2016-1 Computational Structural Analysis Final Exam and Solution

※ 풀이과정을 빠짐없이 적으시오. 모든 과정은 부분점수가 있습니다.

1. Prove the stiffness matrix of the beam (30 pts)

$$K=\frac{H_{33}^c}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$where\ v(x)=a_1x^3+a_2x^2+a_3x+a_4$$

$$Q=\{F_{1y}\,M_1\,F_{2y}\,M_2\}^T\ q=\{v_1\,\phi_1\,v_2\,\phi_2\}^T$$

$$F_{1y}=V(0)=H_{33}^c\frac{d^3v(0)}{dx^3},\ M_1=-M(0)=-H_{33}^c\frac{d^2v(0)}{dx^2}$$

$$F_{2y}=-V(L)=-H_{33}^c\frac{d^3v(L)}{dx^3},\ M_2=M(L)=H_{33}^c\frac{d^2v(L)}{dx^2}$$

solution

$$v(x)=a_1x^3+a_2x^2+a_3x+a_4$$
$$v(0)=v_1=a_4$$
$$\phi(0)=\phi_1=a_3$$
$$v(L)=a_1L^3+a_2L^2+a_3L+a_4=a_1L^3+a_2L^2+\phi_1L+v_1\ \ ......(1)$$
$$\phi(L)=\phi_2=3a_1L^2+2a_2L+\phi_1\ \ ......(2)$$

$$\frac{v_2}{L^3}=a_1+\frac{a_2}{L}+\frac{\phi_1}{L^2}+\frac{v_1}{L^3}$$

$$a_1=\frac{v_2}{L^3}-\frac{a_2}{L}-\frac{\phi_1}{L^2}-\frac{v_1}{L^3}$$

$$\phi_2=\frac{3v_2}{L}-3a_2L-3\phi_1-\frac{3v_1}{L}+2a_2L+\phi_1=\frac{3}{L}(v_2-v_1)-2\phi_1-a_2L$$

$$\therefore a_2=\frac{3}{L^2}(v_2-v_1)-\frac{1}{L}(2\phi_2+\phi_2)$$

$$a_1=\frac{2}{L^3}(v_1-v_2)+\frac{1}{L^2}(\phi_1+\phi_2)$$

$$v(x)=[\frac{2}{L^3}(v_1-v_2)+\frac{1}{L^2}(\phi_1+\phi_2)]^3+[\frac{3}{L^3}(v_2-v_1)-\frac{1}{L}(2\phi_1+\phi_2)]x^2+\phi_1x+v_1$$

$$F_{1y} = H_{33}^c \frac{d^3 v(0)}{dx^3} = \frac{H_{33}^c}{L^3}(12v_1 + 6Lphi_1 - 12v_2 + 6Lphi_2)$$

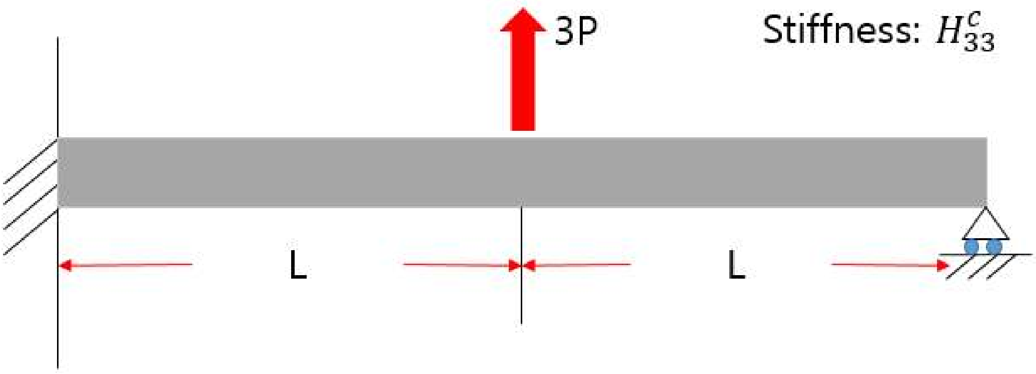
$$M_1 = - H_{33}^c \frac{d^2 v(0)}{dx^2} = \frac{H_{33}^c}{L^3}(6Lv_1 + 4L^2\phi_1 - 6Lv_2 + 2L^2\phi_2)$$

$$F_{2y} = - H_{33}^c \frac{d^3 v(L)}{dx^3} = - \frac{H_{33}^c}{L^3}(-12v_1 - 6Lphi_1 + 12_{v2} - 6Lphi_2)$$

$$M_2 = H_{33}^c \frac{d^2 v(L)}{dx^2} = \frac{H_{33}^c}{L^3}(6Lv_1 + 2L^2\phi_1 - 6Lv_2 + 4L^2\phi_2)$$

[coefficient 정의 각 2점 총 8점, v(x) 정의 3점, Force 및 Moment 각 1점 총 4점]

2. Solve the problem (40 pts)



(1) Construct the stiffness matrix of the beam **(10 pts)**

$$k_{e1} = \frac{H_{33}^c}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}, k_{e2} = \frac{H_{33}^c}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

k

[elemental stiffness 5점, global stiffness 5점]

