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Design Criteria for Reinforced Columns under Axial Load and Biaxial Bending

By BORIS BRESLER

Several design criteria for columns subjected to compression combined with biaxial bending are discussed. The approximate load carrying capacity is defined in terms of easily determined parameters without the cumbersome trial and error procedures.

■ THE CRITERIA GENERALLY proposed[1-5] for determining ultimate strength of reinforced concrete members subjected to compression combined with biaxial bending are based on limiting the maximum strain (or stress) in the concrete to some prescribed value. Ideal non-linear stress-strain laws for steel and concrete, conservation of plane sections, no slip, and no tension resistance by concrete are usually assumed.

The load carrying capacities discussed here apply to relatively short columns for which the effect of lateral deflections on the magnitudes of bending moments is negligible. Furthermore, effects of sustained load and of reversal of bending moments are not considered.

When the position of the neutral axis is known or assumed, the magnitude of the load P_u and the components of bending moments M_x and M_y which result in the prescribed limit strain, can be determined using equations of equilibrium. When the position of the neutral axis is not known, the equations of equilibrium can be solved only by the method of successive approximations. All such procedures involve more or less tedious cycles of numerical calculations.

The criteria proposed in this paper are based on approximations of "surfaces of failure" which are defined as surfaces obtained by plotting the failure load P_u as a function of its eccentricities x and y or of the components of bending moment M_x and M_y (Fig. 1, 2, and 3).

FAILURE SURFACES

The magnitude of the failure load P_u acting on the column with eccentricities x and y depends principally on the column dimensions, amount and distribution of steel reinforcement, stress-strain characteristics of steel and concrete, and on such secondary factors as amount of concrete cover and arrangement and size of lateral ties or spiral. The mathematical expressions required to define the failure load appear

ACI member **Boris Bresler** is professor of civil engineering, Department of Civil Engineering, University of California, Berkeley. He has been on the university faculty since 1946 and has been involved in many and varied research programs. He is a member of ACI-ASCE Committee 326, Shear and Diagonal Tension.

to be so complex that an analytical formulation is not possible at this time. It is assumed here that such an expression would result in a function which would define a surface S_1 (P_u, x, y) as shown in Fig. 1.

This basic surface can be transformed in various ways. For example, a "reciprocal" surface can be derived from S_1, wherein the reciprocal of the failure load P_u is used, so that the surface S_2 $(1/P_u, x, y)$ appears as shown in Fig. 2. Another surface can be obtained by relating the failure load P_u to moments $M_x = P_u y$ and $M_y = P_u x$, so that a surface $S_3 = (P_u, M_x, M_y)$ appears as shown in Fig. 3. It can be seen that the traces of surface S_3 on the $M_x = 0$, and $M_y = 0$ planes are the familiar P-M interaction curves.

While exact mathematical expressions defining the failure surfaces cannot be established, some approximations can be derived. An approximation based on the surface S_3 was recently suggested by Pannell[6] which proposed that an equivalent moment M_g about the major axis y replace the two bending moment components M_x and M_y. The moment M_g is defined as:

$$M_g = K M_y \qquad (1)$$

where K is a coefficient depending on M_x/M_y, section shape, amount and distribution of reinforcement, and steel cover ratio. The derivation of this coefficient has not been included in the reference cited and thus its validity cannot be fully evaluated. Calculation of this coefficient K requires three additional functions which are defined by graphs, as apparently they cannot be readily defined by simple mathematical equations. Furthermore, it appears that the values of K are defined

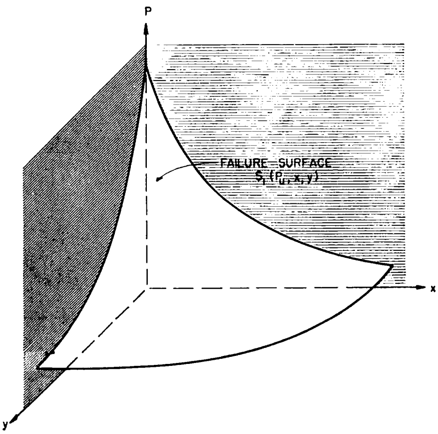

Fig. 1—Failure surface S_1 (P_u, x, y)

