Mid-term exam

1. $\epsilon_0 \vec{\nabla}\cdot\vec{E} = \rho$ $\quad \vec{E} = -\vec{\nabla}\phi$ $\quad$ test particle

$\epsilon_0 \nabla^2 \phi = -\rho$ $\quad \rho = Zen_i - en_e + Q\delta(\vec{r})$

— Assume Boltzmann distribution for n_i, n_e

$n_i = n_{i\infty} e^{-Ze\phi/T_i}$, $\quad n_e = n_{e\infty} e^{e\phi/T_e}$

— Assume $e\phi/T_{e,i} \ll 1$ expand

$$\epsilon_0 \nabla^2\phi = -Zen_{i\infty}(1 - Ze\phi/T_i) + en_{e\infty}(1 + e\phi/T_e) - Q\delta(\vec{r})$$

— Assume quasi neutrality $n_{e\infty} \approx Zn_{i\infty}$ in the background

$$\epsilon_0 \nabla^2 \phi - e^2 n_{e\infty}\left(\frac{Z}{T_i} + \frac{1}{T_e}\right)\phi = -Q\delta(\vec{r})$$

— In spherical geometry

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) - \frac{1}{\lambda_D^2}\phi = -\frac{Q}{\epsilon_0}\delta(\vec{r})$$

$$\phi = \frac{Q}{4\pi\epsilon_0 r} e^{-r/\lambda_D}$$

where $\lambda_D = \sqrt{\frac{\epsilon_0 T_e}{e^2 n_{e\infty}} \frac{1}{1 + ZT_e/T_i}}$

if $T_e \gg T_i$ $\quad \lambda_D \sim \sqrt{\frac{\epsilon_0 T_i}{Ze^2 n_{e\infty}}} \rightarrow$ "cold" ion

not $\sim \sqrt{\frac{\epsilon_0 T_e}{e^2 n_{e\infty}}} \longrightarrow$ "fixed" ion

2. drift velocity by $\vec{\nabla} B$ & $\vec{\kappa}$

due to $\vec{B}(x,z) = B_0 \left[\epsilon z \hat{x} + (1+\epsilon x)\hat{z} \right]$

note, $\vec{\nabla} \times \vec{B} = 0$ i.e) vacuum field

Then,

$$\vec{V}_{\nabla B} + \vec{V}_{curv} = \frac{m(V_\perp^2 + 2V_\parallel^2)}{2qB} \frac{\vec{B} \times \vec{\nabla} B}{B^2}$$

$$\vec{B} \cdot \vec{B} = B_0^2 \left(1 + 2\epsilon x + \mathcal{O}(\epsilon^2)\right)$$

$$B = B_0 \left(1 + \epsilon x + \mathcal{O}(\epsilon^2)\right)$$

$$\vec{\nabla} B = \epsilon B_0 \hat{x} + \mathcal{O}(\epsilon^2)$$

$$\vec{B} \times \vec{\nabla} B = B_0^2 \left((1+\epsilon x)\epsilon \right) \hat{y}$$

$$= \epsilon B_0^2 \hat{y} + \mathcal{O}(\epsilon^2)$$

$$\vec{V}_{\nabla B} + \vec{V}_{curv} = \frac{m(V_\perp^2 + 2V_\parallel^2)}{2qB_0} \epsilon \hat{y} + \mathcal{O}(\epsilon^2)$$

$$= \vec{V}_{E\times B} = \frac{\vec{E} \times B_0 \hat{z}}{B_0^2} + \mathcal{O}(\epsilon^2)$$

if $\vec{E} = -\frac{m(V_\perp^2 + 2V_\parallel^2)}{2q} \epsilon \hat{x} + \mathcal{O}(\epsilon^2)$

3. note $\frac{3}{2}\frac{dp}{dt} = \frac{3}{2}n\frac{dT}{dt} + \frac{3}{2}T\frac{dn}{dt}$

$$= \frac{3}{2}n\frac{dT}{dt} - \frac{3}{2}nT(\vec{\nabla}\cdot\vec{u})$$

So, $\frac{3}{2}\left(\frac{d}{dt}\right)_e P_e + \frac{5}{2}P_e(\vec{\nabla}\cdot\vec{u_e}) + \vec{\nabla}\cdot\vec{q_e}$

$$= \frac{3}{2}n_e\left(\frac{d}{dt}\right)_e T_e + P_e(\vec{\nabla}\cdot\vec{u_e}) + \vec{\nabla}\cdot\vec{q_e} = 0$$

as frequently used.

now just use $\vec{u_e} = \vec{u} - \frac{1}{en_e}\vec{j}$

$$\begin{cases} \left(\frac{d}{dt}\right)_e = \frac{d}{dt} - \frac{1}{en_e}\vec{j}\cdot\vec{\nabla} \\ \vec{\nabla}\cdot\vec{u_e} = \vec{\nabla}\cdot\vec{u} - \cancel{\frac{1}{en_e}\vec{\nabla}\cdot\vec{j}} + \frac{1}{en_e^2}\vec{j}\cdot\vec{\nabla}n_e \end{cases}$$