(2) Solve the deflection at x=L and its rotations at x=L, x=2L **(30pts)**

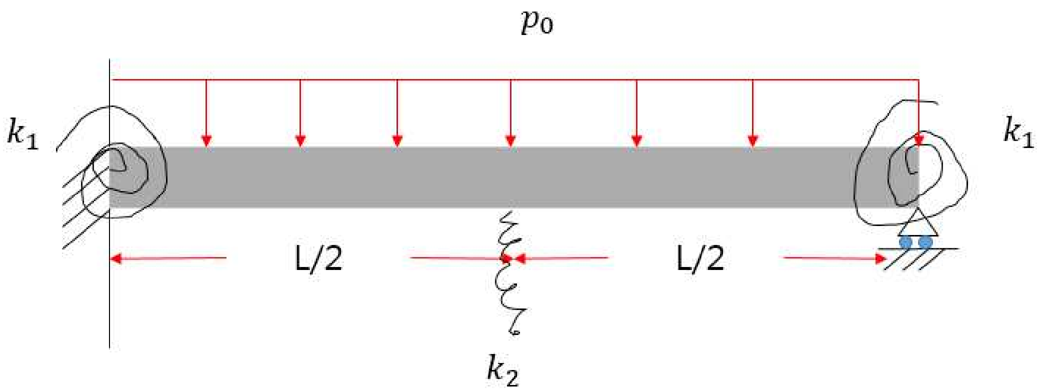
$$k_{uu} = \frac{H_{33}^c}{L^3}\begin{bmatrix} 24 & 0 & 6L \\ 0 & 8L^2 & 2L^2 \\ 6L & 2L^2 & 4L^2 \end{bmatrix}, q_{uu} = \begin{bmatrix} v_2 \\ \phi_2 \\ \phi_3 \end{bmatrix}, \quad Q_{uu} = \begin{bmatrix} 3P \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{Bmatrix} v_2 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = k_{uu}^{-1} q_{uu} = \frac{1}{384 H_{33}^c L} \begin{bmatrix} 28L^4 & 12L^3 & -48L^3 \\ 12L^3 & 60L^3 & -48L^2 \\ -48L^3 & -48L^2 & 192L^2 \end{bmatrix} \begin{Bmatrix} 3P \\ 0 \\ 0 \end{Bmatrix}$$

$$\therefore v_2 = \frac{21L^3P}{96H_{33}^c},\ \phi_2 = \frac{3L^2P}{32H_{33}^c},\ \phi_3 = -\frac{3L^2P}{8H_{33}^c}$$

[k_uu stiffness matrix 5점, q_uu 5점, load array 5점, deflection and rotation 각 5점]

3. Solve the problem (30 pts)



(1) establish the governing equation and its boundary condition for exact solution (strong statement) **(15 pts)**

$$\text{for } 0 \le x \le \frac{L}{2},\ H_{33}^c u_{2_a}^{(4)} = p_o$$

$$\text{for } \frac{L}{2} \le x \le L,\ H_{33}^c u_{2_b}^{(4)} = p_o$$

$$BC1)\ spring1: H_{33}^c u_{2_a}^{(2)}(0) - k_1 u_{2_a}^{(1)}(0) = 0$$

$$BC2)\ spring3: H_{33}^c u_{2_b}^{(2)}(L) - k_1 u_{2_b}^{(1)}(L) = 0$$

$$BC3)\ spring2: -H_{33}^c u_{2_b}^{(3)}(\frac{L}{2}) + H_{33}^c u_{2_a}^{(3)}(\frac{L}{2}) = k_2 u_{2_a}(\frac{L}{2})$$

$$BC4) \geq ometry: u_{2_a}(0) = 0, u_{2_b}(L) = 0, u_{2_a}^{(0,1\,\text{and}\,2)}(\frac{L}{2}) = u_{2_b}^{(0,1\text{and}2)}(\frac{L}{2})$$

[정의역 정의 1점, 각 Boundary condition 2점씩 총 14점]

(2) Construct a 2-term trigonometric approximate solution by using defined [H].

Describe [K]{q} = [Q], do not perform the inverse [K] matrix. **(15 pts)**

$$[H] = \begin{bmatrix} \sin\pi\eta \\ \sin 3\pi\eta \end{bmatrix}$$

$$H(\eta) = \{\sin\pi\eta \;\; \sin 3\pi\eta\}$$
$$B(\eta) = -\frac{\pi^2}{L^2}\{\sin\pi\eta \;\; 9\sin 3\pi\eta\}$$
$$B_s(\eta) = \frac{\pi}{L}\{\cos\pi\eta \;\; 3\cos\pi\eta\}$$

$$k_b = H_{33}^c \int_0^1 B(\eta)B(\eta)^T L dta = \frac{H_{33}^c}{L^3}\pi^4 \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3^4}{2} \end{bmatrix}$$

$$k_{s1} = k_1 B_s(0) B_s(0)^T \frac{1}{L} = k_1 \frac{\pi^2}{L^3}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

$$k_{s3} = k_{s1}$$

$$k_{s2} = k_2 H(\frac{1}{2}) H(\frac{1}{2})^T = k_2 \frac{1}{L^3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