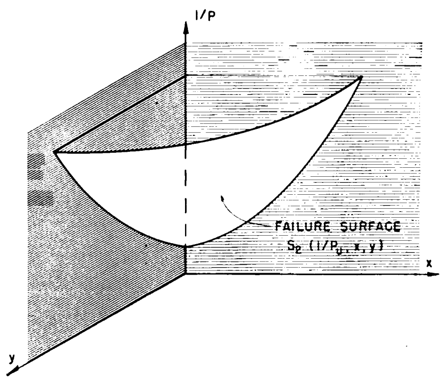

Fig. 2—Failure surface S_2 $(1/P_u, x, y)$

only for cases when nearly equal amounts of steel in each face are used. While the final evaluation of this method must await its full publication, it seems that its use is limited to only those cases for which values of K have been derived and plotted.

Two alternative approximations are discussed here, one of which appears to be remarkably simple and accurate.

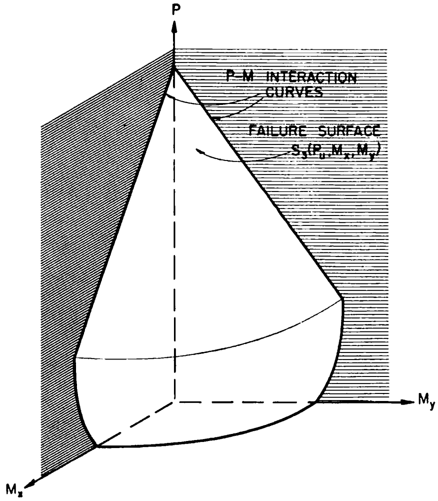

Fig. 3—Failure surface S_3 (P_u, M_x, M_y)

Method A

The ordinate $1/P_u$ on the surface S_2 $(1/P_u, x, y)$ can be approximated by a corresponding ordinate $1/P_i$ on the plane S_2' $(1/P_i, x, y)$, Fig. 4. The plane S_2' is defined as one passing through three characteristic points (Fig. 4):

$$A\left(\frac{1}{P_x}\ x_A,\ 0\ \right),\ B\left(\frac{1}{P_y}\ y_B,\ 0\right),\ \text{and}\ C\left(\frac{1}{P_o},\ 0,\ 0\right)$$

For a particular column, the value of P_o represents the load carrying capacity under pure axial compression; P_x and P_y represent the load carrying capacities under compression with uniaxial eccentricities x and y, respectively. Thus, for a given column, point C is independent of eccentricities, point B depends on eccentricity y only, and point A depends on eccentricity x only. The loads P_o, P_x, and P_y can be determined by established methods.[3,7,8]

For every point on the surface S_2 $(1/P_u, x, y)$, there is a corresponding plane S_2' $(1/P_i, x, y)$. The approximation of S_2 involves an infinite number of planes, each one applicable only for particular values of eccentricities x and y, and thus each plane defines only one point $1/P_i$ which serves as an approximation to $1/P_u$.

The expression for $1/P_i$ can be easily derived as follows. Let $z = 1/P_i$ corresponding to particular values of x and y; then the plane determined by the three characteristic points A, B, and C (Fig. 4) is defined by the following equation:

$$\left[x - x_A + \frac{x_A}{y_B}\, y \right] + \frac{z_B - z_A}{z_C - z_B}\left[x - x_A + \frac{x_A}{z_B - z_A}\,(z - z_A) \right] = 0 \quad \text{....(2)}$$

The ordinate z_i on the plane corresponding to $x = x_A$, $y = y_B$ is found from Eq. (2).

