Engineering Economic Analysis

Fall 2019

Mid-term Exam Solution

2019.04.30

Q.1.)

(a)

The problem can be formulated as the utility maximization problem like below.

$$\begin{aligned} &\max \quad && \ln x_0 + \beta\left(\delta \ln x_1 + \delta^2 \ln x_2\right) \\ &s.t. && (1+r)(x_0 + x_1) + x_2 = w \end{aligned}$$

Lagrangian:

$$L = \ln x_0 + \beta\left(\delta \ln x_1 + \delta^2 \ln x_2\right) - \lambda\left((1+r)(x_0 + x_1) + x_2 - w\right)$$

First order condition:

$$\frac{\partial L}{\partial x_0} = \frac{1}{x_0} - \lambda(1+r) = 0$$

$$\frac{\partial L}{\partial x_1} = \frac{\beta\delta}{x_1} - \lambda(1+r) = 0$$

$$\frac{\partial L}{\partial x_0} = \frac{\beta\delta^2}{x_0} - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = (1+r)(x_0 + x_1) + x_2 - w = 0$$

Solving equations of F.O.C., we get

$$x_0^* = \frac{w}{(1+r)\left(1+\beta\delta+\beta\delta^2\right)}, \quad x_1^* = \frac{\beta\delta w}{(1+r)\left(1+\beta\delta+\beta\delta^2\right)}, \quad x_2^* = \frac{\beta\delta^2 w}{1+\beta\delta+\beta\delta^2}$$

(b) False!

Let the total amount of debt, which depends on β, as $D(\beta)$. Then,

$$D(\beta) = x_0^*(\beta) + x_1^*(\beta) = \frac{w + \beta\delta w}{(1+r)(1+\beta\delta+\beta\delta^2)}$$

(Approach 1 - Comparative statics)

Comparative statics with respect to β can be calculated as follows.

$$\begin{aligned}
\frac{D(\beta)}{d\beta} &= \frac{\delta w\{(1+r)(1+\beta\delta+\beta\delta^2)\} - (w+\beta\delta w)\{(1+r)(\delta+\delta^2)\}}{(1+r)^2(1+\beta\delta+\beta\delta^2)^2} \\
&= \frac{w(1+r)\{\delta(1+\beta\delta+\beta\delta^2) - (1+\beta\delta)(\delta+\delta^2)\}}{(1+r)^2(1+\beta\delta+\beta\delta^2)^2} \\
&= -\frac{w(1+r)\delta^2}{(1+r)^2(1+\beta\delta+\beta\delta^2)^2} < 0 \qquad (\because w \geq 0, r \geq 0)
\end{aligned}$$

So, if the present bias becomes bigger, the total amount of debt becomes smaller. Therefore, the consumer with *present-bias* $(\beta < 1)$ borrow more debt than the consumer without *present-bias* $(\beta = 1)$.

(Approach 2 - Direct comparison)

$$\begin{aligned}
D(1) - D(\beta) &= \frac{w+\delta w}{(1+r)(1+\delta+\delta^2)} - \frac{w+\beta\delta w}{(1+r)(1+\beta\delta+\beta\delta^2)} \\
&= \frac{w(\beta-1)(\delta+\delta^2)}{(1+r)(1+\delta+\delta^2)(1+\beta\delta+\beta\delta^2)} < 0 \qquad (\because \beta < 1)
\end{aligned}$$

Thus, the consumer with *present-bias* borrow more debt than the consumer without *present-bias.*

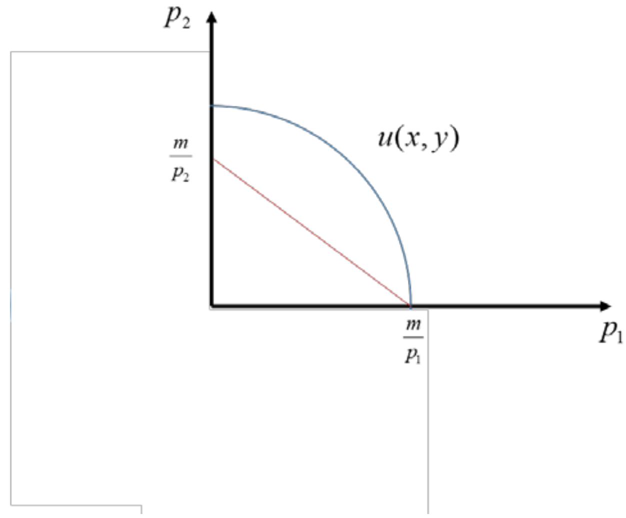
Q.2.)