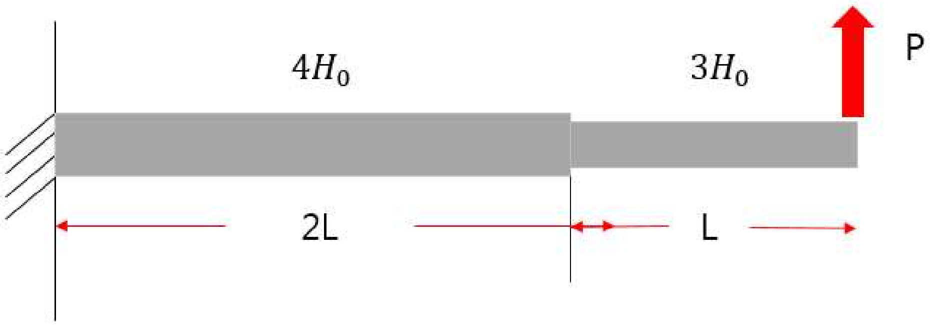
$$replace\, k_1 = \overline{k}_1 H_{33}^c,\, k_2 = \overline{k}_2 H_{33}^c$$

$$k$$

$$Q = p_o \int_0^1 H(\eta) L deta = p_o \int_0^1 \begin{matrix} \sin\pi\eta \\ \sin 3\pi\eta \end{matrix} L deta = p_o L \begin{bmatrix} -\frac{1}{\pi}\cos\pi\eta \\ -\frac{1}{3\pi}\cos 3\pi\eta \end{bmatrix} = p_o L \begin{bmatrix} \frac{2}{\pi} \\ \frac{2}{3\pi} \end{bmatrix}$$

[B matrix 3점, bending stiffness matrix 3점, spring matrix(1 and 3) 각 3점씩 6점, load array 3점]

4. Solve the problem (50 pts)



(1) Construct the stiffness matrix of the beam **(15 pts)**

$$\eta = \frac{x}{L},\ \ H(\eta) = \{\eta^2\ \ \eta^3\}^T$$
$$B(\eta) = \{2\ \ 6\eta\}^T$$

$$k = \int_0^2 B(\eta) 4H_o B(\eta)^T L\, d\eta + \int_2^3 B(\eta) 3H_o B(\eta)^T L\, d\eta$$

$$= \frac{4H_o}{L^3}\begin{bmatrix} 8 & 16 \\ 16 & 96 \end{bmatrix} + \frac{3H_0}{L^3}\begin{bmatrix} 4 & 30 \\ 30 & 228 \end{bmatrix} = \frac{4H_o}{L^3}\begin{bmatrix} 11 & 46 \\ 46 & 267 \end{bmatrix}$$

[B matrix 3점, 각 stiffness matrix 3점씩 총 6점, 답 6점]

(2) Construct the load array of the applied force **(5 pts)**

$$Q(\eta) = PH(\eta = 3) = P\begin{Bmatrix} 9 \\ 27 \end{Bmatrix}$$

[Q matrix 정의하면 2점, 답 3점]

(3) Solve the deflection at x=3L **(15pts)**

$$q = K^{-1}Q = \frac{PL^3}{4H_0}\frac{1}{2937-2116}\begin{bmatrix} 267 & -46 \\ -46 & 11 \end{bmatrix} P\begin{Bmatrix} 9 \\ 27 \end{Bmatrix}$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \frac{PL^3}{3284}\begin{bmatrix} 1161 \\ -117 \end{bmatrix}$$

$$\therefore u_2(\eta) = H(\eta)^T q = \frac{PL^3}{3284H_0}(1161\eta^2 - 117\eta^3)$$

$$u(\eta=3) = \frac{3645}{1642}\frac{PL^3}{H_o} \simeq 2.22\frac{PL^3}{H_o}$$

[Inverse 정의하면 3점, q matrix 변수 맞으면 6점, 답 6점점]

(4) Compare your solution at the tip with the exact solution computed using the unit load method. When you calculate the problem, round off to the nearest thousandths. (Ex 0.0396 = 0.04) **(15pts)**

hint) $\triangle = \int_a^b \frac{\widehat{M}M}{H}dx$

$$\triangle = \int_0^{3L}\frac{\widehat{M_3}M_3}{H_o}dx \quad where \quad M_3 = P(3L-x),\ \widehat{M}_3 = (3L-x)$$

$$= \int_0^{2L}\frac{P(3L-x)^2}{4H_o}dx + \int_{2L}^{3L}\frac{P(3L-x)^2}{3H_o}dx$$

$$= \frac{P}{4H_o}\frac{26}{3}L^3 + \frac{P}{3H_o}\frac{1}{3}L^3 = \frac{41P}{18H_o}L^3 \simeq 2.27\frac{PL^3}{H_o}$$

$$error(\%) = \frac{u(\eta=3)_{exact} - u(\eta=3)_{\approx}}{u(\eta=3)_{exact}} \times 100 = 2.20\%$$

[Moment 맞으면 각 3점씩, 답 9점]