\quad x_2^* = -\frac{5}{4}$ $\quad g_2 = \frac{7}{4} > 0$

not a solution ✓ 6pt

Sufficient Condition

$$\left.\frac{\partial^2 \phi}{\partial \mu_1 \partial \mu_2}\right|_{\mu^*} = \frac{1}{4}\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix} \Rightarrow N-D$$ ✓ 4t

$\underset{\sim}{\mu}^* = (0, 4)$ $\quad \phi_{max} = 4$

$\underset{\sim}{x}^* = (1, -1)$ $\quad f_{min} = 4$ ✓ 8pt

3. $L(\underset{\sim}{x}, \mu) = x_1^2 + x_2^2 + \mu(-(x_1-1)^3 + x_2^2)$

(a) KKT condition

i) stationary condition

$$\frac{\partial L}{\partial x_1} = 2x_1 - \mu 3 (x_1-1)^2 = 0$$

$$\frac{\partial L}{\partial x_2} = 2x_2 + \mu 2 x_2 = 0$$

ii) feasibility condition

$$\frac{\partial L}{\partial \mu} = -(x_1-1)^3 + x_2^2 \le 0$$

iii) orthogonality condition

$$\mu(-(x_1-1)^3 + x_2^2) = 0$$

iv) non-negativity

$$\mu \ge 0$$

10 pt

(b) $\mu \ne 0$

$2x_2(\mu+1) = 0$ $\mu > 0$ 이므로

$x_2 = 0$ $x_1 = 1$ feasibility

$\to$ $2x_1 - \mu 3 (x_1-1)^2 = 2 \ne 0$ 모순

$\mu = 0$

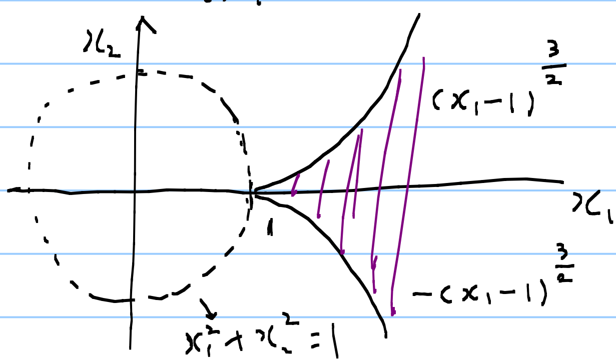
$x_1 = 0$, $x_2 = 0$

$\to$ $-(-1)^3 + 0 \ge 0$ 모순

10 pt

KKT condition 으로는 solution을 구할 수 없다.

$x_2^2 \le (x_1-1)^3 \Rightarrow -(x_1-1)^{\frac{3}{2}} \le x_2 \le (x_1-1)^{\frac{3}{2}}$



5 pt

$(x_1, x_2) = (1, 0)$ 에서 $f_{min} = 1$

KKT condition으로 풀 수 없는 이유

$\rightarrow$ CQ does not hold at $(1, 0)$ 5pt

4. $$SS_T = \sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} \{ (y_i - \hat{y}_i) + (\hat{y}_i - \bar{y}) \}^2$$

$$= \underbrace{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}_{SS_E} + \underbrace{\sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2}_{SS_R} + 2 \sum_{i=1}^{n} (y_i - \hat{y}_i)(\hat{y}_i - \bar{y})$$

5pt

$$e_i = y_i - \hat{y}_i$$

$$\Rightarrow \sum_{i=1}^{n} (\hat{y}_i - \bar{y})(y_i - \hat{y}_i) = \sum_{i=1}^{n} (\hat{y}_i - \bar{y}) e_i$$

5pt

$$L = \underset{\sim}{\varepsilon}^T \underset{\sim}{\varepsilon} = (\underset{\sim}{y} - \underset{\sim}{x}\underset{\sim}{\beta})^T (\underset{\sim}{y} - \underset{\sim}{x}\underset{\sim}{\beta})$$

