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"...equations will greatly simplify the work of the designer and result in more efficient and economical use of reinforced concrete."

Much of ACI's success in its first 100 years can be traced to the development of codes, standards, and reports that translate information gained from experience and from laboratory tests into useful and usable design guidelines and procedures. There are very few papers published by ACI that better exemplify the goals of ACI than the paper that made flexural calculations simple and easily understood. The quote cited above appears in Whitney's introduction to his paper from the March-April 1937 ACI JOURNAL. After carefully reviewing the literature and the available test data, Whitney proposed an equivalent rectangular stress block to represent the real variation of stresses in the concrete above the neutral axis. After showing that the calculations made using the rectangular stress block provided ultimate strength that was nearly identical to that obtained from tests, he concluded that "No further theoretical justification is necessary if the formulas derived therefrom accurately predict the ultimate strength of the member." The code provisions for flexure and combined flexure and axial load in ACI 318-02 are based on Whitney's proposal just as they have been since ultimate strength design was introduced in the 1956 code.

This paper should be a must-read for anyone proposing code provisions and changes. While the rectangular stress block may not be theoretically correct, it provides a tool that engineers can understand and use with confidence. The role of the code is not to be a compendium of research results but to provide structures that will meet the minimum safety requirements desired by the public and the users. Whitney understood this implicitly—his proposal has "stood the test of time" and his argument is as valid today as it was in 1937.

James O. Jirsa
ACI Past President

Design of Reinforced Concrete Members Under Flexure or Combined Flexure and Direct Compression*

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It is the purpose of this paper to suggest a complete revision of the method of designing reinforced concrete members subjected to bending and to present a rational method for the proportioning of arch ribs, rectangular columns under eccentric load, and rectangular beams. This method may be extended to cover the design of T beams, round columns and any other form of concrete members. Simple formulas are given which predict the ultimate strength with remarkable accuracy based on the cylinder strength of the concrete and the yield point of the steel independent of the ratio of their moduli of elasticity.

It is believed that these equations will greatly simplify the work of the designer and result in the more efficient and economical use of reinforced concrete. The suggestions are intended to place structural concrete design on a solid practical foundation which is now justified by the advance in construction practices, the availability of better materials, and the information gained by a great amount of research work.

The present method of designing members under bending and combined bending and direct stress is unsatisfactory for several reasons.

The usual formulas based on assumption of a cracked section and straight line variation of stress are far from correct both under working loads when the concrete is not materially cracked and under ultimate loads when the stress in the concrete is not even approximately proportional to the distance from the neutral axis. They

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do not give a good indication of the conditions under working loads and cannot be used to predict the ultimate strength with any degree of accuracy.

They do not properly recognize the strength of the concrete in compression, and provide a much higher factor of safety against failure of the concrete than of the steel.(2)(6) That may have been justified during the early days of reinforced concrete on the grounds of unreliability but it is certainly not justified today. Properly balanced design will permit the use of a considerably higher percentage of tensile steel with a consequent reduction in the size and weight of members.

The usual flexure formulas are complicated by the use of the value of n which is quite unpredictable under high loads and actually has little effect on the ultimate strength of the beam.(2)(9)(10)

The effective value of n is widely different for dead and live loads. In the case of arch ribs, the dead and live load stresses cannot be computed separately with different values of n and added together, because dead load may produce compression only, while the live load moment alone would require assumption of a cracked section. The calculation in this case would be very complicated with two values of n.

The present method makes no prediction of the loading which will cause cracking and does not give accurate control over the factors of safety under dead loads and live loads.

The method here proposed for the design of members under bending and direct stress, particularly for arches, is direct and simple. First, an analysis of tensile stress in the arch rib should be made assuming the critical combination of live and dead loads, together with temperature, shrinkage and plastic flow effects. This can best be done by using the complete transformed section without cracks. The effective values of n for dead and live loads can be predicted with sufficient accuracy for this purpose at allowable working stresses and the dead and live load stresses for corresponding values of n can be computed separately and added together. The tensile stress in the concrete so determined should not exceed the modulus of rupture of the concrete in an unreinforced beam in order that the arch rib shall not be cracked in service. The compressive stress for this condition is relatively unimportant because the strength of the rib will be controlled by an ultimate strength calculation.