$$z_i = z_A + z_B - z_C \quad \text{........(3)}$$

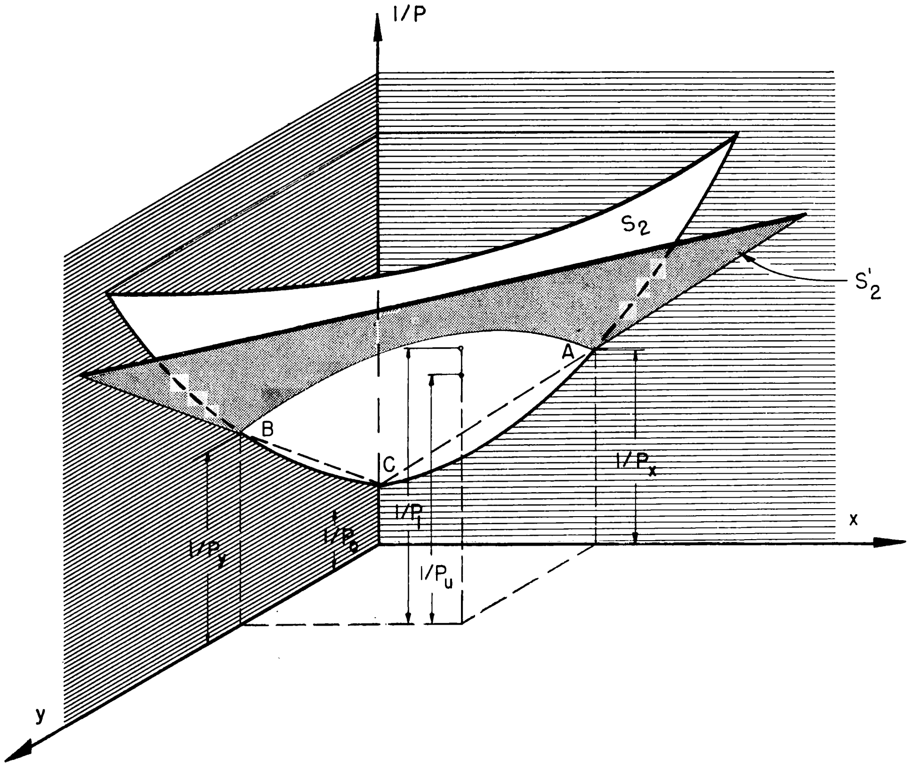

Fig. 4—Graphical representation for Method A

or

$$\frac{1}{P_i} = \frac{1}{P_x} + \frac{1}{P_y} - \frac{1}{P_o} \qquad (4)$$

This approximation has the following advantages:

(1) It is simple in form.

(2) The parameters P_o, P_x, P_y, are determined in a relatively simple manner.

(3) The method appears to be quite general, at least for those common shapes of columns and arrangements of reinforcement for which the point of the surface S_2 can be approximated by a point on the plane S_2' (Fig. 4).

A formula similar to Eq. (4) is given in the Russian specifications,[9] but its derivation could not be found in the Russian textbooks[10,11] or in Russian technical literature available to the author.

Method B

This method is based on approximating surface S_3 (P_u, M_x, M_y) by a family of curves corresponding to constant values of P_u (Fig. 5) which may be thought of as "load contours."

The general form of these curves can be approximated by a non-dimensional interaction equation:

$$\left(\frac{M_x}{M_{xo}}\right)^{\alpha} + \left(\frac{M_y}{M_{yo}}\right)^{\beta} = 1.0 \quad \text{.......... (5)}$$

where

$$M_x = P_u y,\ M_{xo} = P_u y_o \quad \text{when} \quad x = M_y = 0;\ M_y = P_u x,\ M_{yo} = P_u x_o$$

when $y = M_x = 0$; and α and β are exponents depending on column dimensions, amount and distribution of steel reinforcement, stress-strain characteristics of steel and concrete, amount of concrete cover, and arrangement and size of lateral ties or spiral.

