

Scaling laws in the macro-, micro- and nanoworlds

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Abstract

The microworld and the nanoworld are now of interest to the public, scientists, engineers and industrialists. Effects negligible at the macroscopic level become important at the micrometre scale, and *vice versa*. Moreover, at the nanoscale, quantum effects may become dominant. Scaling laws are useful for an understanding of the origin of such differences and/or to generalize results obtained at various scales. It is the aim of this paper to examine scaling laws relevant to mechanics, fluids, electromagnetism, thermodynamics, optics and quantum mechanics. Examples related to biology, micromachines and nanotechnologies are treated to show the usefulness of the scaling laws.

1. Introduction

In recent years, interest in the domains of the microworld and the nanoworld has increased rapidly. The advent of so-called mesoscopic physics, the development of micromachines or micro-electrical mechanical systems (MEMS), the synthesis of nanotubes, the development of nanotechnology, etc... are among the most cited subjects. From both the fundamental and applied points of view, there are interesting questions about these micro- and nanoworlds. In particular, effects negligible at the macroscopic level are important at the micrometre scale, and *vice versa*. Moreover, when one goes from the macroworld to the nanoworld, one passes through two diffuse limits.

When the characteristic dimensions of the elements decrease from the macroscopic to the micrometre size, the effects of gravity become negligible as compared with adhesive and friction effects. Surface tension dominates gravity; other examples are given later. This implies that our reasoning, based on our experience at the macroscopic level, is no longer valid. We have to modify our rules of thinking. The boundary between the macroscopic and microscopic levels is not sharp; it depends on the effect to be considered.

When the characteristic size decreases further to attain the nanometre range, another limit is encountered. While the macroscopic properties of matter remain generally valid at the micrometre size, surface effects become dominant at the nanometre scale. Moreover, when one reaches the interatomic distance range, quantum effects appear.

In order to understand the transition from macroscopic to microscopic dimensions, one obviously needs to consider pluridisciplinary effects. This makes the teaching of, for instance, micromachines, a difficult task [1]. A starting point would be to deal with scaling laws. It is the aim of this paper to show that the discussion of scaling laws and their domains of validity are important in order to understand the origin of the differences between the macroworld, the microworld and the nanoworld. For teaching purposes, the subject has some advantages.

- (1) It shows students that although scaling laws are simple and are generally deduced at the macroscopic level, they may be used to understand the behaviour of the micro- and nanoworlds.
- (2) However, this has to be done with some caution, in particular due to the fact that ‘macroscopic’ scaling laws are deduced under specific ‘experimental’ conditions. These are not always valid at the microscopic and nanoscopic levels.
- (3) The present discussion would help students to better understand the validity limits of well known ‘universal’ laws.
- (4) Although we restrict ourselves to physical parameters, the present discussion is related to various disciplines : physics, biology (the influence of size on the conditions of life), technology and chemistry.
- (5) The discussion also shows that boundaries between the macroscopic and microscopic worlds exist, but are not sharp and depend on the property to be studied.
- (6) The subject is a good introduction to modern scientific and technological subjects, like microphysics and nanotechnology.
- (7) Scaling laws illustrate the fact that shrinking a body does not only lead to size reduction, but also to different modifications of physical effects.

In the following sections, scaling laws relevant to mechanics, fluids, electromagnetism, thermodynamics, optics and quantum physics are considered. We restrict ourselves to physics. It is obvious that chemical effects would also be interesting. In the final section the shape of small animals, some properties of micromachines and nanotechnologies are discussed.

2. Scaling laws, mechanics

Let us consider elements with a typical linear dimension L . In the following, except when explicitly stated, it is assumed that all the linear dimensions vary proportionally to L . This implies that all areas S vary like L^2

$$S \sim L^2 \quad (1)$$

and all volumes V vary like L^3

$$V \sim L^3. \quad (2)$$

This implies that the masses m scale to

$$m \sim L^3. \quad (3)$$

Let us start by discussing how forces vary with L . The most common force is gravitation. At the Earth’s surface, the gravitation force $F_{\text{gr}} = mg$, where g is the gravitational acceleration

$$F_{\text{gr}} \sim L^3. \quad (4)$$

The pressure exerted by a body on the ground $p_{\text{gr}} = F_{\text{gr}}/S$, and