In arch ribs whose center lines follow the dead load pressure lines, the tensile stress can easily be kept somewhat below the modulus of rupture. In the rigid frame type of arch, it may be possible to permit the tensile stress for the extreme case to slightly exceed the 28-day modulus of rupture without introducing objectionable cracking; or if the critical combination of loads includes only a portion of the full live load, the calculated tensile stress can probably be kept below the modulus of rupture without difficulty. This method of design would have the advantage of directing attention to the tensile stresses and would encourage the use of concrete with higher tensile strength.

This calculation using the full strength of the uncracked section might result in the use of too little reinforcing steel and provide an inadequate factor of safety unless an additional calculation is made of the ultimate strength of the rib. Because of the large strains occurring before failure, the effects of temperature change, shrinkage, and plastic flow have no practical effect on the ultimate strength.[1][3][4] It is therefore proposed that these effects be neglected and that the ultimate strength of the member be computed by a new type of formula based on the ultimate strength of the concrete, and on the yield point strength of the steel.

It has been pointed out recently by several investigators[4][9][10] that while the stress variation in the concrete is approximately linear under very light loads, and parabolic under intermediate loads, as the ultimate load is approached it assumes a shape about as shown in Fig. 1.

The stress increases very rapidly near the neutral axis and is nearly uniform for the greater part of the depth of the compression section, probably decreasing slightly toward the edge of the beam. Saliger[10] reports the ultimate strain in the concrete at the outer edge of the beam to be from .003 to .007 while the limit reached by concrete prisms at failure was .002 to .004. The usual "parabolic" formulas have been based on the theory that the stress variation in the beam

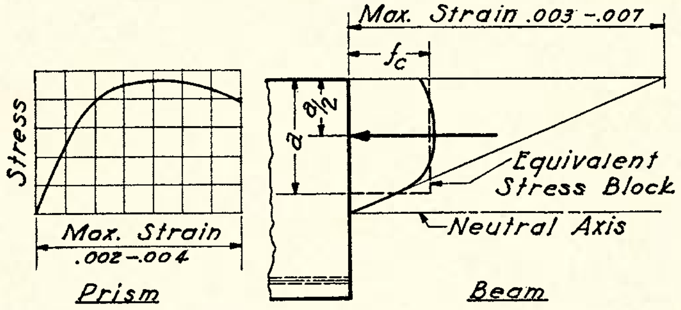

FIG. 1—STRESS-STRAIN CURVE FOR PRISM AND FOR BEAM AT ULTIMATE LOAD

followed the shape of the first part of the stress-strain curve for the concrete cylinder up to the point of maximum load.

This is evidently considerably in error because of the greater ultimate strains and the different behavior of the concrete in the beam. On this account, the writer questions the value of any refinements of beam theory based on an attempt to estimate the ultimate stresses in a beam by comparison with the stress-strain curve for the cylinder or other standards.

It is therefore proposed that a rectangular block of uniform stress as indicated by the dotted lines on Fig. 1 be used to represent whatever stress may exist in the concrete. Whatever it actually is, it must have an average intensity, f_c, and an effective depth, a. The resultant is assumed at the middle of the rectangle. Under ultimate load, Hooke's Law and the theory of elasticity have no significance as far as the internal stresses are concerned. The materials are more nearly in a plastic state but the imperfect and variable action of concrete makes a rigid solution according to the theory of plasticity impractical. No further theoretical justification of the assumption a rectangular compressive stress block is necessary if the formulas derived therefrom accurately predict the ultimate strength of the member. This they appear to do.

This assumption of uniform compressive stress on the concrete in a beam has been suggested by Gebauer and by Copée but their formulas do not appear entirely satisfactory because of their other assumptions. Since the development of the following treatment by the writer, von Emperger[9] and Saliger[10]; in excellent discussions of this subject have recommended the use of simplified ultimate strength formulas for beams independent of n. Their formulas are somewhat less simple and do not appear to check the results of American beam tests quite as well as those presented herein.

SIMPLE FLEXURE

The assumed relations for a rectangular beam under simple flexure are shown in Fig. 2.