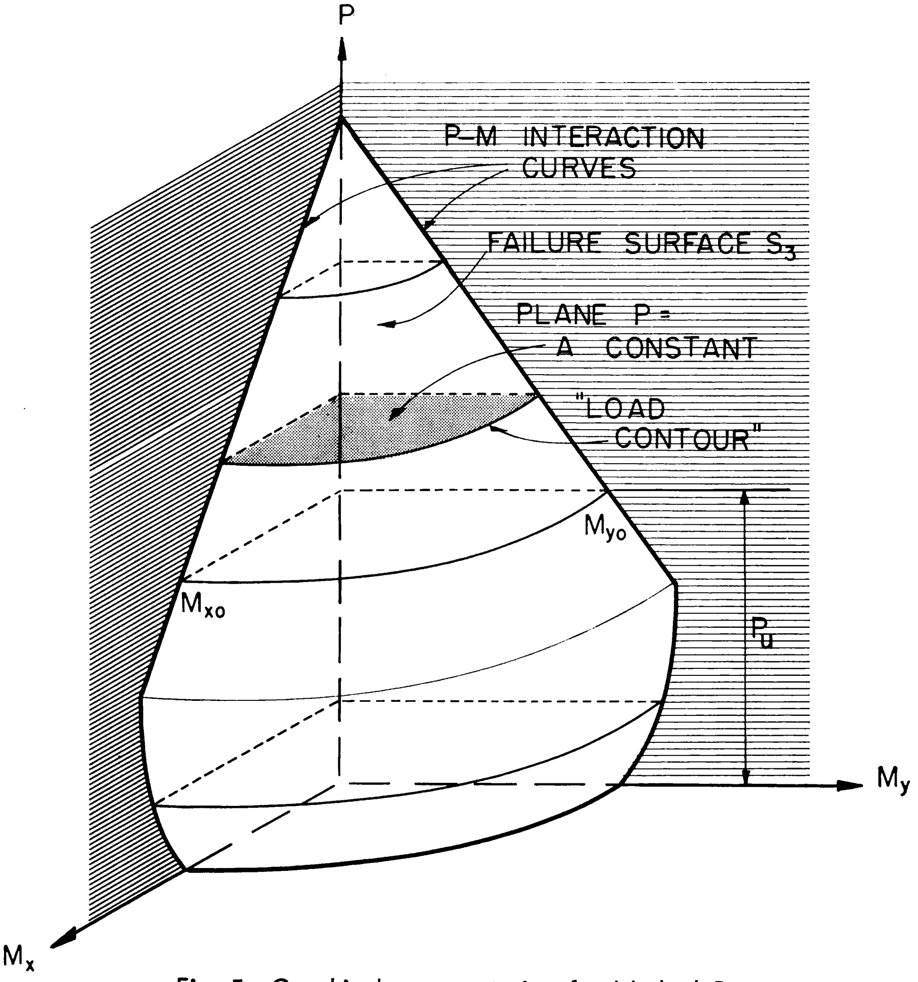

Fig. 5—Graphical representation for Method B

Eq. (5) can be further simplified and transformed into an expression more convenient for design. Dividing all moments in Eq. (5) by P_u results in the following:

$$\left(\frac{y}{y_o}\right)^{\alpha} + \left(\frac{x}{x_o}\right)^{\beta} = 1 \qquad (6)$$

Eq. (6) is written in the form of an interaction equation using eccentricities instead of loads or stresses. Assuming $\alpha = \beta$, the shapes of such interaction curves for different values of α are shown in Fig. 6.

For a given case, the design values of P_u, x, and y are generally known, and for a trial section the values of y_o and x_o corresponding to P_u acting with a single eccentricity can be easily determined. Thus verification of the adequacy of the trial section using Eq. (6) becomes a simple procedure.