$$p_{\text{gr}} \sim L^3/L^2 = L. \quad (5)$$

At the microscopic level, adhesion forces dominate, because the details of the forces at the molecular level are much larger than the gravitational ones [2]. Let us first look at adhesion between two surfaces. It is well known that the adhesion between a solid and another solid (or a liquid) is due to forces between atoms and molecules. The main forces responsible for

adhesion are Van der Waals type forces [3]. The attractive force experienced by an infinite flat slab separated by a distance x from another infinite flat slab is given by [4]

$$F_{\text{vdw}}(x) = H/(6\pi x^3)$$

where H is the so-called Hamaker constant. H depends on the nature of the medium between the slabs. H is of the order of 10^{-19} J in air and of 10^{-20} J in water. This relation is valid for x between around 2 and 10 nm. It is obvious that $F_{\text{vdw}}(x)$ varies like the contact area

$$F_{\text{vdw}}(x) \sim L^2. \quad (6)$$

Since F_{gr} and F_{vdw} behave differently with L , their relative values also vary with L . One obtains easily that

$$F_{\text{vdw}}/F_{\text{gr}} \sim L^{-1} \quad (7)$$

i.e. the adhesion force dominates the gravitational force at low L . The critical value at which both forces are equal depend on x and on the nature of the medium between the two solids. However, below say $L = 1$ mm, F_{gr} is much less than F_{vdw} . Gravitation may then be neglected at such small dimensions, both in the micro- and the nanoworlds.

When two surfaces are sliding about each other, at the macroscopic level, the friction force is given by $F_{\text{fr}} = \mu F_{\text{gr}} = \mu mg$, where μ is the friction coefficient. Provided μ is constant,

$$F_{\text{fr}} \sim L^3. \quad (8)$$

F_{fr} is independent of the contact area. The generally admitted physical reason is that two rough bodies touch each other only at three points. At the microscopic level, things are different due to surface roughness. As seen earlier, the adhesive forces are very large. The striction (i.e. the combination of adhesion and friction) forces F_{str} , have to be taken into account. These forces scale like the contact area [2, 5]

$$F_{\text{str}} \sim L^2. \quad (9)$$

Springs exist in all dimensions. The spring force $F_{\text{spring}} = -k\delta L$, where k is the stiffness or spring constant and δL is the elongation from the equilibrium position. Providing k remains constant, one obtains

$$F_{\text{spring}} \sim L. \quad (10)$$

The corresponding oscillating frequency $\nu_{\text{spring}} = (1/2\pi)(k/m)^{1/2}$ and

$$\nu_{\text{spring}} \sim L^{-3/2}. \quad (11)$$

The period of oscillation then scales as

$$T_{\text{spring}} \sim L^{3/2}. \quad (12)$$

The kinetic energy of a body E_c , is given by the well known relation $E_c = mv^2/2$. Thus, at constant v

$$E_c \sim L^3 \quad (\text{constant } v). \quad (13)$$

When $v \sim L$, one obtains

$$E_c \sim L^5 \quad (\text{when } v \sim L). \quad (14)$$

The gravitational potential energy $E_{\text{pot}} = mgh$ varies, at constant h , as

$$E_{\text{pot}} \sim L^3 \quad (\text{constant } h). \quad (15)$$

When $h \sim L$, one obtains

$$E_{\text{pot}} \sim L^4 \quad (h \sim L). \quad (16)$$

The potential energy of a spring is $E_{\text{pot, spring}} = k\delta L^2$, and

$$E_{\text{pot, spring}} \sim L^2. \quad (17)$$

Rotational bodies are characterized by the inertial momentum $I \sim mL^2$. Then, one obtains

$$I \sim L^5. \quad (18)$$

The rotational kinetic energy $K = (1/2)I\omega^2$, and

$$K \sim L^5. \quad (19)$$

This implies that, at constant ω , the rotational energy of small systems decreases very rapidly with size. Conversely, the kinetic energy stored in a rotating system is much smaller in small systems than in large ones.

In nature, one often sees beams, strings or tubes; such items are characterized by resonance frequencies. The lowest eigenfrequency, ν , corresponds to a state where the length of the device equals a quarter or half the wavelength λ . From the well known relation $v = \lambda\nu$ (where v is the phase velocity of the wave), one obtains that

$$\nu \sim L^{-1}. \quad (20)$$

It turns out that resonance frequencies are large in small systems.

