Micro Fluid Mechanics - Spring 2019

Solution Set 1

March 25, 2019 Prof. Ho-Young Kim

Problem 1

(a) Use the Knudsen number for the judgment.

$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n} = \frac{RT}{\sqrt{2}\pi d^2 N_A P}$$

We have $d = 0.364 \times 10^{-9}$m and $P = 133.3$ N/m^2. Then $\lambda = 5.28 \times 10^{-5}$m thus

$$\mathrm{Kn} = \frac{5.28 \times 10^{-5}\mathrm{m}}{0.01\mathrm{m}} = 5.28 \times 10^{-3} < 0.01$$

Therefore, the flow is modelled to be continuum.

(b) From the formula in the problem, $\rho = 0.11\mathrm{kg/m}^3$. We also use $\rho = mn$, where m is the mass of an oxygen atom and n is the number density of the atom. Thus we get

$$n = \frac{N_A}{16}\rho = 4.1 \times 10^{24}/\mathrm{m}^3$$

Then λ is calculated to be 4.1×10^{-7}m, which yields $\mathrm{Kn} = 4.1 \times 10^{-4} < 0.01$. Therefore, the flow is modelled to be continuum.

Problem 2

(a) In this case, $f(k) = k$. Thus

$$\langle k \rangle = \sum_{k=0}^{\infty} k \frac{n^k e^{-n}}{k!} = e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{(k-1)!}$$

A change of variables, $j = k - 1$ yields

$$\langle k \rangle = e^{-n} \sum_{j=0}^{\infty} \frac{n^{j+1}}{j!} = n e^{-n} \sum_{j=0}^{\infty} \frac{n^j}{j!}$$

Note that the sum is the series definition for e^n, therefore:

$$\langle k \rangle = n$$

(b) This problem is much simpler if we first simplify the expression for the variance:

$$var\{k\} = \langle (k - \langle k \rangle)^2 \rangle = \langle k^2 + \langle k \rangle^2 - 2k\langle k \rangle \rangle$$

It can be seen from the definition that the expectation value has the distributive property. Therefore,

$$var\{k\} = \langle k^2 \rangle + \langle \langle k \rangle^2 \rangle - \langle 2k \langle k \rangle \rangle = \langle k^2 \rangle - \langle k \rangle^2$$

Thus we only need to calculate $\langle k^2 \rangle$ and we are essentially done.

$$\begin{aligned}
\langle k^2 \rangle &= \sum_{k=0}^{\infty} k^2 p(k) = \sum_{k=0}^{\infty} k^2 \frac{n^k e^{-n}}{k!} = e^{-n} \sum_{k=0}^{\infty} \frac{k n^k}{(k-1)!} \\
&= e^{-n} \sum_{j=0}^{\infty} \frac{(j+1) n^{j+1}}{j!} \\
&= n e^{-n} \left[\sum_{j=0}^{\infty} \frac{n^j}{(j-1)!} + \sum_{j=0}^{\infty} \frac{n^j}{j!} \right]
\end{aligned}$$

We've already evaluated both of these sums in part (a):

$$\langle k^2 \rangle = n e^{-n} \left[n e^n + e^n \right] = n^2 + n$$

Thus

$$var\{k\} = \langle k^2 \rangle - \langle k \rangle^2 = (n^2 + n) - n^2 = n$$

(c) We know

$$\frac{N}{V} = \frac{m}{V} \frac{N_A}{\hat{M}} = \frac{\rho N_A}{\hat{M}}$$

Let the deviation in the number of molecules, k, be represented by δk. By definition:

$$\begin{aligned}
\left(\frac{\delta k}{k} \right)_{rms} &= \frac{\sqrt{var\{k\}}}{\langle k \rangle} = \frac{\sqrt{n}}{n} = \left[\frac{N}{V} \delta V \right]^{-1/2} \\
&= \left[\frac{\rho N_A}{\hat{M}} \delta V \right]^{-1/2}
\end{aligned}$$

For water, $\hat{M} \simeq 18\text{g/mol}$ and $\rho \simeq 10^6 \text{g/m}^3$ at 300 K and 10^5 Pa. Therefore, for a cube with sides 100 nm:

$$\left(\frac{k'}{k} \right)_{rms} \simeq \left[\frac{10^6 (6.02 \times 10^{23})}{18} \left(10^{-7} \right)^3 \right]^{-1/2}$$

