5.4.3 Approx. Deflections from PSTPE and Rayleigh Ritz Procedure

For approx. solutions, assume

$$\bar{w}(x,y,z) = \sum_{n=1}^{N\leftarrow \text{finite no.}} a_n \, \phi_n(x,y,z)$$

Unknown amplitude to be determined. (a_n)

ϕ_n: assumed deflection shape over the entire structure

1. Must satisfy the geom. B.C's explicitly (essential to define the problem), but need not satisfy the force B C's.
2. The $\phi_n(x,y,z)$ must be a linearly independent set of fns --- such that none can be form as a linear combination of the others.
3. The $\phi_n(x,y,z)$ must be continuous, single valued, and possess spatial derivatives.

Placing the above into U and V_e, and performing the indicated integration evaluations, we obtain

$$\Pi_p = U + V_e = \Pi_p(a_1, a_2, \ldots, a_N)$$

Application of the PSTPE ($\delta \Pi_p = 0$) gives:

$$\delta \Pi_p = \frac{\partial \Pi_p}{\partial a_1}\delta a_1 + \frac{\partial \Pi_p}{\partial a_2}\delta a_2 + \cdots + \frac{\partial \Pi_p}{\partial a_N}\delta a_N = 0$$

But each displ. variation $\delta a_1, \delta a_2, \ldots$ is independent and arbitrary. Hence $\delta \Pi_p = 0$ if and only if:

$$\left.\begin{aligned} \frac{\partial \Pi_p}{\partial a_1} &= 0 \\ \frac{\partial \Pi_p}{\partial a_2} &= 0 \\ &\vdots \\ \frac{\partial \Pi_p}{\partial a_N} &= 0 \end{aligned}\right\}$$

Set of N algebraic eqs. to solve for the N unknowns $a_1\ a_2 \ldots a_N$.

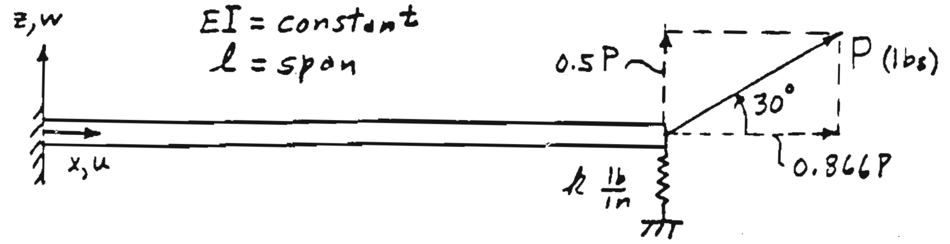
This is the Rayleigh-Ritz Procedure.

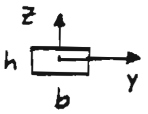
Example: Use of the PSTPE and the Rayleigh-Ritz Method

Bending and Streching of a Bernoulli-Euler Beam

Find the approximate deflections for $u(x)$ and $w(x)$

EI = constant
ℓ = span





Beam Bending BC's: $w(0)=0$ Geometric ; $M(\ell)=EI\,w''(\ell)=0$ Force Type
$w'(0)=0$ Geometric ; $+S(\ell)\equiv EI\,w'''(\ell)=+k\,w(\ell)-0.5P$ Force Type

Beam Stretching BC's: $u(0)=0$ Geometric ; $+F(\ell)\equiv EA\left(\frac{\partial u}{\partial x}\right)_\ell=+.866P$ ~ Force Type

For the Rayleigh-Ritz Method, assume a functional type of solution for $w(x)$ and $u(x)$

$$w(x)=\alpha_0+\alpha_1\left(\frac{x}{\ell}\right)+\alpha_2\left(\frac{x}{\ell}\right)^2+\alpha_3\left(\frac{x}{\ell}\right)^3$$

$$\therefore w'(x)= +\frac{\alpha_1}{\ell}+\frac{2\alpha_2 x}{\ell^2}+\frac{3\alpha_3 x^2}{\ell^3}$$