Cantilever beams are also often seen in natural and constructed systems. The deflection length ζ , of a cantilever beam loaded by its own weight varies as [6]

$$\zeta \sim L^2. \quad (21)$$

A beam a thousand times smaller bends a million times less due to its own weight.

Compressed solid materials may undergo a maximum tension, T_{br} , before breaking. This is the case of animals bones. If they are submitted to their own weight mg , $T_{br} \sim mg/S$, where S is their cross section. Hence

$$T_{br} \sim L^3/L^2 = L. \quad (22)$$

For a given material T_{br} is constant. So, when the dimensions increase, the diameter d of the supporting material is such that $d^2 \sim L^3$, or

$$d \sim L^{3/2}. \quad (23)$$

3. Scaling laws, fluids

Fluids are obviously important, since, except in vacuum, all bodies move in a fluid (air, water, ...). It is then interesting to look at the variation with L of various fluid parameters.

When a body falls 'vertically' in a fluid, viscous friction is such that, after a certain time, it falls with constant velocity v_{lim} . The transient time τ , is proportional to v_{lim} . If the body is spherical, with radius r , the limiting velocity is given by $v_{lim} = 4\rho gr^3/18\eta r$, where η is the viscosity of the fluid and ρ is the density.

$$v_{lim} \sim L^2. \quad (24)$$

$$\tau \sim L^2 \quad (25)$$

Viscous forces then drastically and rapidly damp any motion for small dimensions. A very small body remains immobile in air (when there is no air movement). When air moves, the small body follows the air movement, as experienced by us by looking at dust in sunrays. At high velocities, the hydrodynamics is unstable, and turbulent flow arises. The transition from laminar to turbulent flow is given by the Reynold number Re . $Re = \rho v L / \eta$. If $v \sim L$, then

$$Re \sim L^2 \quad (v \sim L). \quad (26)$$

For flow in pipes, the transition from laminar to turbulent flow occurs when $Re \cong 10^3$ [7]. In this case, turbulence disappears in microsystems in which liquids flow.

A particle travels a distance L by diffusion during a diffusion time given by $\tau_{diff} = L^2/\alpha D$, where α is a geometrical constant and D is the diffusion constant. One obtains

$$\tau_{diff} \sim L^2. \quad (27)$$

This is valid for particle and thermal diffusion.

4. Scaling laws, electromagnetism

A conductor of length L and cross section A has an electrical resistance $R_{el} = \rho_{el}L/A$, where ρ_{el} is the specific electrical conductivity. Thus

$$R_{el} \sim L^{-1}. \quad (28)$$

When one applies a given voltage V_{el} , the electric current I_{el} , is given by Ohm's law, so that

$$I_{el} \sim L. \quad (29)$$

Consequently, the power dissipated by the element is given by Joule's law $W = R_{el}I_{el}^2$, and

$$W \sim L. \quad (30)$$

The number of elements by unit area varies like L^{-2} . So, the electrical power dissipated by unit area varies like

$$W_{un} \sim L^{-1}. \quad (31)$$

This is a well known fact in microelectronics, where it is known that the reduction of size of the components leads to more dissipated power [8]. One way to diminish this effect is to reduce the applied voltage.

When V_{el} remains constant, while L varies, the electric field E_{el} obviously varies

$$E_{el} \sim L^{-1}. \quad (32)$$

When E_{el} increases, it attains a range of values where Ohm's law is no longer valid or, in other words, ρ_{el} varies with E_{el} . A simple derivation of this is given in the appendix.

In microsystems, one also deals with capacitors. Let us consider a capacitance made of two parallel plates of area A , separated by a distance d . The capacitance is $C = \epsilon_0 A/d$, and

$$C \sim L. \quad (33)$$

When one applies a given voltage V_{el} , the charge on each plate $Q = CV_{el}$, and

$$Q \sim L. \quad (34)$$

Under these conditions, the energy stored in the capacitor $E_{cap} = Q^2/2C$, and

$$E_{cap} \sim L. \quad (35)$$

Therefore, the stored energy decreases with the size of the capacitor. One sometimes prefers to deal with capacitors having constant charge density, i.e. $Q \sim L^2$. In this case

$$E_{cap} \sim L^3. \quad (36)$$

Since the electric force between the capacitor plates, F_{cap} , is given by the derivative of E_{cap} relative to the gap distance, one obtains that

$$F_{cap} \sim L^2. \quad (37)$$

The magnetic field in a solenoid, with n turns of wire, of length L , and a current I_{el} through the coil, is given by

