

Fusion Plasma Theory II, 2019 - Midterm

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I.

High frequency wave propagation in plasmas

35 pts

The dispersion relation for the high frequency electromagnetic X-waves which can propagate perpendicular to the magnetic field can be written as:

$$\frac{c^2 k^2}{\omega^2} = \frac{(\omega^2 - \omega_L^2)(\omega^2 - \omega_R^2)}{\omega^2(\omega^2 - \omega_h^2)}$$

Where $\omega_h^2 = \omega_{pe}^2 + \omega_{ce}^2$,

$$\omega_R = \left\{ (\omega_{ce}^2 + 4\omega_{pe}^2)^{1/2} + \omega_{ce} \right\} / 2,$$

$$\omega_L = \left\{ (\omega_{ce}^2 + 4\omega_{pe}^2)^{1/2} - \omega_{ce} \right\} / 2,$$

ω_{pe} and ω_{ce} are the electron plasma frequency and the electron cyclotron frequency respectively.

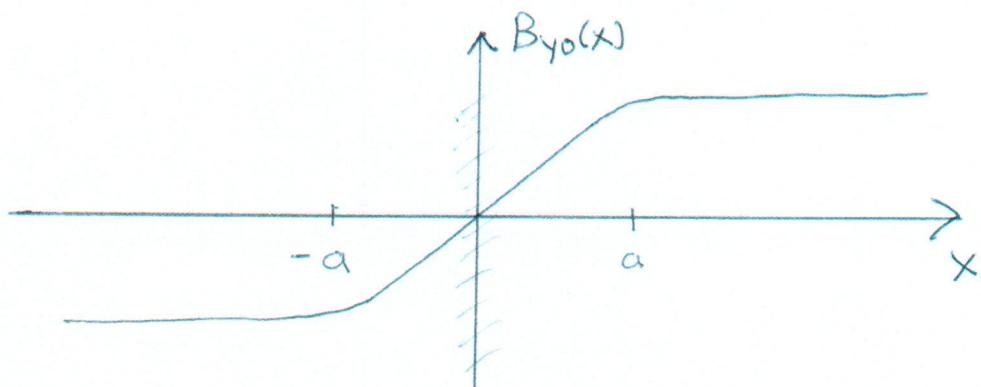
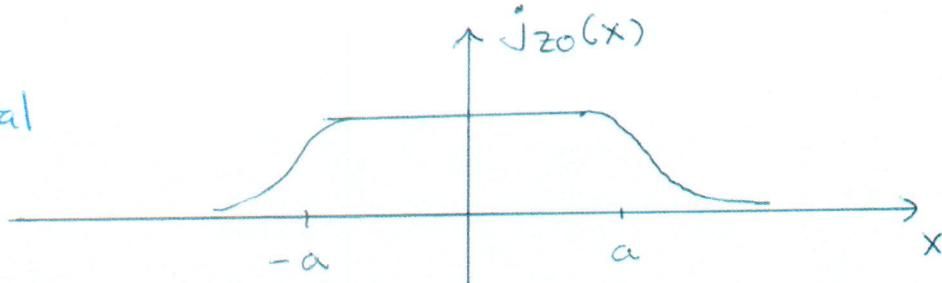
Electromagnetic fields are assumed to vary proportional to $\sim \exp(-i\omega t + ikx)$.

- a) For what values of ω (with respect to ω_h , ω_r and ω_R), can the wave propagate?
- b) State the condition for wave resonance, and find the value of ω (from the dispersion relation) which satisfies that condition.
- c) Do the same for wave cut-off.
- d) Calculate the group velocity for the resonance (case b)) and discuss what happens to the wave energy near resonance.
- e) Do the same for wave cut-off (case c))

Don't Panic. Each sub problem can be solved without solving the previous ones.