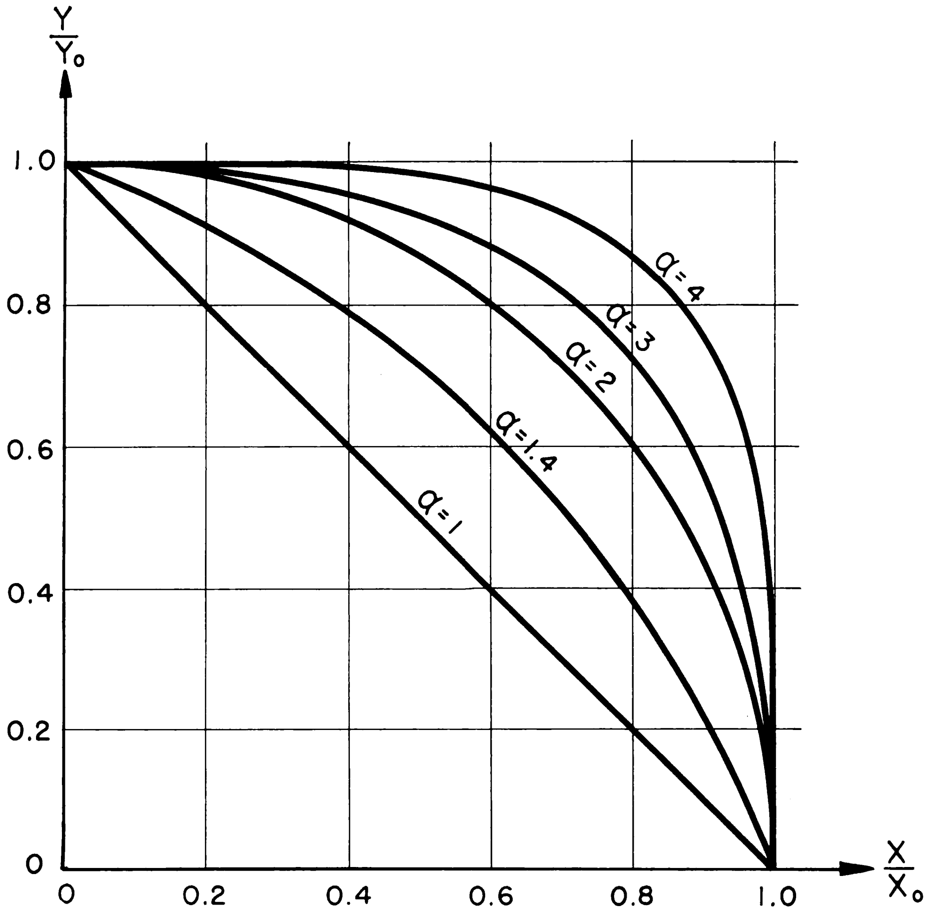

Fig. 6—Interaction curves

ANALYTICAL AND EXPERIMENTAL VERIFICATION

To evaluate the validity of the proposed methods, preliminary calculations and tests were carried out. Values of P_u, M_x, and M_y were calculated for a group of five rectangular columns assuming various positions of neutral axis for each of the columns, and using Jensen's trapezoidal stress-strain law[12] for concrete and the conventional trapezoidal stress-strain law for steel reinforcement. It was found that the strength criteria could be closely approximated by Eq. (6) assuming $\alpha = \beta$. Thus a strength criterion can be defined by Eq. (7).

$$\left(\frac{y}{y_o}\right)^{\alpha} + \left(\frac{x}{x_o}\right)^{\alpha} = 1 \qquad (7)$$