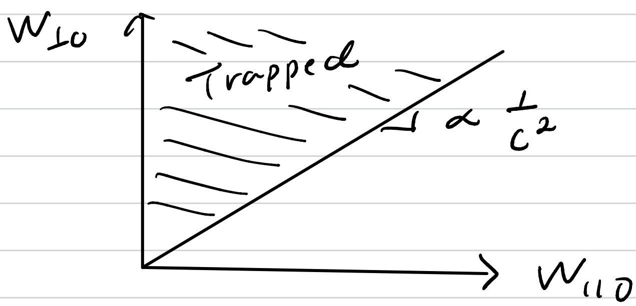
$$\frac{3}{2}n_e\frac{dT_e}{dt} + n_eT_e(\vec{\nabla}\cdot\vec{u}) + \frac{\vec{j}}{en_e}\cdot\left(T_e\vec{\nabla}n_e - \frac{3}{2}n_e\vec{\nabla}T_e\right) + \vec{\nabla}\cdot\vec{q_e} = 0$$

4. (a) $W_\parallel = W - W_\perp = W_{\parallel 0} + (1-R) W_{\perp 0}$ $\quad R \equiv B/B_0$

$1 - R = -z^2/L^2 \quad \rightarrow \quad W_\parallel = W_{\parallel 0} - \frac{z^2}{L^2} W_{\perp 0}$

$\because$ Trapping condition $W_\parallel = 0$ exists

$$W_{\parallel 0} = \frac{z^2}{L^2} W_{\perp 0} < c^2 W_{\perp 0}$$



(b) turning point $W_\parallel = 0 \rightarrow z^2 = L^2 \frac{W_{\parallel 0}}{W_{\perp 0}} \equiv z_T^2$

so rewrite $W_\parallel = W_{\parallel 0}(1 - z^2/z_T^2)$

(c) $B_0 \rightarrow \beta B_0 \quad W_\perp \rightarrow \beta W_\perp, \; W_{\perp 0} \rightarrow \beta W_{\perp 0}$

$$J = \oint v_\parallel \, dz = \oint v_{\parallel 0} (1 - z^2/z_T^2)^{\frac{1}{2}} \, dz$$

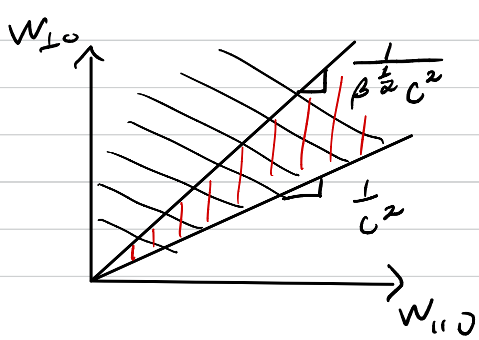
$\propto v_{\parallel 0} z_T \sim$ const

$\Rightarrow \quad v_{\parallel 0}^2 z_T^2 = L^2 \frac{W_{\parallel 0}^2}{W_{\perp 0}} \sim$ const

$\Rightarrow \quad W_{\parallel 0}^2 \rightarrow \beta W_{\parallel 0}^2$

$W_{\parallel 0} \rightarrow \beta^{\frac{1}{2}} W_{\parallel 0}$ and $\frac{W_{\perp 0}}{W_{\parallel 0}} \rightarrow \beta^{\frac{1}{2}} \frac{W_{\perp 0}}{W_{\parallel 0}}$

(d)



so, particle with

$$\frac{1}{c^2} < \frac{W_{\perp 0}}{W_{\parallel 0}} < \frac{1}{\beta^{\frac{1}{2}} c^2}$$

are trapped and then detrapped if $\beta < 1$

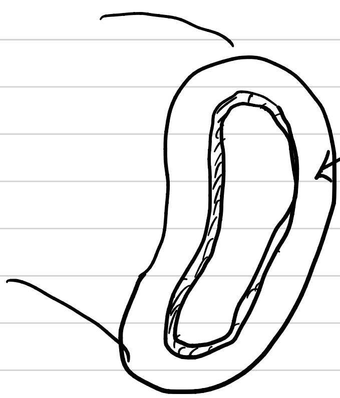
5. $\vec{E} + \vec{u} \times \vec{B} = \eta \vec{j}$

$$\left(\vec{E} + \omega R \hat{\phi} \times \vec{B} = \eta \vec{j}\right) \cdot \vec{j}$$

$$-\vec{j} \cdot \vec{\nabla} \phi + \omega R \, \vec{j} \cdot (\hat{\phi} \times \vec{B}) = \eta j^2$$

$\vec{\nabla} \cdot \vec{j} = 0$

$$\vec{\nabla} \cdot (\phi \vec{j}) + \omega R \hat{\phi} \cdot \underbrace{(\vec{j} \times \vec{B})}_{\text{toroidal torque density}} = -\eta \underbrace{j^2}_{\text{resistive dissipation}}$$



happening at narrow layer

Imagine volume integral over the layer, $d\vec{r}$

$$\cancel{\int d\vec{r} \, \vec{\nabla} \cdot (\phi \vec{j})} + \omega \underbrace{\int d\vec{r} \, R \hat{\phi} \cdot (\vec{j} \times \vec{B})}_{\text{Total torque } T_\varphi} = -\int d\vec{r} \, \eta j^2$$

$$T_\varphi = -\frac{\int d\vec{r} \, \eta j^2}{\omega}$$

← torque by dissipation

↖ rotational screening