The problem can be formulated as the utility maximization problem like below.

$$\begin{aligned}
&\max \quad x_1^2 + x_2^2 \\
&s.t. \quad p_1 x_1 + p_2 x_2 = m
\end{aligned}$$

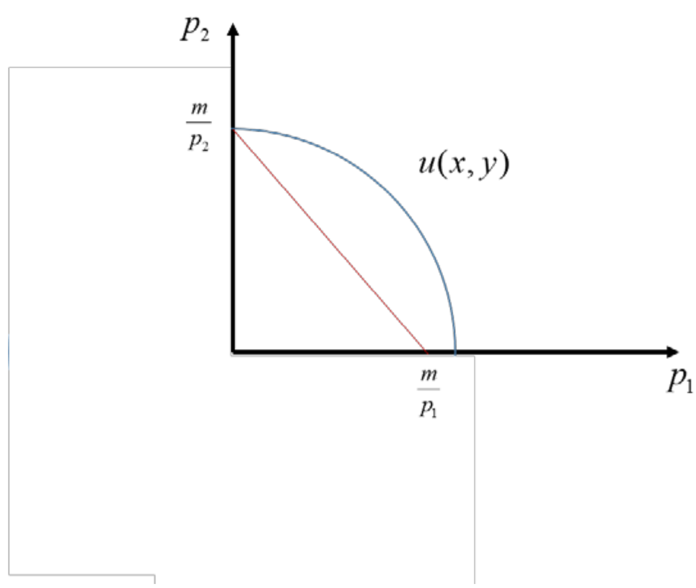
Since the utility function is convex, the solution will be at the corner (corner solutions).

(Case 1) $p_1 < p_2$:



$$x_1^* = \frac{m}{p_1}, \;\; x_2^* = 0$$

(Case 2) $p_2 < p_1$:

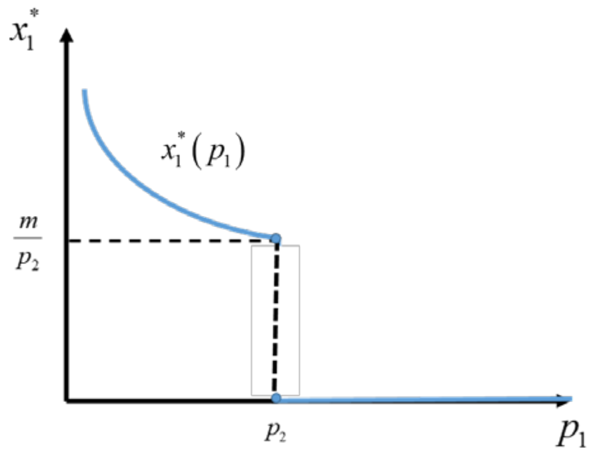


$$x_1^* = 0, \;\; x_2^* = \frac{m}{p_2}$$

(Case 3) $p_1 = p_2$:

$$\left(x_1^*, x_2^*\right) = \left(\frac{m}{p_1}, 0\right) or \left(0, \frac{m}{p_2}\right)$$

Therefore, quantity changes of good 1 with respect to its own price can be depicted like below.



As price becomes higher, the optimal amount of good 1 becomes smaller, if the price of good 1 is cheaper than that of good 2. If the price of good 1 is more expensive than that of good 2, the optimal amount does not change at the level of 0.

Q.3.)

(a)

The problem can be formulated as

$$\begin{aligned} \max \quad & 2x_1^\alpha + x_2 \\ s.t. \quad & p_1x_1 + p_2x_2 = m \end{aligned}$$

Since the utility function is quasi-linear, we can rewrite the problem as

$$\max \quad 2x_1^\alpha + \frac{1}{p_2}(m - p_1x_1)$$

The optimal solution (Marshallian demand function) can be obtained from F.O.C.

$$\frac{du}{dx_1} = 2\alpha x_1^{\alpha-1} - \frac{p_1}{p_2} = 0$$

Therefore, the Marshallian demand functions can be written as

$$x_1^* = \left(\frac{1}{2\alpha}\right)^{\frac{1}{\alpha-1}} \left(\frac{p_1}{p_2}\right)^{\frac{1}{\alpha-1}}, \quad x_2^* = \frac{m}{p_2} - \left(\frac{1}{2\alpha}\right)^{\frac{1}{\alpha-1}} \left(\frac{p_1}{p_2}\right)^{\frac{\alpha}{\alpha-1}}$$

(b)

The indirect utility function can be calculated as

$$v(p_1, p_2, m) = \frac{m}{p_2} - \left(\frac{\alpha - 1}{\alpha}\right)\left(\frac{1}{2\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{p_1}{p_2}\right)^{\frac{\alpha}{\alpha-1}}$$

Then, expenditure function is

$$e(p_1, p_2, u) = p_2\left[u + \left(\frac{\alpha - 1}{\alpha}\right)\left(\frac{1}{2\alpha}\right)^{\frac{1}{\alpha-1}}\left(\frac{p_1}{p_2}\right)^{\frac{\alpha}{\alpha-1}}\right]$$

Therefore, the Hicksian demand functions are obtained as

$$h_1^* = \frac{\partial e}{\partial p_1} = \left(\frac{p_1}{2\alpha p_2}\right)^{\frac{1}{\alpha-1}}$$

$$h_2^* = \frac{\partial e}{\partial p_2} = u - 2\left(\frac{p_1}{2\alpha p_2}\right)^{\frac{\alpha}{a-1}}$$

(c)

Let $m = 10,\ p_1 = p_2 = 2,\ \alpha = 0.5$. The optimal bundles $(x_1^*, x_2^*) = (1, 4)$, with the utility level of 6. If price changes to $p_1 = 1$, The optimal bundles $(x_1^*, x_2^*) = (4, 2)$, with the utility level of 7. If the consumer want to purchase $(x_1, x_2) = (1, 4)$ after the price changes, he only pay \$8 to feel the same utility before the price changed.

$$\therefore \text{ Consumer Surplus } = 10 - 8 = 2$$

Compensating variations can be calculated as follows.

$$\text{CV=} = e(1, 2, 7) - e(1, 2, 6) = 2$$

Equivalent variations can be calculated as follows.

$$\text{EV=} e(2, 2, 7) - e(2, 2, 6) = 2$$

Q.4.)