$$B = \mu I_{el} n/L.$$

When one shrinks the solenoid, n may remain constant, but the cross section of the wire decreases, so that I_{el} varies. It seems reasonable to assume that the current density is constant. Hence, $I_{el} \sim L^2$, and

$$B \sim L. \quad (38)$$

The magnetic energy stored in the solenoid is $E_{magn} = B^2 Vol/2\mu$, where Vol is the volume of the magnetic field. Since $Vol \sim L^3$, one obtains

$$E_{magn} \sim L^5. \quad (39)$$

$$F_{\text{magn}} \sim L^4. \quad (40)$$

When going to small dimensions, one has to allow for technological problems. For instance, the number of wires is not independent of L at low L , because it is not (yet?) possible to manufacture very thin wires. In practice, it seems that solenoids smaller than 1 mm^3 are hardly feasible [6]. Moreover, the maximum current density is limited by the dissipation of energy, i.e. the maximum temperature the system can have. The dissipation is related to size. Hence, the current density depends on L , in a way that depends on the design of the system.

5. Scaling laws, thermodynamics

The energy E_{th} required to heat a system to a temperature T is proportional to the mass,

$$E_{\text{th}} \sim L^3. \quad (41)$$

Heat losses occur by conduction, convection and radiation. For conduction and radiation, the power dissipation, P_{diss} , is proportional to the area of the system, i.e.

$$P_{\text{diss}} \sim L^2. \quad (42)$$

When using this relation, care has obviously to be taken, owing to the fact that thermal losses are related to temperature differences between the interior and the exterior of the system. Heat diffusion coefficients have to be carefully checked. The time τ_{th} needed to homogenize the temperature in a system of given shape is proportional to the square of the linear dimensions of the system [9], i.e.

$$\tau_{\text{th}} \sim L^2. \quad (43)$$

When the size of a system decreases down to the nanometre range, the ratio of surface to volume atoms increases, so that the surface influences the properties of the system. Among others, the melting temperature is known to decrease when L decreases. For spherical particles, the melting temperature T_{m} , decreases like [10]

$$T_{\text{m}} \sim 1 - f/L \quad (44)$$

The constant f is of the order of 1 nm for 3d and 5d elements.

6. Scaling laws, optics

Optics is interesting both for the study of and the manufacturing of small systems. The limitations of optical methods arises from wave optics, in particular diffraction. When a plane wave is shone on an element of linear dimension L , the reflected wave diverges. The divergence angle $\theta \approx \lambda/L$. Hence

$$\theta \sim L^{-1}. \quad (45)$$

This has consequences in various applications [11]. In photolithography, when one irradiates elements with lenses of a fixed numerical aperture ($NA = n \sin \theta$), the minimum diameter of the irradiated zone, $L = 2\lambda/\pi NA$. Hence, the wavelength required to design an element of diameter scales like

$$\lambda \sim L. \quad (46)$$

This is why the specialists of microelectronics go to ultraviolet lithography when they decide to diminish the size of the electronic components on chips. The angular resolution $\Delta\theta$ of an element (like an eye) limited by a circular aperture of diameter L is also given by the same relation. Hence,

$$\Delta\theta \sim L^{-1}. \quad (47)$$

When lenses preserve their shape, their focal length f varies. This is given by the lens-maker's formula

$$(n - 1)(R_1 + R_2)f = R_1 R_2.$$

One thus obtains

$$f \sim L. \quad (48)$$

This also means that the minimum spot diameter at the focus $d_f = f \Delta\theta$, does not vary with L

$$d_f \sim L^0. \quad (49)$$

This remains obviously valid as long as the theory is valid, i.e. the dimensions of the optical elements are much larger than λ .

7. Scaling laws, quantum effects

When the size of elements decreases down to the nanometre scale, quantum effects become important. In most situations they arise in the electronic properties. In other words, quantum effects have to be taken into account at a critical size of the elements L_c , which compares with the wavelength associated with the electrons λ_{el} , i.e. $L_c \approx \lambda_{el} = h/p$, where h is the Planck constant and p is the momentum of the electron. Under these conditions, nanoparticles (also called quantum dots) behave like large atoms. Since the electronic motion is restricted in three dimensions, such a system is sometimes referred to as a zero-dimensional system [12]. Then, quantum mechanical calculations indicate that the electronic levels are discrete as in an atom (and contrary to a solid in which the levels are grouped in energy bands). The spacing of the discrete electronic levels of the quantum dots, ΔE_{dot} , scale as

$$\Delta E_{dot} \sim L^{-2}. \quad (50)$$

When elements come close together, of the order of nanometres or below, electrons can hop from one element to the other by so-called tunnelling. The tunnel current density j_{tun} varies with distance L as [13]

$$j_{tun} \sim \exp(-aL) \quad (51)$$

where a depends on the height of the electronic barrier. This effect is important when L is of the order of a few tenths of a nanometre.

