Engineering Economic Analysis

2019 Final solution

Problem 1

(a) Yes.

Let $\tilde{x}^*$ be the optimal and assume that $p \cdot f\left(\tilde{x}^*\right) - \tilde{w} \cdot \tilde{x}^* = \pi^* > 0$

Scale up production by t>1

Since CRS, $f\left(t\tilde{x}^*\right) = tf\left(\tilde{x}^*\right)$

Then, $p \cdot f\left(\tilde{x}^*\right) - \tilde{w} \cdot \left(t\tilde{x}^*\right) = t\left\{p \cdot f\left(\tilde{x}^*\right) - \tilde{w} \cdot \tilde{x}^*\right\} = t\pi^* > \pi^*$

Contradiction!

(b) Yes.

F.O.C. if profit maximization of a competitive firm is $p = c'\left(y(p)\right)$

Differentiating F.O.C w.r.t p, $1 = c''\left(y(p)\right) \cdot y'(p)$

By S.O.C, we know that $c''(y) > 0$

Thus, $y'(p) > 0$

(c) No

Opt. Profit-Max of third-degree price discrimination in two markets,

$$\max\ \pi = R_1(q_1) + R_2(q_2) - c(q_1 + q_2) \quad \text{when} \quad R_i(q_i) = \text{revenue for } i \text{ market}$$

F.O.C. $R_1'(q_1) - c'(q_1 + q_2) = 0$

$R_2'(q_2) - c'(q_1 + q_2) = 0$

Hessian matrix $H = \begin{pmatrix} R_1'' - c'' & -c'' \\ -c'' & R_2'' - c'' \end{pmatrix}$ should be ND.

Therefore, $R_i'' < c''$ for $i = 1, 2$

$\therefore$ MR in each market must be increasing less rapidly than MC for the output as a whole.

Problem 2

(a)

The average cost curve is $\frac{c(\mathbf{w},y)}{y} = \frac{y^2+1}{y}w_1 + \frac{y^2+2}{y}w_2$

Since it is convex, it has a unique minimum at $y_m = \sqrt{\frac{w_1/w_2+2}{w_1/w_2+1}}$.

The derivative of y_m with respect to w_1/w_2 is negative, so the minimum of the average cost shifts to the left (right) as w_1/w_2 increases (decreases).

(b)

$$\frac{\partial c(\mathbf{w},y)}{\partial y} = 2y(w_1+w_2)$$

$$\therefore y(p) = \frac{p}{2(w_1+w_2)}$$

(c)

Y(p) = arbitrarily large amount (if p>y_m(w1+w2))
any amount (if p=y_m(w1+w2))
0 (otherwise)

Problem 3

a)

Since average cost of j's plant is $AC_j(q_j) = \alpha + \beta_j q_j$,

Total cost is as follows $TC_j(q_j) = \alpha q_j + \beta_j q_j^2$.

Note that cost-min (q_1^*, q_2^*) is same with the profit-max (q_1^*, q_2^*).

We know that the profit-max condition is $MC_1(q_1^*) = MC_2(q_2^*)$.

Thus,

$\alpha + 2\beta_1 q_1 = \alpha + 2\beta_2 q_2 \ldots(1)$

$q_1 + q_2 = Q \ldots(2)$

By solving (1) and (2), optimum quantities can be got as below

$$q_1^* = \frac{\beta_2}{\beta_1 + \beta_2} Q, \quad q_2^* = \frac{\beta_1}{\beta_1 + \beta_2} Q$$

b)

If $\beta_1 < 0$, $\beta_2 > 0$,

$MC_1(q_1) = \alpha + 2\beta_1 q_1 < MC_2(q_2) = \alpha + 2\beta_2 q_2$ for all $q_1, q_2 > 0$

Thus it is optimal to distribute all Q to a plant 1.(reminder : $q_1 + q_2 = Q$)

$\therefore q_1^* = Q, \; q_2^* = 0$

c)

Solved as problem 3-a)

Note that cost-min (q_1^*, q_2^*, q_3^*) is same with the profit-max (q_1^*, q_2^*, q_3^*).

We know that the profit-max condition is $MC_1(q_1^*) = MC_2(q_2^*) = MC_3(q_3^*)$.

Thus,

$\alpha + 2\beta_1 q_1 = \alpha + 2\beta_2 q_2 = \alpha + 2\beta_3 q_3 \ldots(1)$

$q_1 + q_2 + q_3 = Q \ldots(2)$

By solving (1) and (2), optimum quantities can be got as below

$$q_1^* = \frac{\beta_2\beta_3}{\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1} Q, \quad q_2^* = \frac{\beta_1\beta_3}{\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1} Q, \quad q_3^* = \frac{\beta_1\beta_2}{\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1} Q$$

Problem 4

(a)

For firm i

$$\max_{q_i} \ (a - bq_i - b\sum_{j\neq i} q_j)q_i - (F + cq_i)$$

$F.O.C$

$$a - 2bq_i - b\sum_{j\neq i} q_j - c = 0$$

By symmetry in equal output, i.e., $q_1 = q_2 = \cdots = q_N$,

$F.O.C$ becomes

$$a - bq_i - bNq_i - c = 0$$

$\therefore q_i^* = \frac{a-c}{b(N+1)}$ for $\forall i$

Equilibrium market price is $p^* = a - b(N - \frac{a-c}{b(N+1)}) = \frac{a+\mathrm{Nc}}{N+1}$

(b)

In a long run of competitive market, firms must make no profits, $\pi = 0$.

