Review: Summary questions of the last lecture

What is the meaning of eigenvalues and eigenvectors of Laplacian?

What is the meaning of the multiplicity of eigenvalue 0 of Laplacian?

What is the meaning of the smoothness of an eigenvector?

How to get the eigen-spectrum of Laplacian of the complete graph?

What is the relation btw Laplacian and random walks on undirected graphs?

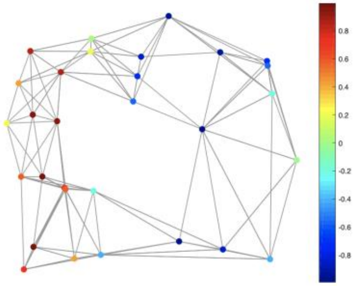
Review: Summary questions of the last lecture

What is the meaning of eigenvalues and eigenvectors of Laplacian?

→ Frequencies and its corresponding graph signals that a graph can have.

A signal $\boldsymbol{f}$ can be written as graph Fourier series:

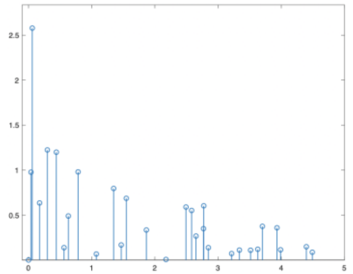
$$\boldsymbol{f} = \sum_{\boldsymbol{i}} \hat{f}_i \boldsymbol{u}_i \qquad \boldsymbol{f}^T \boldsymbol{u}_k = \sum_{\boldsymbol{i}} \hat{f}_i \boldsymbol{u}_i^T \boldsymbol{u}_k = \hat{f}_k \;\; = \boldsymbol{u}_k^T \boldsymbol{f} \qquad \hat{\boldsymbol{f}} = \begin{bmatrix} \hat{f}_{\boldsymbol{1}} \\ \dots \\ \hat{f}_{\boldsymbol{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_1^T \\ \dots \\ \boldsymbol{u}_N^T \end{bmatrix} \boldsymbol{f}$$



$$\hat{\boldsymbol{f}} = \boldsymbol{U}^T \boldsymbol{f}$$

Decompose signal f

Reconstruct signal $\boldsymbol{f}$



Spatial domain: $\boldsymbol{f}$

Spectral domain: $\hat{\boldsymbol{f}}$

Design GCN in spectral domain ↔ Design GCN in spectral domain

Review: Summary questions of the last lecture

What is the meaning of the multiplicity of eigenvalue 0 of Laplacian?

→ The number of connected components in a graph.

[Intuitive Proof]

Letting two eigenvectors be

$\boldsymbol{u}_1 = [\,1\;1\;1\;1\;0\;0\;0\;0]^T$,

$\boldsymbol{u}_2 = [\,0\;0\;0\;0\;1\;1\;1\;1]^T$.

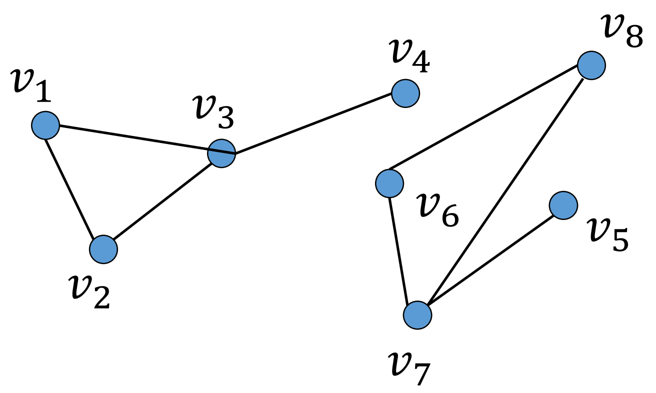
Then

$\boldsymbol{L}\boldsymbol{u}_1 = 0\boldsymbol{u}_1,\ \boldsymbol{L}\boldsymbol{u}_2 = 0\boldsymbol{u}_2.$

Thus

$(0, \boldsymbol{u}_1)$ and $(0, \boldsymbol{u}_2)$ are eigenpairs.

The multiplicity of eigenvalue 0 of L equals to 2.



$$\begin{array}{c} \boldsymbol{L_1} \\ \\ \boldsymbol{L_2} \end{array} \left[\begin{array}{rrrr|rrrr} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{array}\right]$$

$\boldsymbol{L}$

Review: Summary questions of the last lecture

What is the meaning of the smoothness of an eigenvector?