Apply Geom BC's only:

$$\therefore w(0)=\alpha_0=0\rightarrow\alpha_0=0$$

$$w'(0)=\frac{\alpha_1}{\ell}=0\rightarrow\alpha_1=0$$

$$\therefore w(x)=\alpha_2\left(\frac{x}{\ell}\right)^2+\alpha_3\left(\frac{x}{\ell}\right)^3$$

$$w''(x)=\frac{2\alpha_2}{\ell^2}+\frac{6\alpha_3 x}{\ell^3}\quad\therefore[w''(x)]^2=4\frac{\alpha_2^2}{\ell^4}+24\frac{\alpha_2\alpha_3}{\ell^5}x+36\frac{\alpha_3^2}{\ell^6}x^2$$

$$u(x)=\beta_0+\beta_1\frac{x}{\ell}$$

Apply Geom BC only:

$$u(0)=\beta_0=0\rightarrow\beta_0=0$$

$$\therefore u(x)=\beta_1\frac{x}{\ell}\qquad\therefore u'(x)=\frac{\beta_1}{\ell}$$

$$\therefore[u'(x)]^2=\frac{\beta_1^2}{\ell^2}$$

Next, evaluate the internal strain energy: $U=\int_0^\ell\frac{EA}{2}\left(\frac{du}{dx}\right)^2dx+\int_0^\ell\frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2dx+\frac{k}{2}[w(\ell)]^2$

$$U=\int_0^\ell\frac{EA}{2}\left[\frac{\beta_1^2}{\ell^2}\right]dx+\int_0^\ell\frac{EI}{2}\left[4\frac{\alpha_2^2}{\ell^4}+24\frac{\alpha_2\alpha_3}{\ell^5}x+36\frac{\alpha_3^2}{\ell^6}x^2\right]dx+\frac{k}{2}\left[\alpha_2\left(\frac{\ell}{\ell}\right)^2+\alpha_3\left(\frac{\ell}{\ell}\right)^3\right]^2$$

$$=\frac{EA}{2}\frac{\beta_1^2}{\ell}+\frac{EI}{2\ell^3}\left[4\alpha_2^2+12\alpha_2\alpha_3+12\alpha_3^2\right]+\frac{k}{2}\left[\alpha_2^2+2\alpha_2\alpha_3+\alpha_3^2\right]$$

$U_e=-$(Work Fn. of Ext. Prescribed Loads) $=-.866P\,u(\ell)-.5P\,w(\ell)\equiv-.866P\beta_1-0.5P[\alpha_2+\alpha_3]$

PSTPE says $\delta(U+U_e)\equiv\delta\Pi_p=0\rightarrow\frac{\partial\Pi_p}{\partial\alpha_2}\underbrace{\delta\alpha_2}_{arb.}+\frac{\partial\Pi_p}{\partial\alpha_3}\underbrace{\delta\alpha_3}_{arb}+\frac{\partial\Pi_p}{\partial\beta_1}\underbrace{\delta\beta_1}_{arb.}=0\quad\therefore\frac{\partial\Pi_p}{\partial\alpha_2}=0;\ \frac{\partial\Pi_p}{\partial\alpha_3}=0;\ \frac{\partial\Pi_p}{\partial\beta_1}=0$

$$\frac{\partial\Pi_p}{\partial\beta_1}=0:\ \frac{EA}{2}\frac{2\beta_1}{\ell}-.866P=0\quad\therefore\beta_1=\frac{.866P\ell}{EA}$$

$$\therefore\boxed{u(x)=\beta_1\frac{x}{\ell}=\frac{.866P\ell}{EA}\frac{x}{\ell}}\qquad\frac{lb\ in}{\frac{lb}{in^2}\ in^2}=\ ;$$

$$\frac{\partial\Pi_p}{\partial\alpha_2}=0:\ \frac{EI}{2\ell^3}[8\alpha_2+12\alpha_3]+\frac{k}{2}[2\alpha_2+2\alpha_3]-0.5P=0$$

$$\frac{\partial\Pi_p}{\partial\alpha_3}=0:\ \frac{EI}{2\ell^3}[12\alpha_2+24\alpha_3]+\frac{k}{2}[2\alpha_2+2\alpha_3]-0.5P=0$$

$$\alpha_2=+0.5P\left(\frac{EI}{\ell^3}\right)/D;\ \alpha_3=0.5P\left(-2\frac{EI}{\ell^3}\right)/D$$

