# Fusion Plasma Theory I

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## Mid-Term Exam

April 21, 2021

**1. [Ion Debye Length]** In some literature, the Debye length is introduced only with electrons by assuming "cold" ions. Show this is misleading by deducing an expression for the Debye potential for a test particle of charge $+Q$ immersed in a plasma consisting of electrons and ions of charge $Ze$, the temperature of the electrons and the ions being $T_e$ and $T_i$, but $T_e \gg T_i$.

**2. [Guiding Center Drift]** Consider a spatially nonuniform magnetostatic field given by

$$\vec{B}(x,z) = B_0\left[\epsilon z\hat{x} + (1+\epsilon x)\hat{z}\right] \tag{1}$$

with $\epsilon > 0$, $|\epsilon x| \ll 1$ and $|\epsilon z| \ll 1$. Assume that a particle with the mass $m$ and charge $q$ is moving initially $\vec{v}_0 = v_\perp\hat{y} + v_\parallel\hat{z}$ from the origin $(x,y,z) = (0,0,0)$. Estimate the guiding center drift of the particle other than the free streaming along with $\hat{z}$, in leading order $\epsilon$. Give also an electric field $\vec{E}$ that could remove this drift.

**3. [Electron Temperature Evolution]** The isotropic pressure evolution of electrons ignoring energy exchange can be described by

$$\frac{3}{2}\left(\frac{d}{dt}\right)_e p_e + \frac{5}{2}p_e(\vec{\nabla}\cdot\vec{u}_e) + \vec{\nabla}\cdot\vec{q}_e = 0. \tag{2}$$

where $(d/dt)_e \equiv \partial/\partial t + \vec{u}_e\cdot\vec{\nabla}$ and other definitions are conventional. However for two fluids with electrons and main ions, an equation based on the single fluid flow, or approximately ion flow, $\vec{u} \approx \vec{u}_i$, respectively, is often favored as they are the measurable quantities. Also, the evolution of the electron temperature is of more interest than the electron pressure. Assume $n_i = n_e$, $\vec{\nabla}\cdot\vec{j} = 0$ where $\vec{j}$ is the electric current, derive the following expression accordingly;

$$\frac{3}{2}n_e\frac{dT_e}{dt} + n_eT_e(\vec{\nabla}\cdot\vec{u}) + \frac{\vec{j}}{en_e}\cdot\left(T_e\vec{\nabla}n_e - \frac{3}{2}n_e\vec{\nabla}T_e\right) + \vec{\nabla}\cdot\vec{q}_e = 0, \tag{3}$$

where the total time derivative is now following the ion flow.

**4. [Magnetic Mirror]** Consider electrons in a magnetic mirror machine with the magnetic field near the axis given by

$$\mathbf{B} = \begin{cases} B_0(1 + z^2/L^2)\hat{z}, & \text{if } z^2 < c^2L^2 \\ B_0(1 + c^2)\hat{z}, & \text{if } z^2 > c^2L^2. \end{cases}$$

The electrons are sufficiently dilute that collisions are completely ignorable. Similarly, any self-generated electric or magnetic fields within the plasma are completely ignorable.

(b) Derive the trapping condition in midplane ($z = 0$) energy coordinates, $W_{\perp 0}$ and $W_{\| 0}$, respectively the perpendicular and parallel energies as the midplane is crossed. Sketch in $W_{\perp 0}$-$W_{\| 0}$ space the region of trapped electrons.

(b) Show that the turning points for trapped electrons obey: $z_T^2/L^2 = W_{\| 0}/W_{\perp 0}$.

(c) Now suppose that $B_0 = B_0(t)$ is a very slowly varying function of time, such that it is monotonically increasing or decreasing in going from $B_0(t = 0) = B_{0i}$ to $B_0(t = t_f) = B_{0f}$, where $B_{0f}/B_{0i} = \beta$. Calculate the changes in the perpendicular and parallel midplane energies. Note: by slowly varying we mean slow enough that particle motion can be considered to be adiabatic.

(d) For what midplane coordinates (as a function of $\beta$) will electrons initially trapped become untrapped as a result of the slowly changing $B_0 = B_0(t)$? Sketch in $W_{\perp 0}$-$W_{\| 0}$ space the region of initially trapped electrons that was initially trapped that then became untrapped.

**5. [Ohm's Law in Resistive MHD]** Suppose that one is investigating a narrow radial region in a tokamak where an electromagnetic toroidal torque appears to be generated due to resistive dissipation (with resistivity $\eta$) but under some shielding due to toroidal rotation $\vec{u} = \omega R\hat{\phi}$, where $\omega$ is the angular frequency of toroidal rotation and $R$ is the major radius of the region. Simply write down the resistive MHD Ohm's law without the Hall term and see if you can quantify this. Assume the electric field is static, i.e. $\vec{E} = -\vec{\nabla}\Phi$ and the electric current is divergentless, that is, $\vec{\nabla} \cdot \vec{j} = 0$.

<u>Possibly useless information:</u>

$$\vec{A} \cdot (\vec{B} \times C) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$
$$\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot \vec{\nabla} f$$