(a)

- Type-A (100 consumers) have the problem as follows.

$$\max \quad \min\{x, 2y\}$$
$$s.t. \quad p_1 x + p_2 y = m = 3,600$$

Solving it, the optimal solution can be obtained as

$$x^* = \frac{2m}{2p_1 + p_2} = \frac{7,200}{2p_1 + p_2}, \; y^* == \frac{m}{2p_1 + p_2} = \frac{3,600}{2p_1 + p_2}$$

- Type-B (200 consumers) have the problem as follows.

$$\max \quad \sqrt{xy}$$
$$s.t. \quad p_1 x_1 + p_2 y = m = 3,600$$

Solving it, the optimal solution can be obtained as

$$x^* = \frac{m}{2p_1} = \frac{1,800}{p_1}, \; y^* = \frac{m}{2p_2} = \frac{1,800}{p_2}$$

Therefore, the market demand function for good x can be calculated as

$$D_x = 100 * \left(\frac{7,200}{2p_1 + p_2}\right) + 200 * \left(\frac{1,800}{p_1}\right) = 720,000\left(\frac{4p_1 + p_2}{4p_1^2 + 2p_1 p_2}\right)$$

(b) True!

Price elasticity can be calculated as

$$\varepsilon = \frac{d \ln D_x}{d \ln p_1} = \frac{d \ln D_x}{dp_1} \cdot \frac{dp_1}{d \ln p_1} = \frac{d \ln D_x}{dp_1} \cdot \frac{1}{\frac{d \ln p_1}{dp_1}} = \frac{d \ln D_x}{dp_1} \cdot p_1$$

$$= p_1 \frac{d}{dp_1}\left\{\ln\left(4p_1 + p_2\right) - \ln\left(4p_1^2 + 2p_1 p_2\right)\right\}$$

$$= p_1\left[\left(\frac{4}{4p_1 + p_2}\right) - \left(\frac{8p_1 + 2p_2}{4p_1^2 + 2p_1 p_2}\right)\right]$$

$$= -\frac{8p_1^2 + 4p_1 p_2 + p_2^2}{8p_1^2 + 6p_1 p_2 + p_2^2} = \frac{41}{45}$$

It means good x is inelastic with respect to its own price, since $|\varepsilon| < 1$. So, the revenue will increase if a producer of good x raises the price of good x.

(c)

If the price of good x = \$20 and that of y = \$10:

- type-A will purchase (144, 72), and the utility level is 144.
- type-B will purchase (90, 180), and the utility level is $90\sqrt{2}$.

If the price of good x becomes \$10, and lump-sum tax as t

(Type A)

$$\begin{aligned} &\max \quad \min\{x, 2y\} \\ &s.t. \quad 10x + 10y = 3{,}600 - t \end{aligned}$$

the optimal solution can be obtained as

$$x^* = \frac{7{,}200 - 2t}{30} = 240 - \frac{2}{30}t, \quad y^* = \frac{3{,}600 - t}{30} = 120 - \frac{1}{30}t$$

His utility level is $240 - \frac{2}{30}t$, thus he would pay lump-sum tax if $240 - \frac{2}{30}t \geq 144$.

$$\therefore t \leq 1{,}440$$

(Type B)

$$\begin{aligned} &\max \quad \sqrt{xy} \\ &s.t. \quad 10x_1 + 10y = 3{,}600 - t \end{aligned}$$

the optimal solution can be obtained as

$$x^* = \frac{3{,}600 - t}{20} = 180 - \frac{1}{20}t, \quad y^* = 180 - \frac{1}{20}t$$

His utility level is $180 - \frac{1}{20}t$, thus he would pay lump-sum tax if $180 - \frac{1}{20}t \geq 90\sqrt{2}$.

$$\therefore t \leq 1800\left(2 - \sqrt{2}\right)$$

Type A => 1440, Type B => $1800\left(2 - \sqrt{2}\right)$

This is the same value of compensating variations.

(cf.) EV

- EV of type A

If price of A = \$10, price of B = \$10, the utility is 240 with the optimal bundle of (240, 120).

Therefore, EV = $20*240+10*120-3,600=2,400$.

- EV of type B

If price of A = \$10, price of B = \$10, the utility is 180 with the optimal bundle of (180, 180).

To feel same utility level when price of A = \$20, price of B = \$10, $\frac{m}{20\sqrt{2}}=180$ should holds.

Therefore, EV= $3600\left(\sqrt{2}-1\right)$