Micro Fluid Mechanics - Spring 2019

Solution Set 2 (Part II)

April 4, 2019 — Prof. Ho-Young Kim

Problem 3

We start with

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\Psi=0$$

The boundary conditions are such that $u_r=0$ or $\partial\Psi/\partial\theta=0$ on $r=a$, $\tau_{r\theta}=2\mu e_{r\theta}=0$ on $r=a$, and $\Psi\to\frac{1}{2}Ur^2\sin^2\theta$ as $r\to\infty$. In the second boundary condition, a component of the rate-of-strain tensor $e_{r\theta}$ is written as

$$2e_{r\theta}=r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial\theta}$$

The third boundary condition suggests that we try a solution of the form $\Psi=f(r)\sin^2\theta$. The free-stream boundary condition further gives

$$f(r)=\frac{A}{r}+Br+\frac{1}{2}Ur^2$$

Substituting this form into the first and second boundary conditions, after tedious calculation, we get

$$\Psi=\frac{1}{2}U(r^2-ar)\sin^2\theta$$

The stress component on the bubble in the radial direction τ_r is

$$\tau_r=-p+2\mu\frac{\partial u_r}{\partial r}$$

on $r=a$. To find p, we note that

$$\frac{\partial p}{\partial r}=\frac{\mu}{r^2\sin\theta}\frac{\partial}{\partial\theta}E^2\Psi$$

Then we get

$$p=p_\infty-\frac{\mu Ua}{r^2}\cos\theta$$

This leads to

$$\tau_r=-p_\infty+\frac{3\mu U}{a}\cos\theta$$

The stress component in the direction of net force is $\tau=\tau_r\cos\theta$. Then the drag force is given by

$$D=\int_0^{2\pi}\int_0^{\pi}\tau a^2\sin\theta d\theta d\phi=4\pi\mu Ua$$

Problem 4

We first take the curl on both sides of

$$\nabla p = \mu \nabla^2 \overline{u}$$

By a vector identity, $\nabla \times \nabla p = 0$, thus $\nabla \times \left(\nabla^2 \overline{u}\right) = 0$. Now we only need to show that $\nabla \times \left(\nabla^2 \overline{u}\right) = \nabla^2 \left(\nabla \times \overline{u}\right)$. A vector identity tells us that $\nabla^2 \overline{u} = \nabla \left(\nabla \cdot \overline{u}\right) - \nabla \times \left(\nabla \times \overline{u}\right)$. Due to continuity, we are left with

$$\nabla \times \left(\nabla^2 \overline{u}\right) = -\nabla \times \nabla \times \left(\nabla \times \overline{u}\right) \tag{1}$$

Using the same vector identity but replacing $\overline{u}$ by $\nabla \times \overline{u}$, we get

$$\nabla^2 \left(\nabla \times \overline{u}\right) = \nabla \left[\nabla \cdot \left(\nabla \times \overline{u}\right)\right] - \nabla \times \left[\nabla \times \left(\nabla \times \overline{u}\right)\right] \tag{2}$$

The first term in the righthand side of Eq.(2) vanishes by a vector identity. Comparing Eqs.(1) and (2), we find $\nabla \times \left(\nabla^2 \overline{u}\right) = \nabla^2 \left(\nabla \times \overline{u}\right)$. Therefore, $\nabla^2 \left(\nabla \times \overline{u}\right) = 0$.

Problem 5

Kinesins are the motor proteins that drive cell motility such as secretory vesicle transport and mitosis. This problem implies that the fundamental understanding of life is deeply related to microhydrodynamics.

(a) $F_{drag} = c_{\|} L v = 2\pi \mu L v / \cosh^{-1}(h/r) \sim 6 \times 10^{-14}$ N ~ 0.1 pN

(b) In the low-Re flows, the drag force is linearly proportional to the product of the viscosity and the velocity. Therefore, assuming that the kinesin exerts the same force as in the previous medium, the viscosity should be increased four-fold. *Note*: In reality, the kinesin can exert much stronger forces than calculated, i.e. it exerts more force in more viscous media. It was found that the viscosity must be increased 100-fold to slow the movement of a 10-μm-long microtubule to 0.25 μm/s, which means that the kinesin exerts $\sim$ 1.4 pN in this case. It was reported that a kinesin molecule could exert forces up to $\sim$ 5 pN against a viscous load. For more information on motor proteins, consult general molecular biology textbooks or J. Howard's *Mechanics of motor proteins and the cytoskeleton*.