(II.) Resistive Tearing Modes

65 pts total



Consider a current slab model in which

$$B_{y0}(x) = B_0 \tanh\left(\frac{x}{a}\right)$$

and

$$j_{z0}(x) = \frac{B_0}{a} \operatorname{sech}^2\left(\frac{x}{a}\right)$$

$$\left(\tanh x \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{and} \quad \operatorname{sech} x \equiv \frac{1}{\cosh x} \right)$$

We'll be looking for perturbations of a form

$$\psi_i(x, t) = \psi_i(x) e^{ik_y y - i\omega t}, \quad k = k_y$$

All notations are the same as those in the class and the text book unless specified otherwise.

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The equation of motion can be written as

$$\rho_0 \frac{d}{dt} \vec{u} = -\vec{\nabla} p + \vec{j} \times \vec{B}, \quad (1)$$

5) 1) Linearize this equation

10) 2) Take $\hat{z} \cdot \vec{\nabla} \times$ of the equation you get
and show that

$$-\frac{\omega \mu_0 \rho_0}{k} \frac{\partial^2}{\partial x^2} \delta u_x = \frac{\partial}{\partial x} B_{y0}^2 \frac{\partial}{\partial x} \left(\frac{\delta B_x}{B_{y0}} \right) - k^2 B_{x0} \delta B_x \quad (2)$$

is satisfied near $x \approx 0$, where $\frac{\partial^2}{\partial x^2} \gg k^2$,

You can use the incompressibility condition $\vec{\nabla} \cdot \vec{\delta u} = 0$.

10) 3) Show that in the ideal MHD exterior region
sufficiently far away from $x=0$,

$$\frac{\partial^2}{\partial x^2} \delta B_x - \left(k^2 + \frac{\partial j_{0z}(x)}{\partial x} \right) \delta B_x = 0 \quad \text{is satisfied.} \quad (3)$$

5) 4) Show that $\delta B_x = e^{-k|x|} \left(1 + \frac{1}{ka} \tanh \left(\frac{x}{a} \right) \right)$
is a solution of Eq.(3) (4)

5) Calculate the delta prime Δ' of the solution in

10 Eq.(4); i.e.,

$$\Delta' \equiv \frac{1}{SB_{x(0)}} \left\{ \left[\frac{\partial}{\partial x} SB_x \right]_{+0} - \left[\frac{\partial}{\partial x} SB_x \right]_{-0} \right\},$$

and ~~the~~ discuss the stability of resistive tearing mode as a function of wavelength in y direction

6) Starting from the Ohm's law

5 $\vec{E} + \vec{u} \times \vec{B} = \gamma \vec{j}$, using appropriate equations, approximations, and linearization, show that

$$\omega SB_x = -k B_{y0} S_{ux} + \frac{i\gamma}{\mu_0} \frac{\partial^2}{\partial x^2} SB_x \quad (6)$$

is satisfied near $x=0$.

7) Using Eqs (2) and (6), show that the large $-x$ asymptotic

10 form of S_{ux} must be in the form

$$S_{ux} \sim \exp(-x^2/2\delta^2) \text{ where}$$

$$\delta \propto (8\gamma \rho_0)^{1/4} / (k B_{y0}')^{1/2}. \quad (B_{y0}' = B_{y0}'(x=0))$$

where $\text{Im}(\omega) \equiv \gamma$

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- 8) Finally, perform the asymptotic matching
10 between the resistive layer solution
and the exterior region solution
to show that

$$\gamma \propto \eta^{3/5} \Delta^{4/5}.$$

Note that

$$\left[\frac{\partial S B_x}{\partial x} \right]_{\infty} - \left[\frac{\partial S B_x}{\partial x} \right]_0 \quad \begin{matrix} \text{from the exterior} \\ \text{region} \end{matrix}$$

should be matched to

$$\left. \begin{cases} \frac{d}{dx} S B_x \\ \frac{\partial^2}{\partial x^2} S B_x \end{cases} \right|_{-\infty} \quad \begin{matrix} \text{from the resistive region} \\ \text{region} \end{matrix}$$