where α is a numerical constant for given column characteristics. The

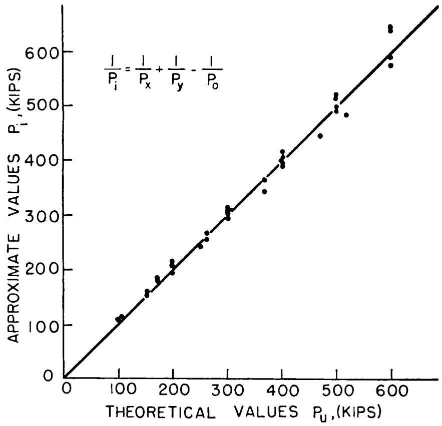

Fig. 7—Comparison of P_i and P_u

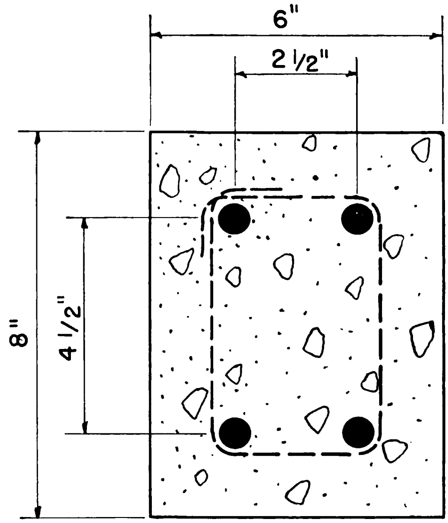

Fig. 8 — Column details. All columns were 4 ft long reinforced with four #5 bars; $f_y = 53.5$ ksi. Ties were 1/4-in. plain bars spaced at 4 in. on centers. Special loading assemblies were used at ends of columns

TABLE 1 — COLUMN CHARACTERISTICS AND COMPUTED VALUES OF α

Column	Size, in.	Reinforcing steel			Concrete f_c', ksi	Coefficient α†
		Bars*	p, percent	f_y, ksi		
A	15x15	4-#7	1.07	50	3	1.48
B	15x15	8-#7	2.14	50	3	1.35
C	15x15	8-#10	4.49	50	3	1.15
D	12x20	4-#7	1.0	50	3	1.55
E	12x20	8-#7	2.0	50	3	1.45

*Concrete cover for all columns is taken as 2 in. clear for main reinforcing bars.
†See Eq. (7).

TABLE 2 — COMPARISON OF COMPUTED VALUES OF P_i AND P_u

Column (Table 1)	x, in.	y, in.	P_u,* kips	P_x,* kips	P_y,* kips	P_o,* kips	P_i,† kips	P_i/P_u
A	8.1	8.1	105	200	200	795	115	1.09
	5.8	5.8	170	300	300		185	1.09
	4.0	4.0	260	400	400		267	1.03
	2.8	2.8	365	500	500		365	0.03
	1.7	1.7	515	600	600		485	0.94
	11.1	4.2	100	125	395		108	1.08
	8.4	3.5	150	190	470		162	1.08
	7.0	2.7	200	245	515		211	1.06
	4.9	1.8	300	355	585		306	1.02
	3.5	1.3	400	445	640		393	0.98
	2.4	0.9	500	540	690		490	0.98
	1.5	0.6	600	620	725		575	0.96
B	6.8	6.8	170	300	300	915	180	1.06
	4.9	4.9	260	400	400		256	0.98
	3.6	3.6	365	500	500		344	0.94
	2.5	2.5	475	600	600		447	0.94
	10.5	4.8	150	190	405		151	1.01
	7.8	3.6	200	260	500		210	1.05
	5.6	2.5	300	360	600		298	0.99
	4.6	2.1	400	460	650		381	0.95
	3.0	1.2	500	555	765		496	0.99
C	6.8	6.8	250	400	400	1179	241	0.96
	5.3	5.3	300	490	490		310	1.03
	3.9	3.9	400	600	600		404	1.01
	2.9	2.9	500	715	715		513	1.03
	2.1	2.1	600	835	835		645	1.08
	9.1	4.3	250	310	565		242	0.97
	7.2	3.4	300	385	655		305	1.02
	5.1	2.4	400	505	780		415	1.04
	3.8	1.8	500	610	885		531	1.04
	2.8	1.3	600	730	960		640	1.07
E	11.2	3.3	200	250	450	960	193	0.96
	7.5	2.2	300	375	580		299	0.99
	5.4	1.6	400	495	675		407	1.02
	4.1	1.2	500	585	745		502	1.00
	3.0	0.9	600	675	810		590	0.98
	6.8	5.7	200	270	405		194	0.97
	4.5	3.8	300	550	395		303	1.01
	3.2	2.7	400	640	505		400	1.00
	2.4	2.1	500	715	600		492	0.98
	1.8	1.5	600	780	690		592	0.99

