Viscous Flow
Midterm Exam
2019. 10. 31 Hyungmin Park

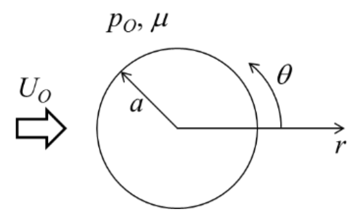
- Write CLEARLY and describe ALL details of your work.
- Without a proof, you may use the information provided together.

1. (20 points) Derive the vorticity equation (eq2), shown in the below. Hint: (i) start by taking the curl of the Navier-Stokes equation (eq1); (ii) you may need to use the vector identities provided together in the last page.

$$\frac{D\bar{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\bar{u} \text{ (eq1)}; \qquad \frac{D\bar{\omega}}{Dt} = \bar{\omega}\cdot\nabla\bar{u} + \nu\nabla^2\bar{\omega} \text{ (eq2)}$$

2. (30 points) Stream function (ψ) of the Stokes flow around a sphere is obtained as below, where a is the radius of the sphere, and U_o is the free-stream velocity.

$$\psi(r,\theta) = U_O r^2 \sin^2\theta\left(\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right)$$

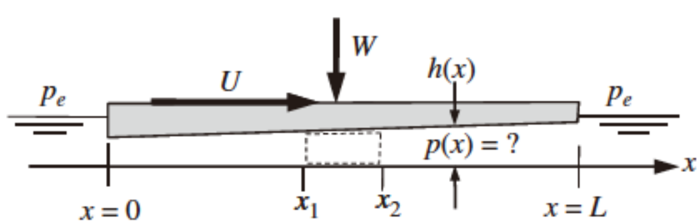


(a) (10 points) Obtain the pressure field as a function of r and θ.

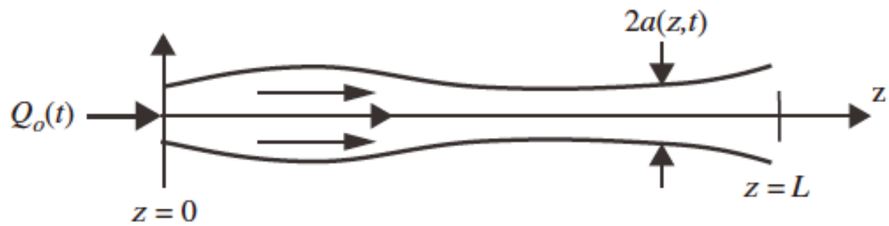
(b) (10 points) Calculate the pressure drag (per unit area) on the sphere.

(c) (10 points) If you calculate the pressure drag on a sphere in a potential (i.e., inviscid and irrotational) flow, how would you obtain the pressure field (assuming that the stream function for the potential flow around a sphere is known)?

3. (20 points) A sloped bearing pad of width B into the page moves horizontally at a steady speed of U on a thin layer of oil with density ρ and viscosity μ. The gap between the bearing pad and a stationary rigid, flat surface located at $y = 0$ is given by $h(x) = h_o(1 + \alpha x/L)$ where $\alpha \ll 1$. If P_e is the exterior pressure and $p(x)$ is the local pressure in the oil under the bearing pad, determine the load W (per unit width into the page) that the bearing can support.



4. (30 points) As a simple model of small-artery blood flow, consider slowly varying viscous flow through a round flexible tube with inlet at z = 0 and outlet at z = L. At z = 0, the volume flux entering the tube is $Q_o(t)$. At z = L, the pressure equals the exterior pressure p_e. The radius of the tube, a(z, t), expands and contracts in proportion to pressure variations within the tube so that: 1) $a - a_e = \gamma(p - p_e)$, where a_e is the tube radius when the pressure, p(z, t), in the tube is equal to p_e, and γ is a positive constant. Assume the local volume flux, Q(z, t), is related to $\partial p/\partial z$ by 2) $Q = -(\pi a^4/8\mu)(\partial p/\partial z)$.

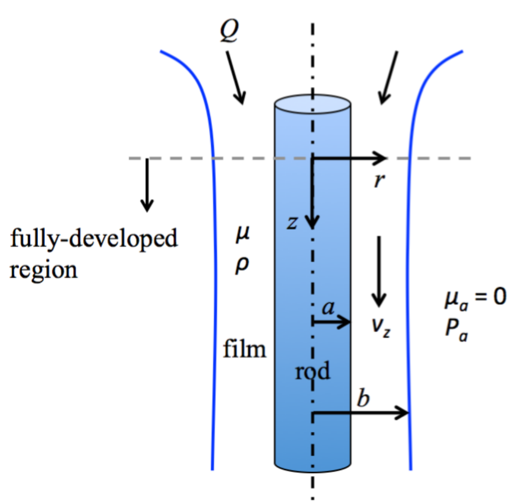


(a) (10 points) By conserving mass, find a partial differential equation that relates Q and a.

(b) (10 points) Combine 1), 2), and the result of part a) into one partial differential equation for a(z, t).

(c) (10 points) Determine a(z) when Q_o is a constant and the flow is perfectly steady.

5. (20 points) Consider a film of liquid draining at volume flow rate Q down the outside of a vertical circular rod (radius a). Some distance down the rod, a fully developed region is reached where the fluid shear stress balances gravity and the film thickness remain constant. Assuming incompressible laminar flow and negligible shear interaction with the atmosphere, find an expression for $v_z(r)$ and a relation between Q and film radius b.



These might be useful….

Stokes stream function (ψ) in spherical coordinates

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta},\ u_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$$

Some derivatives in spherical coordinates

$$\frac{\partial p}{\partial r} = \mu\left(\nabla^2 u_r - \frac{2}{r^2}u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{2}{r^2}u_\theta \cot\theta\right)$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2 \sin^2\theta}\right)$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right)$$

Continuity and momentum conservation equations for incompressible flows in cylindrical coordinates (no body force)

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial z} + \mu\left(\nabla^2 u_z\right)$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Vector Identities

$$\frac{1}{2}\nabla(A\cdot A) = (A\cdot\nabla)A + A\times(\nabla\times A)$$

$$\nabla\cdot(A\times B) = (\nabla\times A)\cdot B - A\cdot(\nabla\times B)$$

$$\nabla\times(A\times B) = A(\nabla\cdot B) + B(\nabla\cdot A) - (B\cdot\nabla)A - (A\cdot\nabla)B$$