It is assumed that in an under-reinforced beam, that is, one which will fail in the tensile steel, the concrete will crack as the steel stretches and the depth of the beam in compression a will be reduced until the concrete unit stress reaches the ultimate or

$$a = \frac{A_s f_s}{b f_c} \qquad \text{Eq. (1)}$$

in which, A_s = area of tensile steel
f_s = yield point stress in steel
b = width of beam
f_c = ultimate strength of concrete

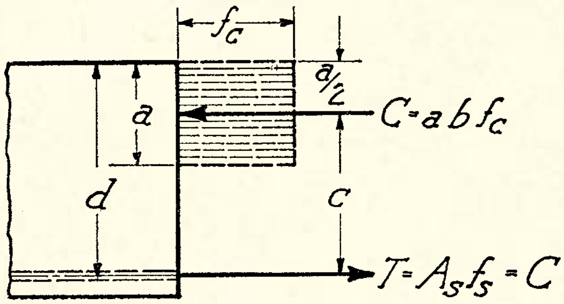

Fig. 2

This determines the lever arm of the steel reinforcement since

$$c = d - \frac{a}{2}$$

It will be assumed that the ultimate compressive strength of the concrete in the beam is equal to 85 per cent of the cylinder strength in order to be consistent with the results of tests on concentrically loaded columns and to have a treatment which can be applied to the full range of cases from simple flexure to direct load.

The values of a and c are derived as follows for any particular bending moment:

$$M = \left(d - \frac{a}{2}\right) abf_c \quad \text{.......... Eq. (2)}$$

from which

$$a = d - \sqrt{d^2 - \frac{2M}{bf_c}}, \text{ or, } \frac{a}{d} = 1 - \sqrt{1 - \frac{2M}{bd^2f_c}} \quad \text{.......... Eq. (3)}$$

and

$$c = d - \frac{a}{2} = \frac{1}{2}\left(d + \sqrt{d^2 - \frac{2M}{bf_c}}\right) \quad \text{.......... Eq. (4)}$$

These expressions are independent of the area of steel and the value of $\frac{E_s}{E_c}$.

The required steel area is simply,

$$A_s = \frac{M}{cf_s} \quad \text{.......... Eq. (5)}$$

Since the assumed compressive stress distribution has no exact theoretical basis, the limiting value of the depth of compression a for equal concrete and steel strengths in flexure must be determined experimentally. If the beam has at least sufficient steel to fully

TABLE 1—BEAM TESTS REPORTED BY SLATER AND LYSE

Group No.	d (in.)	b (in.)	p	Cylinder strength (lbs. per sq. in.)	Maximum Moment (in lbs.)	Value of a from Eq (3)	$\frac{a}{d}$	$\frac{c}{d}$	Maximum Moment from Eq (6)	Actual Maximum Moment Divided by Maximum Moment as Given by		
										Equation (6)	A. C. I. Formula	Parabolic Formula
1	10.2	8.2	.021	1390	597,000	$>d$	>1.0*		396,000	1.508*	2.06*	1.50*
2	10.3	8.2	.028	2790	818,000	5.6	0.544	.728	811,000	1.007	1.51	1.03
3	10.3	8.2	.037	4070	1,152,000	5.3	0.515	.742	1,183,000	0.973	1.50	0.98
4	10.1	8.2	.047	4800	1,360,000	5.6	0.554	.723	1,342,000	1.013	1.54	1.00
5	10.2	8.3	.056	5740	1,584,000	4.9	0.481	.759	1,660,000	0.955	1.46	0.95
6	14.2	8.2	.030	2590	1,805,000	12.9	0.910*	.545*	1,430,000	1.262*	1.85*	1.27*
6A	14.1	8.2	.039	4130	2,196,000	7.3	0.518	.741	2,250,000	0.976	1.50	0.97
7	12.2	8.3	.028	2950	1,240,000	6.75	0.553	.723	1,220,000	1.016	1.56	1.03
8	8.0	8.1	.031	2760	543,000	5.36	0.670*	.665*	478,000	1.135*	1.66*	1.13*
9	5.9	7.9	.032	2900	310,000	4.17	0.708*	.646*	266,000	1.166*	1.69*	1.14*
10	4.1	8.0	.030	2820	129,000	2.26	0.552	.724	127,000	1.016	1.48	1.00
10A	4.1	8.0	.040	3810	179,000	2.37	0.576	.712	171,000	1.046	1.55	1.03
Average							.537 ± .024	.732 ± .012		1.00 ± .024	1.51 ± .028	1.00 ± .024

*These values not included in average.

develop the strength of the concrete, additional steel does not materially increase the strength of the beam.

