Fusion Plasma Theory I

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Final Exam

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1. [**MHD Equilibrium in Pinch**] Consider a cylindrical circular pinch device where plasma is confined by axial magnetic field in the z-direction only, with the fluid pressure given by $p = p_0 e^{-r^2/a^2}$. Let the magnetic field in vacuum be $B_\infty = B_z(r \to \infty)$.

(a) Find $B_z(r)$ and $j_\theta(r)$ in equilibrium.

(b) Estimate the conventional $\beta_t = 2\mu_0 \langle p \rangle / B_0^2$ in terms of $\hat{\beta} \equiv 2\mu_0 p_0 / B_\infty^2$. Here $\langle\ \rangle$ is the volume average and B_0 is the magnetic field at the center $r = 0$.

2. [**Grad-Shafranov Equation**] In this problem you will be asked to describe some of basic features contained in MHD equilibrium for tokamak geometry.

(a) First just write down the equation for tokamak equilibrium that *Grad* and *Shafranov* derived for poloidal flux $\psi = RA_\phi$, where R is the major radius and A_ϕ is the toroidal component of vector potential. Explain each term as much as you know.

(b) Imagine a solution rescaling by $\psi \to \alpha\psi$. How would you change the two flux functions $p(\psi)$ and $F(\psi) = RB_\phi$ where B_ϕ is the toroidal field, to make your new ψ still satisfying the Grad-Shafranov equation? Is this rescaled solution changing any of β_t or β_p, or $q(\psi)$, or plasma inductance?

(c) The rescaling in (b) results in a change of toroidal plasma current j_ϕ. Can you imagine another simple way to construct a different solution similar to (b) but without changing j_ϕ? This method is frequently used to numerically change $q(\psi)$ without changing j_ϕ or $p(\psi)$, although unfortunately it changes β_t.

3. [**Magnetic Islands**] Suppose a perturbed magnetic field $\vec{B}_1 = \hat{x}\delta\sin(ky)$ is added onto an equilibrium magnetic field $\vec{B} = \hat{y}B'_{y0}x + \hat{z}B_0$. Show that the field lines projected in (x, y) compose magnetic islands, and that the width of the magnetic islands is proportional to $\sqrt{\delta}$.

4. [Energy Principle with Simple Force] Suppose a MHD equilibrium given by $\rho\vec{g} = \vec{\nabla}p$ where ρ is plasma density, p is plasma pressure, and $\vec{g}$ is gravity vector.

(a) Show that the potential energy due to a fluid displacement $\vec{\xi}$ is given by

$$\delta W = \frac{1}{2}\int d\vec{x}\left[\gamma p(\vec{\nabla}\cdot\vec{\xi})^2 + (\vec{\xi}\cdot\vec{\nabla}p)(\vec{\nabla}\cdot\vec{\xi}) + (\vec{\xi}\cdot\vec{g})\vec{\nabla}\cdot(\rho\vec{\xi})\right]. \tag{1}$$

(b) What value should plasma compressibility $\vec{\nabla}\cdot\vec{\xi}$, take to minimize δW?

(c) For stability, what sign should $\vec{\nabla}(p/\rho^\gamma)$ take with respect to $\vec{g}$? This is called *Schwarzschild* condition.

(d) For low pressure cases, the equilibrium by gravity is often described by $\rho\vec{g} = \vec{\nabla}p_B$ where $p_B = B^2/2\mu_0$ and with $\gamma = 2$. Then one can assess the stability condition just by replacing p by p_B in (b) and (c). Show that this can be justified since the perturbed magnetic field $\vec{Q} = \vec{\nabla}\times(\vec{\xi}\times\vec{B})$ becomes zero for $\vec{\xi}$ minimizing δW, in particular for the interchange instability where line-bending term is not allowed, i.e. $(\vec{B}\cdot\vec{\nabla})\vec{\xi} = 0$.

5. [Electrostatic Drift Wave] The basic dispersion relation for the electron drift wave without any dissipation is given by

$$\omega^2 - \omega_{*e}\omega - k_z^2 c_s^2 = 0, \tag{2}$$

where $\omega_{*e} = k_y v_{*e}$, $v_{*e} = -(T_{e0}/en_{e0}B_0)dn_{e0}/dx$, $c_s = (T_{e0}/M)^{1/2}$, with electric charge e, ion mass M, the wave numbers in y and z directions, k_y and k_z, through constant magnetic field $\vec{B} = B_0\hat{z}$, constant electron temperature T_{e0}, but inhomogeneous electron density $n_{e0}(x)$, without any flow in equilibrium.

Derive the dispersion relation simply by linearizing

- Electron parallel momentum balance ignoring electron inertia and electron temperature fluctuation ($T_{e1} = 0$)
- Ion parallel momentum balance entirely ignoring ion pressure ($T_{i0,i1} = 0$)
- Ion continuity equation

Assume quasi-neutrality condition ($n_e = n_i$). Another critical assumption is that the perturbation is electrostatic, i.e. without magnetic fluctuation $\vec{B}_1 = 0$, and the perturbed electric field $\vec{E}_1 = -\vec{\nabla}\phi_1$. Note that the perturbed flow becomes $u_1 = (\hat{z}\times\vec{\nabla}\phi_1)/B_0 + u_{\parallel 1}\hat{z}$, constituting a closed set of equations only with n_{e1}.

Possibly useless information:

$$\vec{A}\cdot(\vec{B}\times C)=\vec{C}\cdot(\vec{A}\times\vec{B})=\vec{B}\cdot(\vec{C}\times\vec{A})$$
$$\vec{\nabla}\cdot(f\vec{A})=f(\vec{\nabla}\cdot\vec{A})+\vec{A}\cdot\vec{\nabla}f$$
$$\vec{\nabla}\times(\vec{A}\times\vec{B})=\vec{A}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{A})+(\vec{B}\cdot\vec{\nabla})\vec{B}-(\vec{A}\cdot\vec{\nabla})\vec{B}$$

In cylindrical coordinates,

$$(\vec{\nabla}\times\vec{A})_r=\frac{1}{r}\frac{\partial A_z}{\partial\theta}-\frac{\partial A_\theta}{\partial z}$$
$$(\vec{\nabla}\times\vec{A})_\theta=\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}$$
$$(\vec{\nabla}\times\vec{A})_z=\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)-\frac{1}{r}\frac{\partial A_r}{\partial\theta}$$