→ The smoothness of a eigenvector is its eigenvalue (frequency).

$$S_G(f) = \boldsymbol{f}^T \boldsymbol{L} \boldsymbol{f} = \boldsymbol{f}^T \boldsymbol{U} \boldsymbol{\Lambda} \boldsymbol{U}^T \boldsymbol{f} = \boldsymbol{\alpha}^T \boldsymbol{\Lambda} \boldsymbol{\alpha} = \|\boldsymbol{\alpha}\|_{\boldsymbol{\Lambda}}^2 = \sum_{1 \le i \le N} \lambda_i \alpha_i^2, \quad \boxed{\boldsymbol{\alpha} = \boldsymbol{U}^T \boldsymbol{f}}$$

Spectral coordinate (unique vector) of eigenvector $\boldsymbol{u}_k$: $\boldsymbol{\alpha}_k = \boldsymbol{U}^T \boldsymbol{u}_k = \boldsymbol{e}_k$.

$$S_G(\boldsymbol{u}_k) = \boldsymbol{u}_k^T \boldsymbol{L} \boldsymbol{u}_k = \boldsymbol{u}_k^T \boldsymbol{U} \boldsymbol{\Lambda} \boldsymbol{U}^T \boldsymbol{u}_k = \boldsymbol{e}_k^T \boldsymbol{\Lambda} \boldsymbol{e}_k = \|\boldsymbol{e}_k\|_{\boldsymbol{\Lambda}}^2 = \sum_{1 \le i \le N} \lambda_i e_{k,i}^2 = \lambda_k$$

Review: Summary questions of the last lecture

How to get the eigen-spectrum of Laplacian of the complete graph?

→ The 1-st eigen-pair is $(0, \mathbf{1}_N)$ and compute the second eigen-pair of which eigenvector is orthogonal to 1-st eigenvector. The remaining ones can be computed to be orthogonal to the previous ones.

If $\boldsymbol{u} \neq \mathbf{0}$ and $\boldsymbol{u} \perp \mathbf{1}_N \;\Rightarrow\; \sum_i u_i = 0$. To get the other eigenvalues, we compute $(\boldsymbol{L}_{\boldsymbol{K}_N}\boldsymbol{u})_1$ and divide by u_1 (letting $u_1 \neq 0$).

$$(\boldsymbol{L}_{\boldsymbol{K}_N}\boldsymbol{u})_1 = (N - 1)u_1 - \sum_{2 \le i \le N} u_i = N u_1$$

→ $(0, \mathbf{1}_N)$, $(N, [1 \;\; -1 \;\; 0 \;\; \ldots 0]^T), \ldots$

Review: Summary questions of the last lecture

What is the relation btw Laplacian and random walks on undirected graphs?

→ The random walks is a stochastic process with a transition probability $p_{ij} = \frac{w_{ij}}{d_i}$ between node i and j of a graph with a Laplacian $L = D - W$.

Transition matrix: $\boldsymbol{P} = [p_{ij}] = \boldsymbol{D}^{-1}\boldsymbol{W}$ (notice $\boldsymbol{L}_{rw} = \mathbf{I} - \boldsymbol{P}$).

Unique stationary distribution $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_N)$ where $\pi_i = \frac{d_i}{vol(\boldsymbol{V})}$.

$$\leftarrow vol(\boldsymbol{G}) = vol(\boldsymbol{V}) = vol(\boldsymbol{W}) \triangleq \sum_i d_i = \sum_{ij} w_{ij}.$$

$\boldsymbol{\pi} = \frac{\mathbf{1}^T\boldsymbol{W}}{vol(\boldsymbol{W})}$ verifies $\boldsymbol{\pi}\boldsymbol{P} = \boldsymbol{\pi}$ as

$$\boldsymbol{\pi}\boldsymbol{P} = \frac{\mathbf{1}^T\boldsymbol{W}\boldsymbol{P}}{vol(\boldsymbol{W})} = \frac{\mathbf{1}^T\boldsymbol{D}\boldsymbol{P}}{vol(\boldsymbol{W})} = \frac{\mathbf{1}^T\boldsymbol{D}\boldsymbol{D}^{-1}\boldsymbol{W}}{vol(\boldsymbol{W})} = \frac{\mathbf{1}^T\boldsymbol{W}}{vol(\boldsymbol{W})} = \boldsymbol{\pi}.$$