The limiting value of a as computed from tests reported by Slater and Lyse[2] is given in Table 1. The value of f_c assumed in Eq.(3) is 85 per cent of the corresponding cylinder strength, of, $f_c = 0.85\, fc'$. Eliminating four groups of tests which appear to be erratic, the average value of $\frac{a}{d}$ is 0.537. This is independent of p, d, and f_c.

The flexural strength of a fully reinforced rectangular beam with tensile steel only is then, from Eq.(2).

$$M = (d - 0.2685\, d)\, 0.537\, d\, b\, f_c$$
$$= 0.393\, b\, d^2 f_c$$

$$\text{or } \frac{M}{bd^2} = 0.393\, f_c = \frac{f_c'}{3} \quad \text{.......... Eq. (6)}$$

in which f_c is 85 per cent of the compressive strength of standard cylinders, f_c'. Column 10 of Table 1 gives the maximum moment for the test beams as predicted by Eq.(6). It may be noted that the percentage of error is practically the same for Eq.(6) as for the parabolic formula given by Slater and Lyse using the measured value of n.

Fig. 3 shows the value of $\frac{M}{bd^2 f_c'}$ given by the 36 beams tested by Slater and Lyse and by 33 beams tested by Humphrey[11] which had cylinder strengths between 1000 and 2500 p.s.i. The Humphrey beams with higher strength concrete were not included because it appears that there was not sufficient reinforcement to fully develop such higher concrete strength even though they may have been reported as compression failures. Some confusion no doubt exists because it is difficult to differentiate between primary steel or concrete failure unless the amount of steel is considerably below the critical percentage required to develop the full concrete strength. A slight stretching of the steel may reduce the concrete compression area and cause what looks like a concrete failure in an under-reinforced beam.

The value of $\frac{M}{bd^2 f_c'}$ shown by Fig. 3 is unaffected by the concrete strength between 3000 and 6000 p.s.i. although it rises about 50 per cent for weaker concretes from 3000 to 1000 p.s.i. With the present methods and materials, it is not probable that much structural con-

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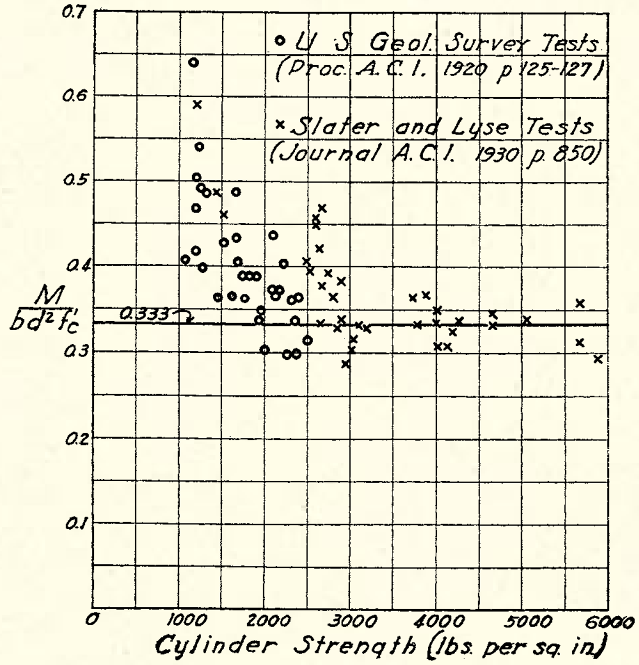

FIG. 3—VALUE OF $\frac{M}{bd^2f_c'}$ FROM TESTS OF BEAMS

crete will be used with a strength of less than 2500 or 3000 p.s.i., and if it is used, a greater factor of safety is not inappropriate. Therefore it appears that a value of 0.333 for $\frac{M}{bd^2f_c'}$ can be used generally.

