1. Equilibrium $\frac{d}{dr}\left(p + \frac{B_z^2}{2\mu_0}\right) = 0$

(a) $$p_0 e^{-r^2/a^2} + \frac{B_z^2}{2\mu_0} = \frac{B_\infty^2}{2\mu_0}$$

$$\rightarrow \quad B_z = B_\infty \left(1 - \hat{\beta} e^{-r^2/a^2}\right)^{\frac{1}{2}}$$

Ampere's law $\mu_0 \vec{j} = \vec{\nabla} \times (B_z(r) \hat{z})$

$$\mu_0 j_\theta = -\frac{\partial B_z}{\partial r}$$

$$\rightarrow \quad j_\theta = -\frac{B_\infty \hat{\beta}}{\mu_0 a^2} r \left(1 - \hat{\beta} e^{-r^2/a^2}\right)^{-\frac{1}{2}}$$

(b) $$B_0 = B_\infty (1 - \hat{\beta})^{\frac{1}{2}}$$

$$\langle p \rangle = 2\pi \int_0^a r\, p(r)\, dr \Big/ \pi a^2$$

$$= p_0 \int_0^a 2(r/a^2)\, e^{-r^2/a^2}\, dr$$

$$= p_0 \left[-e^{-r^2/a^2}\right]_0^a = p_0 \left(1 - \frac{1}{e}\right)$$

$$\therefore \quad \beta_t = \frac{2\mu_0 p_0 \left(1 - \frac{1}{e}\right)}{B_\infty^2 (1 - \hat{\beta})} = \left(1 - \frac{1}{e}\right) \frac{\hat{\beta}}{1 - \hat{\beta}}$$

2. (a) G-S equation

$$\Delta^* \psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \psi}{\partial R}\right) + \frac{\partial^2 \psi}{\partial z^2} = -\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}$$

$p(\psi)$, $F(\psi) = RB_\phi$: Flux fcns.

Note
$$\vec{B} = \frac{1}{R}\vec{\nabla}\psi \times \hat{\phi} + \frac{1}{R}F(\psi)\hat{\phi} = \vec{B_p} + \vec{B_\phi}$$
$$\mu_0 \vec{J} = \frac{1}{R}\frac{dF}{d\psi}\vec{\nabla}\psi \times \hat{\phi} - \frac{1}{R}\Delta^*\psi\hat{\phi} = \vec{J_p} + \vec{J_\phi}$$
$$q = \frac{\oint \vec{B}\cdot\vec{\nabla}\phi \frac{dl_p}{B_p}}{\oint \vec{B}\cdot\vec{\nabla}\theta \frac{dl_p}{B_p}}, \quad l_i = \frac{4}{\mu_0 R I^2}\int \frac{B_p^2}{2\mu_0} d\vec{x}$$

toroidal current $\sim \vec{J}_\phi$

(b) $\psi \to \alpha\psi$, $p \to \alpha^2 p$ $F \to \alpha F$

makes $\alpha \Delta^* \psi = \alpha\left(-\mu_0 R^2 \frac{dp}{d\psi} - \frac{1}{2}\frac{dF^2}{d\psi}\right)$

without geometry or profile changes.

Then, $\beta_t \sim \frac{p}{B_\phi^2} \sim \frac{p}{F^2} \to \frac{\alpha^2 p}{\alpha^2 F^2} \sim \beta_t$

$$\beta_p \sim \frac{p}{B_p^2} \sim \frac{p}{(\vec{\nabla}\psi)^2} \to \frac{\alpha^2 p}{\alpha^2 B_p^2} \sim \beta_p$$

$$q \sim \frac{B_\phi}{B_p} \sim \frac{F}{|\vec{\nabla}\psi|} \to \frac{\alpha F}{\alpha B_p} \sim q$$

$$l_i \sim \frac{B_p^2}{I^2} \sim \frac{(\vec{\nabla}\psi)^2}{(\Delta^*\psi)^2} \to \frac{\alpha^2 B_p^2}{\alpha^2 I^2} \sim l_i$$

(c) (b) makes $j_\phi \sim \Delta^* \psi \to \alpha \Delta^* \psi \sim \alpha j_\phi$

So, total current I is changed.