8. Some examples

The previous scaling relations may be applied in a large number of problems. In this section, a few are given, in order (a) to show how to use them, and (b) to demonstrate their usefulness in biology and technology. They deal with the life of small animals, the behaviour of micromachines and some properties of nanosystems.

The first point to be considered is that, when one tries to compare the relative values of effects characterized by different n of L^n , the larger the value of n , the more important the effect at large L . Or, at the opposite extreme, the larger n , the less important at small L .

8.1. The Fantastic Voyage

In a famous movie, *The Fantastic Voyage* by Richard Fleischer in 1966, a submarine and its human occupants are shrunk to the size of a few hundred micrometres. They are then injected into the body of a famous Czech scientist, in order to repair some brain problem. From a scientific point of view, this is obviously unrealistic. However, this poses many questions. In particular, what would our shape have to be to survive under such conditions? It is obvious that scaling relations would be helpful. Let us address the problems of bones, feet, general shape and vision.

When a body is shrunk so that its linear dimensions L , decrease by say a factor of 10^3 , the mass decreases by 10^9 (equation (3)), as does the weight (equation (4)). In order to support this weight, the diameters of the bones have to decrease by a factor up to $10^{9/2} = 3 \times 10^4$ (equation (23)). Hence the diameter of the legs might be comparatively much smaller than at our size. In which case, the legs would be like thin tubes. It is obvious that if the diameter of the bones decreases like L , the bones would support the weight of the body without breaking. At that size, the adhesion force of the feet to the ground would be more important (equation (6)) than the gravitational one. For instance the adhesion force between two identical spheres is larger than the gravitational force (in the Earth's field) when the diameter is less than about 1 mm [2]. Therefore, if the relative dimensions of the feet were equal to ours, we would adhere to the ceiling as do insects. In order to move on the ground, the dimensions of the feet would have to be like small points. It is then obvious that this would lead to problems of stability. Therefore one concludes that, instead of walking on two legs, it would be more adequate to have (at least) four legs.

At small size, the heat power dissipation is relatively much more rapid than at our actual size. In other words, in order to maintain our temperature, we would have to decrease the area of our body (equation (1)). But the amount of chemical energy necessary for compensating these losses is proportional to the volume (equation (2)). This implies that the more adequate shape for our body would be a sphere! However, let us note that, in order to maintain a constant temperature, the quantity of food to be eaten during a fixed period would be much larger than at our dimensions. In other words, we would need to eat continuously. This is why small animals are not warm-blooded animals.

Another difficult point concerns vision. Indeed, the angular resolution of the eye $\Delta\theta$, is limited by the circular aperture of the eye (equation (47)). If the relative dimensions of the eye remain unchanged, $\Delta\theta$ would be such that the scene behind us would occupy one 'pixel'. In order to see like us, our eye would have to look at light of wavelength 10^{-3} times that of visible light (equation (46)), i.e. x-rays. Since it is known that the absorption length of x-rays in living matter is of the order of a few centimetres, while it is of the order of a few micrometres for visible light, the thickness of the eye would need to be larger than the length of the body!

From the previous reasoning, it is the obvious that: (a) it would be impossible for shrunken men to survive at the small dimensions as imagined by Richard Fleisher; (b) in order to survive, the canons of beauty would be different at micrometre sizes. The sexy looking girls and boys would look like dust mites; (c) the shape of dust mites and other small animals is the one best adapted to their size.