For beams with less steel than required to develop the full compressive strength the allowable bending moment is given by the formula:

$$\frac{M}{bd^2} = pf_s\left(1 - \frac{pf_s}{1.7f_c'}\right) \qquad \text{Eq. (7)}$$

The critical percentage of steel required to develop the full compressive strength of the concrete is:

$$p_o = 0.456\frac{f_c'}{f_s} \qquad \text{Eq. (8)}$$

Table 2 shows the results of the application of Eq.(5) to 72 beams of the series tested by Humphrey[(5)] except beams 489-490-491 where Eq.(6) controls on account of the low strength concrete. The average ratio of actual maximum moment to that given by the formulas is 1.071 to 1, and, eliminating the low strength cinder concrete beams, the ratio is 106.24 to 1. Classified according to age, the ratio is 1.052 to 1 at 4 weeks and 1.09 to 1 at 52 weeks. According to percentage of steel, it is 111.7 for $p = 0.0049$, 105.44 for $p = 0.0098$, and 104.02 for $p = 0.0196$. There appears to be no correlation between modulus of elasticity of the concrete and the accuracy of the formula.

The value of the steel lever arm, $\frac{c}{d}$, from Eq. (4) is given in Table 2 for comparison with the standard value, j. The value given by Eq. (4) is greater and appears to be more satisfactory than j by the usual method.

Equations (5) (6) and (7) appear to check the results of well controlled tests within the limits of the variability of the materials. They appear to be as accurate as any treatment which can be devised from information available although more complete tests may make possible a slight improvement in the empirical constant.

The effect of these formulas is shown in Fig. 4 based on 3000 p.s.i. concrete cylinder strength and 50,000 p.s.i. steel yield point with a factor of safety of 2½ for both concrete and steel.. The allowable value of $\frac{M}{bd^2}$ is plotted against the percentage of steel and compared with the value by the usual formula with $f_c = 1200$ p.s.i., $f_s = 20{,}000$ p.s.i. and $n = 10$. The new formulas raise the critical percentage of steel from 0.0113 to 0.0273 and the allowable $\frac{M}{bd^2}$ from 197.5 to 400. For under-reinforced beams, they give a higher value of the steel lever arm and a closer approximation to the true steel stresses.

It may be noted that the same method can be applied to T beams if it is determined from tests what the limiting effective proportions should be. The writer has not as yet made any examination of test data but there are probably enough on record to indicate how widely the uniform compressive stress in the concrete can be assumed to be distributed.

BEAMS REINFORCED FOR COMPRESSION

If there is steel in the compression side of the beam as in the case of a symmetrically reinforced arch rib, the compression steel will be stressed up to its elastic limit if the beam fails in compression. If it fails in the tensile steel, the presence of the compressive steel will have comparatively little effect on its ultimate strength. In the latter