Thus,

$$\pi_i = (a - bQ)q_i - (F + cq_i)$$

$$= \frac{(a-c)^2}{b(N+1)^2} - F = 0$$

Thus equilibrium number of firms

$$N = \frac{a-c}{\sqrt{bF}} - 1$$

$\therefore N^*$ = The highest integer which is smaller or equal to N .

Problem 5

(a)

Let the firms who sell output goods $f_i = 1,2$ and the firms who sell input $m_i\ (i=1,2)$

Then, f_1 and f_2 compete with output price simultaneously (Bertrand model), and m_1 and f_i competes sequentially (Stackelberg game).

So, based on backward induction, best response of f_i should be determined.

(Stage 1 – Bertrand model)

Let the profit function of firm f_i as

$$\Pi_1 = (a - p_1 + bp_2)(p_1 - c_1)$$
$$\Pi_1 = (a - p_2 + bp_1)(p_2 - c_2)$$

First order condition can be written as

$$\frac{\partial \Pi_1}{\partial p_1} = 2p_1 - bp_2 - a - c_1 = 0 \quad \& \quad \frac{\partial \Pi_2}{\partial p_2} = 2p_2 - bp_1 - a - c_2 = 0$$

Then, the best response function of each firm f_i is

$$p_1 = \frac{2(a+c_1)+b(a+c_2)}{4-b^2} \quad \& \quad p_2 = \frac{2(a+c_2)+b(a+c_1)}{4-b^2}$$

(Stage 2 – Stackelberg model)

The profit function of each firm m_i, given the best response function of f_i can be written as

$$\Pi_{m_1} = (a - p_1 + bp_2)c_1 = \left(a - \left(\frac{2(a+c_1)+b(a+c_2)}{4-b^2}\right) + b\left(\frac{2(a+c_2)+b(a+c_1)}{4-b^2}\right)\right)c_1$$

$$\Pi_{m_2} = (a - p_2 + bp_1)c_2 = \left(a - \left(\frac{2(a+c_2)+b(a+c_1)}{4-b^2}\right) + b\left(\frac{2(a+c_1)+b(a+c_2)}{4-b^2}\right)\right)c_2$$

From the first order conditions ($\frac{\partial \Pi_{m_i}}{\partial c_i} = 0$), the optimal c_1, c_2 can be calculated as

$$c_1^* = c_2^* = -\frac{a(b+2)}{2b^2+b-4}$$

By symmetry, optimal price of each firm m_i is equal, i.e., $c_1^* = c_2^* = c^*$.

(b)

Since the optimal $c^* = -\frac{a(b+2)}{2b^2+b-4}$ for firm m_1, m_2, the optimal price of each firm f_i can be calculated as

$$p_n^* = \frac{(a+c^*)(2+b)}{(4-b^2)} = -\frac{2a(b^2-3)}{(b-2)(2b^2+b-4)}$$

If f_1 integrates its system, the problem is changed. The formulation can be written as below.

(Stage 1 – Bertrand model)

$$\Pi_1 = (a - p_1 + bp_2)\, p_1$$
$$\Pi_1 = (a - p_2 + bp_1)(p_2 - c_2)$$

First order condition can be written as

$$\frac{\partial \Pi_1}{\partial p_1} = 2p_1 - bp_2 - a = 0 \quad \& \quad \frac{\partial \Pi_2}{\partial p_2} = 2p_2 - bp_1 - a - c_2 = 0$$

Then,

$$p_1 = \frac{2a + b(a+c_2)}{4-b^2} \quad \& \quad p_2 = \frac{2(a+c_2)+ab}{4-b^2}$$

(Stage 2 – Stackelberg model)

$$\Pi_{m_2} = (a - p_2 + bp_1)c_2 = \left(a - \left(\frac{2(a+c_2)+ab}{4-b^2}\right) + b\left(\frac{2a+b(a+c_2)}{4-b^2}\right)\right)c_2$$

From the first order condition, the optimal $c_2^* = \frac{a(2+b)}{2(2-b^2)}$.

Then, $p_1^* = \frac{a(4+b-2b^2)}{2(2-b^2)(2-b)} = p_v$.

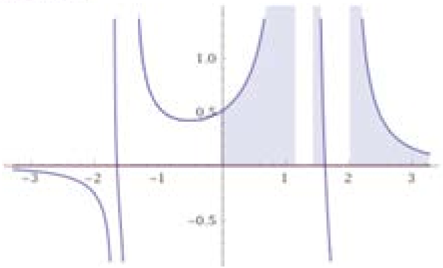
Therefore, $p_v < p_n$ is as below

$$p_v - p_n = \frac{a(4+b-2b^2)}{2(2-b^2)(2-b)} - \left(-\frac{2a(b^2-3)}{(b-2)(2b^2+b-4)}\right) = \frac{-a(3b^2-8)}{2(2-b^2)(2-b)(2b^2+b-4)} < 0$$

Since $a > 0, b > 0$

$$\therefore \quad \frac{\left(3b^2 - 8\right)}{\left(b^2 - 2\right)\left(b - 2\right)\left(2b^2 + b - 4\right)} > 0$$

Inequality plot:



Therefore, The condition for $p_v < p_n$ is

$0 < b < \frac{\sqrt{33} - 1}{4}$,

$\sqrt{2} < b < \frac{2\sqrt{6}}{3}$,

$2 < b$