Thus

$$\left(\frac{k'}{k} \right)_{rms} \simeq 1.7 \times 10^{-4}$$

(d) Rearranging the expression from above yields

$$\delta V = \frac{\hat{M}}{\rho N_A} \left(\frac{k'}{k} \right)_{rms}^{-2}$$

Therefore, for a relative rms deviation of 10%:

$$\begin{aligned}\delta V &= \frac{18}{(10^6)6.02\times10^{23}}(0.1)^{-2}\\ &= 3.0\times10^{-27}\text{m}^3\end{aligned}$$

Thus the cube will be approximately 1.4 nm on a side.

(e) We need only worry about orders of magnitude at this point. First we must estimate the total number of air molecules. Assume that the air surrounding the earth is a shell with a thickness of $t = 10$ km. Also assume that the air in this shell is of constant density $\rho \simeq 1.32\text{kg/m}^3$. Therefore, the total mass of air surrounding the earth is

$$M \simeq 4\pi\rho R_{earth}^2 t \simeq 6.7\times10^{18}\text{kg}$$

Thus the total number of air molecules surrounding the earth is

$$\begin{aligned}N &= \frac{m}{\hat{M}}N_A = \frac{6.7\times10^{18}}{0.028}6.02\times10^{23}\\ &= 1.45\times10^{44}\text{molecules}\end{aligned}$$

Now assume that in General Yi's dying moment he used one lungful of air, approximately one liter. Therefore,

$$\begin{aligned}N_{Yi} &= \frac{\rho V}{\hat{M}}N_A = \frac{1.32(10^{-3})}{0.028}6.02\times10^{23}\\ &= 2.8\times10^{22}\text{molecules}\end{aligned}$$

Thus assuming that no air has been created or destroyed since General Yi, the probability of picking "one" air molecule out of the air that is NOT one that General Yi also breathed during his final moment is

$$P_1 = 1 - \frac{N_{Yi}}{N} = 1 - 2\times10^{-22}$$

Now, assuming that with each breath I breathe in one liter of air as well, the probability that none of the molecules will be one that General Yi breathed is

$$\begin{aligned}P_{not} &= P_1^{N_{breath}} = (1-2\times10^{-22})^{2.8\times10^{22}}\\ &= (1-x)^{5.5/x} \approx e^{-5.5}\end{aligned}$$

where the identity

$$\lim_{x\to0}(1-x)^{1/x} = e^{-1}$$

was used. Thus the probability that with each breath I breathe in an air molecule breathed by General Yi is

$$P = 1 - P_{not} = 1 - e^{-5.5} \simeq 1$$

Therefore, it is very "unlikely" that in my lifetime I will not breathe in an air molecule breathed by General Yi during his last moment.

Problem 3

The continuum hypothesis breaks down when the Knudsen number exceeds 0.01:

$$\mathrm{Kn} = \frac{\lambda}{L} = \frac{RT}{\sqrt{2}\pi d^2 N_A P L} > 0.01$$

where $L = 50\mu\mathrm{m}$. Then the pressure P should satisfy the following:

$$\begin{aligned} P &< \frac{100RT}{\sqrt{2}\pi d^2 N_A L} \\ &= 14\mathrm{kPa} \end{aligned}$$

Problem 4

The flow is being driven only by the motion of the boundary, thus the Navier-Stokes equation reduces to

$$\frac{d^2u}{dy^2} = 0$$

with the boundary conditions: $u = \beta du/dy$ at $y = 0$ and $u = U$ at $y = h$. Solving the differential equation, we get

$$u = \frac{U}{h+\beta}(y+\beta) = \frac{U}{h}y\left(\frac{1+\beta/y}{1+\beta/h}\right)$$

Problem 5

You may use the conventional Hagen-Poiseuille flow solution but with a slip boundary condition at $r = R$, where R is the tube radius: $u_s = -\beta \partial u/\partial r$. Then you find

$$\beta = \frac{R}{4}\left(\frac{U_{obs}}{U_{exp}} - 1\right),$$

where U_{obs} and U_{exp} are the observed flow velocity and the expected flow velocity, respectively. For case 2, we get $\beta = 67.4\ \mu\mathrm{m}$ which is very close to 68 $\mu\mathrm{m}$ in Table 1.