8.2. Micromachines

As said earlier, photolithography is used to manufacture elements in microelectronics and micromachines. The minimum dimension L , to be treated scales with λ (equation (46)). $L = 2\lambda/\pi NA$. Under realistic conditions $NA \approx 0.4\text{--}0.5$, so that $L \approx \lambda$. Ultraviolet light has $\lambda \approx 400$ nm. If one wishes to attain $L \approx 100$ nm (as expected by the microelectronics roadmap in 2007), it would be necessary to use ultraviolet light with $\lambda \approx 100$ nm. Excimer lasers (ArF; $\lambda = 193$ nm) could be used. For any further decrease of L , other technologies are required. The design of micromachines is made difficult by the fact that many effects often contribute to their behaviour, each scaling differently with L . Since our goal is to show how scaling laws may help to understand how micromachines differ from macromachines, we restrict ourselves to simple 'idealized' cases. Readers interested by practical cases may refer to [6, 14].

Let us first consider simple cases of energy conversion. Among the first micromachines, steam engines have been demonstrated. A small quantity of water is embedded in a cavity, where one wall is a moving piston. The system is heated by the Joule effect, the water evaporates and the pressure is sufficient to push the piston. Let us compare the values of the applied voltage needed to heat the water, neglecting heat losses, if one assumes that the

system is shrunk linearly. The thermal energy necessary to evaporate water is $E_{\text{th}} \sim L^3$ (equation (41)), while the Joule energy (assuming that heating lasts for the same time) varies as $W \sim L$ (equation (30)), when V_{el} is fixed. This implies that the value of W necessary for evaporating water is L^2 times the value obtained by simply reducing all sizes. Since $I_{\text{el}} \sim L$, at constant V_{el} (equation (29)), the applied voltage has to vary as L . Hence, decreasing the size of this ‘ideal’ steam engine by a factor of ten would require a reduction of the applied voltage to the heating element by the same proportion.

Let us consider that the electric energy stored in a capacitor is converted entirely into rotation of a wheel. How does the angular velocity vary with L when all dimensions vary simultaneously. Given equations (19) and (35), one obtains that $\omega \sim L^{-5/2}$. It turns out that ‘free’ spinning in the microworld is achieved far more easily than in the macroworld.

However, one has also to consider friction effects. Since, at small sizes $F_{\text{str}} \sim L^2$ (equation (9)), while $F_{\text{fr}} \sim L^3$ (equation (8)) at macroscopic sizes, the power dissipated by friction forces P_{fr} , is relatively more important in the microworld. Assuming a situation where the linear velocity v of the body scales with L , $P_{\text{fr}} = F_{\text{fr}}v \sim L^4$ at large size and $\sim L^3$ at small sizes. This is one of the main difficulties encountered in the design of micromachines. The problem is further complicated by the fact that viscous forces counter the movement more at small dimensions, since $v_{\text{lim}} \sim L^2$ (equation (24)). Moreover, due to the fact that the Reynolds number $\text{Re} = \rho v L / \eta$ decreases with decreasing L , turbulent effects are less effective. Our current feeling about motions (free, friction and fluids) has then to be revisited.

A problem related to micromachines deals with the accuracy of measurements. Are microdetectors more or less accurate than macrodetectors? There is no general response to this. For instance, let us consider the hypothetical case of a cantilever beam suited for the determination of the gravitational field. As seen from equation (21), the deformation of the beam is proportional to L^2 . Thus the deflection is much less in microscopic systems than in macroscopic ones. Provided the deflection resolution is the same, it is obvious that micromachines are much less accurate than normal ones. Meanwhile, for some applications, good accuracy is not necessary, so that the price/performance ratio might be beneficial to microsystems.

8.3. Nanoparticles

Nanoparticles have dimensions of the order of some nanometres. In this size range, quantum effects are particularly important. Moreover, it is possible to work with movements of single electrons. Hence their interest for new computing and physical mechanisms.

The energy for charging a small particle with one electron is given by $E_{\text{cap}} = e^2/2C$, where e is the electron charge and $C \sim L$ (equation (33)). Hence $E_{\text{cap}} \sim L^{-1}$. For instance, in a CdSe dot, $E_{\text{cap}} \sim 0.1$ eV for a diameter of 5.5 nm [15].

Coincidentally, in CdSe, the spacing of the lower energy levels is also of the order of 0.1 eV for the same diameter. This diameter corresponds to the wavelength of the electron, via $L_c \approx \lambda_{\text{el}} = h/p$. Indeed, $\lambda_{\text{el}} = 4$ nm when the electron energy is equal to 0.1 eV.