TABLE 2—COMPARISON OF CALCULATED AND ACTUAL ULTIMATE STRENGTH OF BEAMS TESTED BY HUMPHREY AND LOSSE

Beam Numbers	Concrete			Steel				Max. Moment Actual (in. lbs.)	$\frac{c}{d}$	Max. Moment Calculated From Eq (5) (in. lbs.)	Maximum Moment Actual Divided by Calculated
	Aggregate	Age weeks	Cylinder Strength (lbs. per sq. in.)	A_s (sq. in.)	d (in.)	p	Yield Point (lbs. per sq. in.)				
162–163–164	Granite	4	3241	.393	10	.49	42,490	177,000	.9621	160,500	1.103
186–187–188		4	2807	.785	10	.98	43,043	321,000	.9115	307,500	1.043
237–238–239		4	3158	1.571	9¼	1.96	41,857	492,000	.8350	507,000	.970
171–172–173		52	5589	.393	10	.49	42,573	193,600	.978	163,500	1.183
195–196–197		52	5086	.785	10	.98	42,320	340,000	.952	319,000	1.066
234–235–236		52	4957	1.571	9¼	1.96	41,877	543,000	.895	545,000	.996
255–256–257	Limestone	4	2087	.393	10	.49	41,653	164,000	.9425	154,000	1.064
273–274–275		4	2600	.785	10	.98	40,797	291,000	.9095	289,000	1.006
321–322–323		4	2725	1.571	9¼	1.96	38,037	471,000	.815	451,000	1.044
246–247–248		52	4069	.393	10	.49	41,003	183,000	.9708	157,000	1.165
282–283–284		52	4360	.785	10	.98	41,413	317,000	.9451	307,000	1.033
330–331–332		52	4230	1.571	9¼	1.96	37,673	532,000	.890	487,000	1.092
333–334–335	Gravel	4	3441	.393	10	.49	43,447	169,000	.9635	164,000	1.030
357–358–359		4	3440	.785	10	.98	42,600	321,000	.9285	310,000	1.035
405–406–407		4	3376	1.571	9¼	1.96	38,633	504,000	.857	482,000	1.045
342–343–344		52	5186	.393	10	.49	44,140	173,000	.9754	169,000	1.022
366–367–368		52	5242	.785	10	.98	40,107	356,000	.9559	300,000	1.186
414–415–416		52	5467	1.571	9¼	1.96	38,643	533,000	.9120	512,000	1.040
417–418–419	Cinders	4	1629	.393	10	.49	37,357	163,200	.9337	138,000	1.182
441–442–443		4	1555	.785	10	.98	41,170	281,000	.8475	273,000	1.030
489–490–491		4	1643	1.571	9¼	1.96	37,887	403,000	.7315	376,000*	1.071
426–427–428		52	2944	.393	10	.49	38,343	172,600	.9624	145,000	1.188
450–451–452		52	2619	.785	10	.98	41,103	306,000	.9090	293,000	1.044
498–499–500		52	2763	1.571	9¼	1.96	38,047	488,000	.827	458,000	1.064
										Average	1.071 ± .05

*Determined by Concrete Strength, Eq. (6)

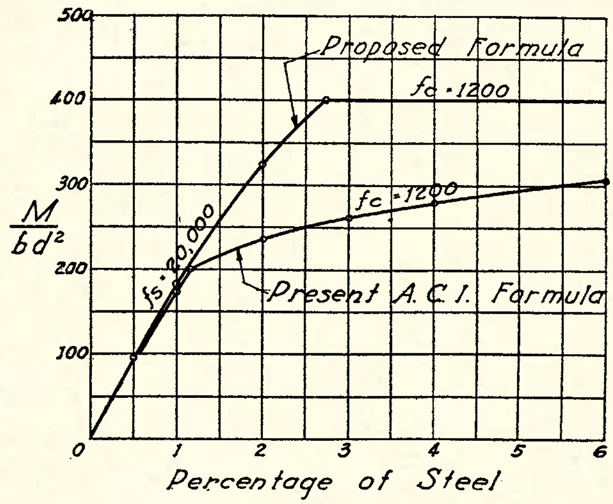

FIG. 4—COMPARISON OF RESISTING MOMENT OF RECTANGULAR BEAMS AS GIVEN BY PROPOSED FORMULA AND PRESENT A. C. I. FORMULA

case, the compressive steel can still be figured at its elastic limit if it comes within the compression zone and the lever arm of the tensile steel can be computed on that basis.

With compressive steel, the ultimate strength in compression is computed by adding the moment of the steel compressive stress to that of the concrete stress computed before. This condition is shown in Fig. 5.

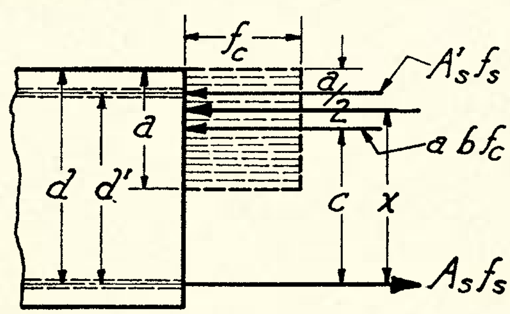

FIG. 5

The ultimate compressive moment will be

$M_c = 0.333\, bd^2 f_c' + d'A'_s f_s$ .... Eq. (9)

The ultimate tensile moment will be

$$M_t = A_s f_s \frac{cabf_c + d'A'_s f_s}{abf_c + A'_s f_s} = x\, A_s f_s \quad \text{.......... Eq. (10)}$$

in which the value of c is obtained from Equation (4) using $(M - d'A'_s f_s)$ in place of M, and x is the distance from the center of tensile steel to the center of gravity of the compressive stresses in the beam. Also note that $f_c = 0.85f'_c$. Equations (9) and (10) have also been checked against tests with satisfactory results provided the failure is not through shear or bond.