For another method, note G-S equation depends on p' and $(F^2)'$

$\boxed{\text{So, for example } F^2 \to F^2 + \gamma^2}$

will not change the solution ψ.

nor $j_\phi \sim \Delta^* \psi$.

but now $R^2 B_{\phi,new}^2 = R^2 B_{\phi,old}^2 + \gamma^2$

$\therefore B_{\phi,new} \uparrow,\ q \uparrow$.

3. Field line trajectory (x, y)

$$\frac{dx}{B_x} = \frac{dy}{B_y} \qquad \frac{dx}{\delta \sin(ky)} = \frac{dy}{x B_{y0}'}$$

Integrate. $\frac{1}{2}x^2 = c - \frac{\delta}{k B_{y0}'} \cos(ky)$

$$x = \pm w \sqrt{c^2 - \sin^2(ky/2)}$$

$$w = \left(\frac{-4\delta}{k B_{y0}'}\right)^{\frac{1}{2}} \quad (k B_{y0}' < 0)$$

$\propto \delta^{\frac{1}{2}}$: half-width of magnetic islands

4. Let $\vec{F} = \rho_1 \vec{g} - \vec{\nabla} P_1$

(a) perturbed velocity $\vec{u}_1 = \frac{\partial \vec{\xi}}{\partial t}$

perturbed pressure $P_1 = -\vec{\xi}\cdot\vec{\nabla}P - \gamma P(\vec{\nabla}\cdot\vec{\xi})$

perturbed density $\rho_1 = -\vec{\nabla}\cdot(\rho\vec{\xi})$

perturbed force $\vec{F}[\vec{\xi}] = -\vec{g}\,\vec{\nabla}\cdot(\rho\vec{\xi}) + \vec{\nabla}\left(\vec{\xi}\cdot\vec{\nabla}P + \gamma P(\vec{\nabla}\cdot\vec{\xi})\right)$

perturbed energy $\delta W = -\frac{1}{2}\int \vec{\xi}\cdot\vec{F}[\vec{\xi}]\,d\vec{x}$

$$= \frac{1}{2}\int (\vec{\xi}\cdot\vec{g})\vec{\nabla}\cdot(\rho\vec{\xi})$$

$$-\frac{1}{2}\int\left[\cancel{\vec{\nabla}\cdot\left\{\vec{\xi}\left(\vec{\xi}\cdot\vec{\nabla}P + \gamma P(\vec{\nabla}\cdot\vec{\xi})\right)\right\}} - (\vec{\nabla}\cdot\vec{\xi})(\vec{\xi}\cdot\vec{\nabla}P) - \gamma P(\vec{\nabla}\cdot\vec{\xi})^2\right]d\vec{x}$$

surface term

(b) $$\delta W = \frac{1}{2}\int\left[\gamma P(\vec{\nabla}\cdot\vec{\xi})^2 + \left\{\vec{\xi}\cdot\vec{\nabla}P + \rho(\vec{\xi}\cdot\vec{g})\right\}(\vec{\nabla}\cdot\vec{\xi}) + (\vec{\xi}\cdot\vec{g})(\vec{\xi}\cdot\vec{\nabla}\rho)\right]d\vec{x}$$

using $\vec{g} = \vec{\nabla}P/\rho$,

$$= \frac{1}{2}\int\left[\gamma P(\vec{\nabla}\cdot\vec{\xi})^2 + 2(\vec{\xi}\cdot\vec{\nabla}P)(\vec{\nabla}\cdot\vec{\xi}) + \frac{(\vec{\xi}\cdot\vec{\nabla}P)(\vec{\xi}\cdot\vec{\nabla}\rho)}{\rho}\right]d\vec{x}$$

$$= \frac{1}{2}\int\left[\gamma P\left(\vec{\nabla}\cdot\vec{\xi} + \frac{\vec{\xi}\cdot\vec{\nabla}P}{\gamma P}\right)^2 - \frac{(\vec{\xi}\cdot\vec{\nabla}P)^2}{\gamma P} + \frac{(\vec{\xi}\cdot\vec{\nabla}P)(\vec{\xi}\cdot\vec{\nabla}\rho)}{\rho}\right]d\vec{x}$$

$\therefore$ $\vec{\nabla}\cdot\vec{\xi} = -\frac{\vec{\xi}\cdot\vec{\nabla}P}{\gamma P}$ minimizing δW