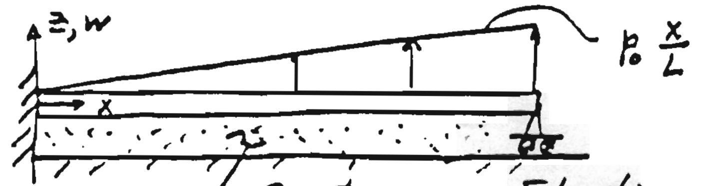
where $D=\left[4\frac{EI}{\ell^3}+k\right]\left[12\frac{EI}{\ell^3}+k\right]-\left[6\frac{EI}{\ell^3}+k\right]^2$

$$\therefore\boxed{w(x)=\alpha_2\left(\frac{x}{\ell}\right)^2+\alpha_3\left(\frac{x}{\ell}\right)^3=+\frac{3EIP}{\ell^3D}\left(\frac{x}{\ell}\right)^2-\frac{EIP}{\ell^3D}\left(\frac{x}{\ell}\right)^3}$$

$$\frac{\frac{lb}{in^2}\,in^4\,lb}{in^3\,\frac{lb^2}{in^2}}\left(\frac{in}{in}\right)^{2+3}=in\quad OK.$$

Comments on Examples : PSTPE and Rayleigh–Ritz

1. Uniform B-Euler beam on a continuous elastic foundation.



EI = constant
L = span

Continuous Elastic Foundation: $k_f \; \dfrac{\text{lb}}{(\text{in of defl.})\ \text{in of span}}$

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx + \int_0^L \frac{k_f}{2}\,[w(x)]^2\,dx$$

$$\frac{\text{lb}}{\text{in}^2}\ \text{in}^4\left(\frac{\text{in}}{\text{in}^2}\right)^2 \text{in} \qquad \frac{\text{lb}}{\text{in}^2}\ (\text{in})^2\ \text{in}$$

$$U_e = -\int_0^L p_z(x)\, w(x)\, dx$$

A. Find approx. defls. by R-R procedure.

B. Can derive Euler DE and BC's from PSTPE : $EI\dfrac{d^4w}{dx^4} + k_f w = p_z(x)$

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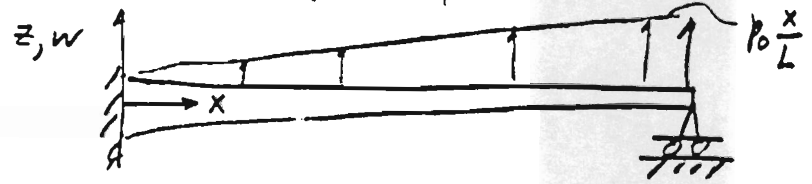
2. Bern.-Euler beam with non-uniform $EI = (EI)_0\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)\right]$.



L = span

Assume : $w(x) = \alpha_1 + \alpha_2 \frac{x}{L} + \alpha_3\left(\frac{x}{L}\right)^2 + \alpha_4\left(\frac{x}{L}\right)^3$

Apply 3 Geom. BC's : $w(0) = w'(0) = w(L) = 0 \Rightarrow \alpha_1 = \alpha_2 = 0;\ \alpha_3 = -\alpha_4$

$\therefore\ w(x) = \alpha_3\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right]$

$$U = \int_0^L \frac{EI}{2}\left(\frac{d^2w}{dx^2}\right)^2 dx = \frac{(EI)_0}{2}\int_0^L \left[1 - 2\frac{x}{L}\right]\left[\alpha_3\left(\frac{2}{L^2} - \frac{6x}{L^3}\right)\right]^2 dx$$

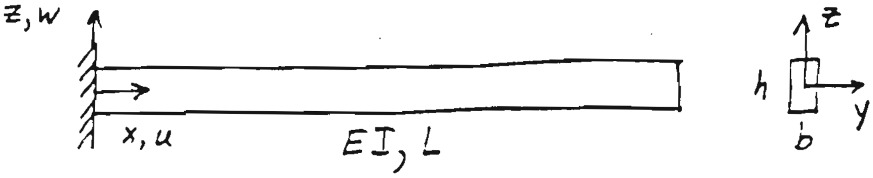
$$U_e = -\int_0^L p_z(x)\, w(x)\, dx = -\int_0^L p_0 \frac{x}{L}\left[\alpha_3\left\{\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right\}\right] dx$$

Then $\Pi_p = \Pi_p(\alpha_3)$

Use PSTPE : $\delta \Pi_p = 0 \;\Rightarrow\; \dfrac{\partial \Pi_p}{\partial \alpha_3} = 0$ solve for α_3.