These devices work well at very low temperature, $T = 4.2$ K. One of the reasons is that the thermal electronic excitation rate to highest energy levels is so low that this does not perturb the behaviour of the device, via the electronic transfer to an upper level. This rate, R_{el} , is proportional to $\exp(-\Delta E/kT)$, where ΔE is the energy separation of the electronic levels. What then would be the size of the dot necessary for operation at room temperature, $T_{\text{RT}} = 300$ K? For correct behaviour R_{el} has to be equal to its value at low T . In other words, $\Delta E/kT$ remains unchanged. Since $\Delta E_{\text{dot}} \sim L^{-2}$ (equation (50)), one must have $T \sim L^{-2}$ or $L \sim T^{-1/2}$. If $L = 5.5$ nm at $T = 4.2$ K, one obtains $L = 0.65$ nm.

In order to apply such small devices, it would be necessary to inject a small number of electrons. This is generally done by injecting electrons through narrow barriers via tunnelling (so-called tunnel barriers). One drawback of these small systems is that spurious electrons might transfer from one element to the other through such small barriers. In other words the

noise due to this effect might become relatively large.

9. Conclusions

Scaling laws relevant to mechanics, fluids, electromagnetism, thermodynamics, optics and quantum physics have been given. They were applied to questions relevant to biology, micromachines and nanotechnologies. It turns out that the reasoning based on our own experience has to be modified when dealing with the microworld and the nanoworld, due to the fact that various physical effects scale differently with size.

Appendix

When one applies a voltage difference of 5 V across an element 1 μm wide it means the electric field $E_{\text{el}} = 5 \times 10^6 \text{ V m}^{-1}$. At such high electric fields, the current is no longer proportional to the electric field. Let us give a simple demonstration of this effect.

Let J be the electron current density, i.e. the amount of charge crossing a unit area of the conductor (perpendicular to J) per unit time. As is given in most textbooks (i.e. see [16])

$$J = qn v_d \quad (\text{A1})$$

where q is the charge of the electron, n is the electron density and v_d is the so-called drift velocity of the carriers (the electrons). This is valid when no gradient of carrier density exists, in which case a diffusion term has to be added to the equation.

In a solid, the electrons are in continuous motion in all directions. The motion is random and, in the absence of external effects, there is no net transport of charge. Meanwhile, there are collisions between particles. In a solid, electrons are characterized by their kinetic energy. Given Fermi–Dirac statistics, it is known that electrons possess kinetic energies between zero and E_F , the Fermi energy. In other words, they have velocities between 0 and v_F , the Fermi velocity

$$E_F = m v_F^2 / 2. \quad (\text{A2})$$

If the mean free path is l , one can define a mean time between collisions (at E_F)

$$\tau = l / v_F. \quad (\text{A3})$$

In the presence of an electric field E_{el} , the drift velocity is given by the well known relation

$$v_d = q \tau E_{\text{el}} / m = \mu E_{\text{el}}. \quad (\text{A4})$$

μ is the mobility of electrons.

Under usual conditions, when E_{el} is not too large, μ is independent of E_{el} . Then one obtains Ohm's law

$$J = q n \mu E_{\text{el}} = \sigma_{\text{el}} E_{\text{el}} \quad (\text{A5})$$

where σ_{el} is the conductivity.

When E_{el} increases, as long as l remains constant, τ decreases since the mean velocity of the electrons increases

$$\tau = [-v_F + (v_F^2 + 2a E_{\text{el}} l)^{1/2}] / a E_{\text{el}} \approx \tau_0 (1 - b E_{\text{el}}) \quad (\text{A6})$$

where b is constant. Introducing this in (A.5), one obtains

$$J \approx q n \mu_0 (1 - b E_{\text{el}}) E_{\text{el}}. \quad (\text{A7})$$

Since J is not directly proportional to E_{el} , Ohm's law is not valid at high E_{el} . Moreover, since b is positive, J tends to saturate at high fields, as observed experimentally [17].

What is E_c , the order of magnitude of E_{el} at which this effect becomes effective? It is such that the value of v_d compares with that for v_F . In other words, between two successive

collisions, the kinetic energy of the electrons increases by a quantity of the order of E_F . This implies

$$E_F \approx q E_c l. \quad (\text{A8})$$

Typical values for semiconductors are [18] $l \sim 10^{-6}$ m; $E_F \sim 10$ eV. One obtains $E_c \sim 10^7$ V m $^{-1}$, in agreement with experimental data.

This reasoning is based on simple principles. More detailed calculations including the true band structure of the solids are required to obtain exact results.

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