*Values computed using Jensen's stress-strain law for concrete.
†Values computed using Eq. (4).

sections used in the preliminary numerical studies and the results of the calculations are shown in Table 1.

Values of P_u, P_o, P_x, and P_y corresponding to selected eccentricities x and y were determined for a group of four columns. Using Eq. (4) values of P_i were calculated for these columns and the results compared with values of P_u computed directly on the basis of stress-strain laws. This comparison is shown in Table 2 and in Fig. 7.

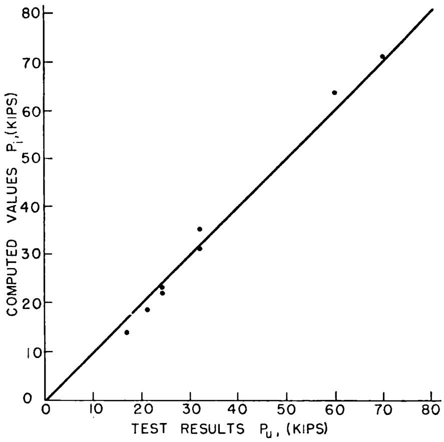

Fig. 9—Comparison of P_i and P_u

Eight columns were tested in the laboratory to determine values of P_x, P_y, and P_u. The details of the columns are shown in Fig. 8 and the test results are shown in Table 3. Values of P_i corresponding to the test values of P_x and P_y were calculated using Eq. (4), and these are compared with the test values in Table 3 and in Fig. 9.

SUMMARY

Two criteria for design of columns subjected to compression combined with biaxial bending were proposed. One, defined by Eq. (7), with calculated values of α varying from 1.15 to 1.55 (see Table 1), was found to provide a good approximation of analytical results. Greater variation in values of α is expected for columns with a wider range of variation in shape and in values of f_c', f_y, and p.

Preliminary verification indicated that ultimate strength P_i predicted by Eq. (4) is in excellent agreement with calculated theoretical values and with test results, the maximum deviation being 9.4 percent, and average deviation being 3.3 percent.

TABLE 3 — COMPARISON OF TEST RESULTS WITH COMPUTED VALUES

Column*	f_c', ksi	x, in.	y, in.	Test P_u	Computed				P_i/P_u
					P_x†	P_y†	P_o‡	P_i§	
B-1	3.7	6	0	24	23.6	—	—	23.6	0.98
B-2	3.9	3	0	60	63.8	—	—	63.8	1.06
B-3	3.7	0	4	70	—	71.3	—	71.3	1.02
B-4	4.6	0	8	32	—	31.1	—	31.1	0.97
B-5	3.2	3	4	32	55.5	65.2	191.4	35.3	1.10
B-6	3.7	6	8	17	23.6	30.2	210.8	14.2	0.84
B-7	3.5	6	4	21	23.4	69.0	203.3	18.8	0.90
B-8	3.6	3	8	24	59.5	28.3	205.5	21.6	0.89

*f_y = 52.3 ksi for all columns.
†Values computed using Eq. (A9) or (A11) of the appendix in ACI Building Code ACI 318-56.
‡Values computed using Eq. (A6) of the ACI Building Code.
§Values computed using Eq. (4).

Based on the preliminary studies outlined above, Eq. (4) appears to provide a simple, direct, and accurate approximation of ultimate strength of a reinforced concrete column subjected to axial compression and biaxial bending.

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