$(\underset{\sim}{\beta}^T \underset{\sim}{x}^T \underset{\sim}{y} = \underset{\sim}{y}^T \underset{\sim}{x} \underset{\sim}{\beta})$ scalar

$$= \underset{\sim}{y}^T \underset{\sim}{y} - \underset{\sim}{\beta}^T \underset{\sim}{x}^T \underset{\sim}{y} - \underset{\sim}{y}^T \underset{\sim}{x} \underset{\sim}{\beta} + \underset{\sim}{\beta}^T \underset{\sim}{x}^T \underset{\sim}{x} \underset{\sim}{\beta}$$

$$= \underset{\sim}{y}^T \underset{\sim}{y} - 2 \underset{\sim}{\beta}^T \underset{\sim}{x} \underset{\sim}{y} + \underset{\sim}{\beta}^T \underset{\sim}{x}^T \underset{\sim}{x} \underset{\sim}{\beta}$$

to minimize error L

$$\left. \frac{\partial L}{\partial \underset{\sim}{\beta}} \right|_{\underset{\sim}{\beta} = b} = -2 \underset{\sim}{x}^T \underset{\sim}{y} + 2 \underset{\sim}{x}^T \underset{\sim}{x} \underset{\sim}{b} = 0$$

$$\rightarrow (\underset{\sim}{x}^T \underset{\sim}{x}) \underset{\sim}{b} = \underset{\sim}{x}^T \underset{\sim}{y} \quad \text{-----} \quad (1)$$

$$\begin{bmatrix} \boxed{n \quad \sum_{i=1}^{n} x_{i1} \quad \cdots \quad \sum_{i=1}^{n} x_{ik}} \\ \sum_{i=1}^{n} x_{i1} \quad \sum_{i=1}^{n} x_{i1}^2 \quad \cdots \quad \sum_{i=1}^{n} x_{i1} x_{ik} \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ \sum_{i=1}^{n} x_{ik} \quad \sum_{i=1}^{n} x_{ik} x_{i1} \quad \cdots \quad \sum_{i=1}^{n} x_{ik}^2 \end{bmatrix} \boxed{\begin{Bmatrix} b_0 \\ b_1 \\ \vdots \\ b_k \end{Bmatrix}} = \begin{Bmatrix} \boxed{\sum_{i=1}^{n} y_i} \\ \sum_{i=1}^{n} x_{i1} y_i \\ \vdots \\ \sum_{i=1}^{n} x_{ik} y_i \end{Bmatrix}$$

$$\Rightarrow n b_0 + \sum_{j=1}^{k} b_j \sum_{i=1}^{n} x_{ij} = \sum_{i=1}^{n} y_i \quad \text{----} \quad (2)$$

$$\hat{\underset{\sim}{y}} = \underset{\sim}{x} \underset{\sim}{b} \quad \text{-------} \quad (3)$$

20pt

$$\sum_{i=1}^{n}(\hat{y}_i - \bar{y})e_i = 0$$ 증명

$$\sum_{i=1}^{n}\hat{y}_i e_i = \hat{\mathbf{y}}^T\mathbf{e} = (\mathbf{X}\mathbf{b})^T(\mathbf{y} - \hat{\mathbf{y}}) \quad \leftarrow (3)$$

$$= \mathbf{b}^T\mathbf{X}^T\mathbf{y} - \mathbf{b}^T\mathbf{X}^T\mathbf{X}\mathbf{b}$$

$$= \mathbf{b}^T(\mathbf{X}^T\mathbf{y} - \mathbf{X}^T\mathbf{X}\mathbf{b}) \quad \leftarrow (1)$$

$$= 0$$

10 pt

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n}(y_i - \hat{y}_i)$$

$$= \sum_{i=1}^{n}\left\{y_i - \left(b_0 + \sum_{j=1}^{k} b_j x_{ij}\right)\right\}$$

$$= \sum_{i=1}^{n} y_i - \left(n b_0 + \sum_{j=1}^{k} b_j \sum_{i=1}^{n} x_{ij}\right) \quad \leftarrow (2)$$

$$= 0$$

$$\Rightarrow \sum_{i=1}^{n}\bar{y}e_i = 0 \Rightarrow \sum_{i=1}^{n}(\hat{y}_i - \bar{y})e_i = 0$$

10pt

$$\Rightarrow SS_T = SS_R + SS_E$$