The results of tests of beams with compressive steel made by Bach and Graf at Stuttgart and reported by von Emperger [7, page 225] are of interest because they show the danger of bond failure. The full yield point stress of 35000 p.s.i. was developed in 20 mm. round bars but the beams with bars of steel with a 60,000 lb. yield point failed in bond before the full strength was developed. Glanville[12] advises against counting on the compression reinforcement but his tests were made on very small beams which evidently failed in shear before the full moment was developed. He says, "In the simple beams, serious cracking developed towards the end of the tests, and it is possible that the higher shear stresses in the continuous beams with compression reinforcement may have been the reason for the low moment carried over the central support."

FLEXURE AND DIRECT LOAD

The case of bending and direct stress can be treated in the same manner as bending alone. The strength of the compressive side will be the same as before and the steel tension will be reduced by the amount of the direct compression, P, as shown in Fig. 6.

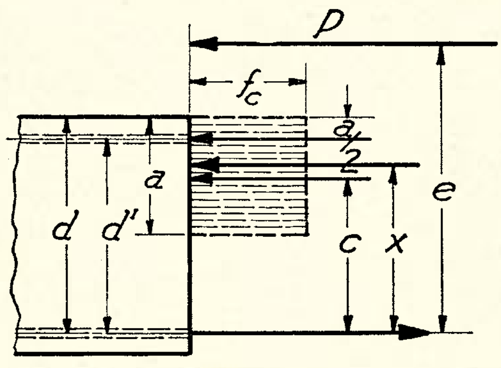

FIG. 6

The ultimate compressive moment is given by Eq. (9)

$$M_c = Pe = 0.333bd^2f'_c + d'A'_sf_s$$

The ultimate tensile moment is

$$M_t = Pe = A_s f_s \frac{x}{1 - \frac{x}{e}} \quad \text{Eq. (11)}$$

When there is no steel on the compressive side Equation (11) becomes

$$M_t = Pe = A_s f_s \frac{c}{1 - \frac{c}{e}} \quad \text{Eq. (12)}$$

and the value of c can be obtained from the formula

$$c = \frac{1}{2}\left[d + e - \sqrt{(d + e)^2 - 4de + 2Ye}\right] \quad \text{Eq. (13)}$$

in which

$$Y = \frac{A_s f_s}{bf_c} = \frac{A_s f_s}{0.85\, bf'_c}.$$

When there is steel on the compression side but the load is not sufficient to develop the full compressive strength of both the steel and concrete, the required tensile steel area can best be computed by considering the effect of the compression steel and concrete separately. First compute the area of tensile steel required to develop the moment $(d'A'_s f_s)$ of the compressive stress with the formula:

$$A_s = A'_s\left(1 - \frac{d'}{e}\right) \quad \text{Eq. (14)}$$

Then deduct the moment $(d'A'_s f_s)$ from the external moment, Pe, and compute the additional area of tensile steel required to balance the compression stress in the concrete using Eq. (12).

When the eccentricity is small compared with d, the ultimate strength of the member can be computed as twice the strength of the weaker side of the section from the formula (see Fig. 7):

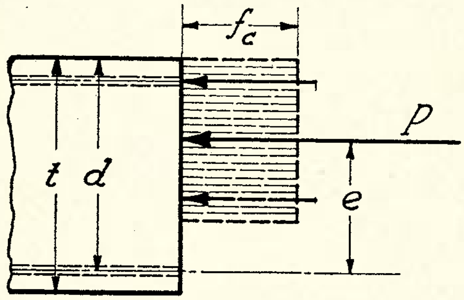

FIG. 7

$P = 2A'_s f_s + 1.7b\,(d - e)\,f'_c$ .... Eq. (15)

which becomes for a concentric load on a symmetrical member:

$P = 2A'_s f_s + 0.85btf'_c$ .... Eq. (16)

The value of e for which the Equation (9) and (15) will give equal values of P is given by Equation (17) and for smaller eccentricities, Equation (15) should be used.