For a better solution, use more and/or more admissible assumed deflection shapes.

Rayleigh-Ritz Example for Bernoulli-Euler Beam
(small deflection linear-elastic with thermal loading)



$A = bh \; ; \; I = \frac{1}{12} b h^3$

$\Delta T = A_0 + A_1\left(\frac{2z}{h}\right) + A_2\left(\frac{2z}{h}\right)^2$

Use the Rayleigh-Ritz procedure via PSTPE to obtain an approximate solution for $u(x)$ and $w(x)$; both u and w are displacements of the CB origin. Let E and α be temperature independent

NOTE: $\gamma_{11} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}$ and $\gamma_{11} = \frac{\sigma_{11}}{E} + \alpha \Delta T$

$\therefore \sigma_{11} = E[\gamma_{11} - \alpha \Delta T]$

Then: $\bar{U} = \int_0^{\gamma_{11}} \sigma_{11}\, \delta\gamma_{11} = \int_0^{\gamma_{11}} E[\gamma_{11} - \alpha\Delta T]\,\delta\gamma_{11} = \frac{E}{2}\gamma_{11}^2 - E\alpha\Delta T\,\gamma_{11} \; ; \; U = \iiint \bar{U}\, dV$

Assume deflection shapes for u and w

$u = a + bx$ but $u(0) = 0 \; \therefore \; \boxed{u = bx}$

$w = c + dx + ex^2$ but $w(0) = 0 \Rightarrow c = 0$; $w'(0) = d + 2e\underbrace{x}_{0} = 0 \therefore d = 0 \therefore \boxed{w = ex^2}$

Have satisfied Geom BC's explicitly.

$\gamma_{11} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} = b - z(2e)$

$$U = \iiint \frac{E}{2}(b - 2ez)^2\, dx\,dy\,dz - \iiint E\alpha\left[A_0 + A_1\left(\frac{2z}{h}\right) + A_2\left(\frac{2z}{h}\right)^2\right](b - 2ez)\, dx\,dy\,dz$$

$$= \iiint \frac{E}{2}(b^2 - 4bez + 4e^2z^2)\, dx\,dy\,dz$$

$$- \iiint E\alpha\left[A_0 b + bA_1\frac{2z}{h} + bA_2\left(\frac{2z}{h}\right)^2 - 2eA_0 z - 2eA_1\left(\frac{2z^2}{h}\right) - 2eA_2\left(\frac{4z^3}{h^2}\right)\right] dx\,dy\,dz$$

$$= \frac{E}{2}\left[b^2AL - 0 + 4e^2IL\right] - E\alpha\left[bA_0AL + 0 + \frac{4bA_2IL}{h^2} - 0 - \frac{4A_1eIL}{h} - 0\right]$$

and $U_e = 0$. $\Pi_p = U + U_e = \Pi_p(b, e)$. Next, set $\delta\Pi_p = 0$:

$$\frac{\partial \Pi_p}{\partial b} = \frac{E}{2}(2bAL) - E\alpha\left(A_0AL + 4\frac{IL}{h^2}A_2\right) = 0 \;\therefore\; b = \alpha\left(A_0 + 4\frac{I}{Ah^2}A_2\right) = \alpha\left[A_0 + \frac{A_2}{3}\right]$$

$$\frac{\partial \Pi_p}{\partial e} = \frac{E}{2}(8ILe) - E\alpha\left(-\frac{4A_1IL}{h}\right) = 0 \;\therefore\; e = -\frac{A_1\alpha}{h}$$

$$\therefore \quad u = \left[\alpha\left(A_0 + \frac{A_2}{3}\right)\right]x$$

$$w = \left[-\frac{A_1\alpha}{h}\right]x^2$$

Students:

Try $u(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$

$w(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$