(c) with the choice for $\vec{\nabla}\cdot\vec{\xi}$ in (b)

$$\delta W = \frac{1}{2}\int \left[-\frac{(\vec{\xi}\cdot\vec{\nabla}P)^2}{\gamma P} + \frac{(\vec{\xi}\cdot\vec{\nabla}P)(\vec{\xi}\cdot\vec{\nabla}\rho)}{\rho} \right] d\vec{x}$$

$$= \frac{1}{2}\int \left[-\left(\frac{\vec{\xi}\cdot\vec{\nabla}P}{\gamma}\right) \vec{\xi}\cdot\vec{\nabla}\ln\left(\frac{P}{\rho^\gamma}\right) \right] d\vec{x}$$

$\therefore$ $\nabla(P/\rho^\gamma)$ should have the opposite sign of $\vec{\nabla}P$ (or $\vec{g}$) for stability ($\delta W > 0$)

(d) The revised stability condition is

$\vec{\nabla}(P_B/\rho^2) \propto \vec{\nabla}(B/\rho)$ to have the opposite sign of $\vec{g}$

and $\vec{\nabla}\cdot\vec{\xi} = -\dfrac{\vec{\xi}\cdot\vec{\nabla}P_B}{2P_B} = -\dfrac{\vec{\xi}\cdot\vec{\nabla}B}{B}$

This makes

$$\vec{Q} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}) = \cancel{\vec{\xi}(\vec{\nabla}\cdot\vec{B})} - \vec{B}(\vec{\nabla}\cdot\vec{\xi}) + \cancel{(\vec{B}\cdot\vec{\nabla})\vec{\xi}} - (\vec{\xi}\cdot\vec{\nabla})\vec{B}$$

interchange

$$= -\vec{B}(\vec{\nabla}\cdot\vec{\xi}) - (\vec{\xi}\cdot\vec{\nabla})\vec{B} = 0$$

justifying the assumption a "posteriori".

5. Electron parallel momentum balance

$$\cancel{\left(m_e n_e \frac{du_{\parallel,e}}{dt}\right)_1} = e n_{e0} \nabla_\parallel \phi_1 - \nabla_\parallel P_{e1}$$

$$e n_{e0} \nabla_\parallel \phi_1 = T_{e0} \nabla_\parallel n_{e1} \qquad (1)$$

Ion parallel momentum balance

$$M n_{e0} \frac{\partial u_\parallel}{\partial t} = - e n_{e0} \nabla_\parallel \phi_1 - \cancel{\nabla_\parallel P_{i1}} \qquad (2)$$

Ion continuity equation

$$\frac{\partial n_{e1}}{\partial t} + \frac{(\hat{z}\times\vec{\nabla}\phi_1)}{B_0}\cdot\vec{\nabla} n_{e0} + n_{e0} \nabla_\parallel u_\parallel = 0 \qquad (3)$$

since $\nabla_\parallel n_{e0} = 0$, $n_{e0} = n_{e0}(x)$ since $\vec{\nabla}\cdot\left(\frac{\hat{z}\times\vec{\nabla}\phi_1}{B_0}\right) = 0$

Let $\phi_1 (u_\parallel, n_{e1}) \propto e^{i(k_y \hat{y} + k_z \hat{z} - \omega t)}$

$(1) \rightarrow i e n_{e0} k_z \phi_1 = i T_{e0} k_z n_{e1} \qquad n_{e1} = e n_{e0} \phi_1 / T_{e0}$

$(2) \rightarrow -i\omega M n_{e0} u_\parallel = -i e n_{e0} k_z \phi_1 \qquad u_\parallel = \dfrac{e k_z}{\omega M}\phi_1$

$(3) \rightarrow -i\omega n_{e1} - i\dfrac{k_y \phi_1}{B_0}\left(\dfrac{dn_{e0}}{dx}\right) + i k_z n_{e0} u_\parallel = 0$

$(1),(2) \rightarrow (3) \quad \left(\dfrac{-i\omega e n_{e0}}{T_{e0}} - i\dfrac{k_y}{B_0}\left(\dfrac{dn_{e0}}{dx}\right) + \dfrac{i k_z^2 e n_{e0}}{\omega M} = 0\right) \times \dfrac{i\omega T_{e0}}{e n_{e0}}$

$$\rightarrow \omega^2 + \omega \frac{k_y T_{e0}}{e n_{e0} B_0}\frac{dn_{e0}}{dx} - k_z^2 \frac{T_{e0}}{M} = 0$$

$$\rightarrow \omega^2 - \omega_{*e}\,\omega - k_z^2 c_s^2 = 0$$