$$e = \frac{1}{2}\left[d + Y' + \sqrt{(d + Y')^2 - 0.786d^2 - 2Y'd'}\right] \text{ .... Eq. (17)}$$

in which

$$Y' = \frac{A'_s f_s}{bf_c} = \frac{A'_s f_s}{0.85bf'_c}.$$

The formulas for ultimate strength of members under flexure and direct load have been verified by comparison with the results of tests on 46 columns made by Bach and Graf [7][8]. The ratio of actual ultimate load to calculated ultimate strength is given in Table 3 which shows an average error of three per cent for the series. The columns were about 8 ft. 3 in. long and 16 in. square and were loaded with eccentricities varying from zero to about 20 in. The average 30 cm. cube strength of the concrete was 225 kg. per sq. cm. and

TABLE 3—COMPARISON OF CALCULATED AND ACTUAL ULTIMATE LOADS FOR COLUMNS TESTED UNDER ECCENTRIC LOADS BY BACH AND GRAF

(See JOURNAL Amer. Concrete Inst. *Proceedings* Vol. 26, p. 661. April, 1930.)

Column	Eccentricity	Calculated Ultimate Load		Actual Ultimate Load—Kg.	Actual Load Divided by Calculated Load
		Formula No.	Load—Kg.		
Type I	0	16	271,000	280,333	1.035
	.50	12	94,000	93,000	.990
	.75	12	57,000	60,333	1.057
	1.25	12	29,000	29,967	1.016
Type II	0	16	331,600	338,333	1.021
	.25	15	196,100	202,500	1.032
	.50	9	128,000	124,000	.968
	.75	12 & 14	69,800	69,600	.998
	1.25	12 & 14	32,800	32,350	.986
Type III	0	16	379,100	404,700	1.067
	.25	15	243,800	225,000	.923*
	.50	9	150,000	157,500	1.050
	.75	12 & 14	106,500	105,000	.987
	1.25	12 & 14	56,000	53,500	.955
Plain Columns	0	16	270,400	276,167	1.020
	.25	15	135,200	136,000	1.005
				Average	1.007
				Average Error	.031

*Average of two tests only, 0.882 and 0.965.

the cylinder strength was taken as $\frac{225}{1.13}$ or 199 kg. per sq. cm. The yield point of the steel averaged 3773 kg. per sq. cm. for Types I and II and 3572 for Type III.

The maximum variation in the strength of the cubes was about $\pm 9\frac{1}{2}$ per cent. The errors shown by the ultimate strength formulas are well within the limits of the variability of the materials and it would appear that the use of more elaborate formulas is not warranted.

For small eccentricities, Equation 15 agrees with test results much better than the usual formula based on straight line stress variation. Such agreement is more important than theoretical verification but it can be supported on the basis of actual stress variation. It is further supported by the results of tests on two plain concrete prisms reported by Slater and Lyse [2, page 859]. The 8 x 8 x 12 in. prisms were loaded with $e = \frac{t}{6}$ and the ultimate load averaged 155,520 lbs. The cylinder strength of the concrete was 4060 p.s.i. Eq. (15) gives

$$P = 1.7 \times 8 \times \frac{8}{3} \times 4060 = 147{,}000 \text{ lbs.}$$

or about 94.6 per cent of the actual load.

The foregoing equations predict the untimate strength of a member but cannot be used to compute the actual stresses below the ultimate. Being based on 85 per cent of the cylinder strength, they are consistent with the results of tests of columns under concentric load. It is hoped that they will provide a satisfactory method which is now lacking for the design of members of elastic frames and arches after the external moments have been determined by the theory of elasticity. They indicate that a higher percentage of tensile steel may be used than is now general practice without overstressing the concrete and they will permit lighter members where bond and shear can be properly taken care of.

It is further proposed that the dead load moments and thrusts be multiplied by a suitable factor of safety and added to the live load moments and thrusts multiplied by an appropriate factor, probably larger than that used for dead load, equating the total to the ultimate strength given by the equations. The factors of safety could also be introduced by using reduced values of the stresses if it proves desirable. It may be that the ultimate strength formulas alone will be sufficient for ordinary building design without the necessity for making any calculation of the tensile stresses under working loads. A thorough study should be made to determine what factors of safety should be used for different types of structures.

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