

Structural Dynamics Overview

- Modeling
- Continuous and Discrete Systems
- Modal methods
 - Eigenmodes
 - Rayleigh - Ritz
 - Galerkin
- Discrete Point Methods
 - Finite Difference
 - Finite Element
- Solution of Dynamic Problems
 - Mass Condensation - Guyan Reduction
 - Component Mode Synthesis

o Modeling Levels

• Real structural dynamics system (structures)

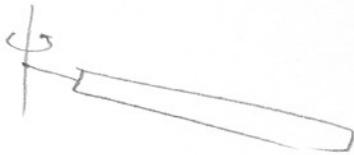


Real structures, in 3-D space, composed of different material, and subject to external excitation

↓ Assumption: - material (linear elastic)
- geometry
- loads

• Continuous representation of the structure

- Idealized model (infinite d.o.f.)



1-D (continuous beam) representation of the blade

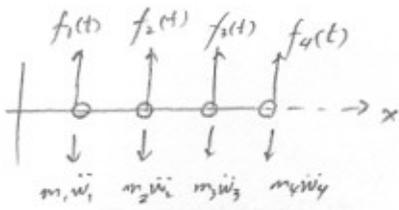
↓ More assumptions, for example: - low frequency behavior

• Discrete representation of the structure

- Idealized model (finite d.o.f.)



1-D finite element representation of the blade



Total force: $F_{i \text{ TOT}} = f_i - m_i \ddot{w}_i$ (D'Alembert's Principle)

\uparrow Applied force \uparrow Inertial force

$$\{F_i\}_{\text{TOT}} = \{f_i\} - \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & m_3 & \\ & & & m_4 \end{bmatrix} \begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{w}_3 \\ \ddot{w}_4 \end{Bmatrix}$$

\downarrow
[M]

Deflection w_i is flexibility influence coefficient, Deflection @ i due to a unit load @ j

$$\{w_i\} = \{C_{ij}\} \{F_j\}_{\text{TOT}}$$

$$= \{C_{ij}\} [\{f_j\} - [M] \{\ddot{w}_i\}]$$

C_{ij}
deflection load

Repose

$$\boxed{[M] \{\ddot{w}\} + [K] \{w\} = \{f\}}$$

This can also be extended to a full 2-D, 3-D structures

$$[M] \begin{Bmatrix} \ddot{u} \\ \vdots \\ \ddot{v} \\ \vdots \\ \ddot{w} \end{Bmatrix} + [K] \begin{Bmatrix} u \\ \vdots \\ v \\ \vdots \\ w \end{Bmatrix} = \begin{Bmatrix} F_u \\ \vdots \\ F_v \\ \vdots \\ F_w \end{Bmatrix}$$

Note: Generally both [M] and [K] have coupled structures (off-diagonal components), symmetric

$$[M] \ddot{\underline{w}} + [K] \underline{w} = \underline{F}$$

set of simultaneous, coupled DE subject to IC's @ $t = 0$

$$\left. \begin{aligned} w_i &= w_i^0 \\ \dot{w}_i &= \dot{w}_i^0 \end{aligned} \right\} @ t = 0$$

- First solve homogeneous equations for the lowest (few) eigenvalues (ω) and eigenvectors ($[\phi]$: mode shape matrix)

$$[M] \ddot{\underline{w}} + [K] \underline{w} = \underline{0}$$

$$\text{set } \underline{w} = \underline{\tilde{w}} e^{i\omega t}$$

$$\underbrace{[-\omega^2 [M] + [K]]}_{\text{characteristic eqn.}} \underbrace{\underline{\tilde{w}}}_{\text{eigenvector}} e^{i\omega t} = \underline{0} \quad \dots (*)$$

4 Eigenvalue $\lambda_i = \omega_i^2$, natural frequency $f_i = \frac{\omega_i}{2\pi}$

Eigenvectors are obtained by placing any root into (*)

$$\begin{bmatrix} (k_{11} - m_{11}\omega_i^2) & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} \underline{\phi}^{(i)} = \underline{0}$$

Need to set at least one value of $\underline{\phi}^{(i)}$



A N D.O.F. system has N natural frequency and N mode shapes associated these natural frequencies.

- orthogonality relations

$\omega_j, \phi_i^{(j)}$ set of free vibration mode shapes

Each satisfies $-\omega^2 [M] \underline{\phi} + [K] \underline{\phi} = 0$

$$\omega_r^2 [M] \underline{\phi}^{(r)} = [K] \underline{\phi}^{(r)} \quad \dots (1)$$

$$\omega_s^2 [M] \underline{\phi}^{(s)} = [K] \underline{\phi}^{(s)} \quad \dots (2)$$

Multiply (1) by $\underline{\phi}^{(s)T}$ and (2) by $\underline{\phi}^{(r)T}$

Take transpose of both sides

$$\begin{aligned} \omega_r^2 \underline{\phi}^{(s)T} [M] \underline{\phi}^{(r)} &= \underline{\phi}^{(s)T} [K] \underline{\phi}^{(r)} \\ \omega_s^2 \underline{\phi}^{(r)T} [M] \underline{\phi}^{(s)} &= \underline{\phi}^{(r)T} [K] \underline{\phi}^{(s)} \quad \dots (3) \\ \omega_r^2 \underline{\phi}^{(r)T} [M]^T \underline{\phi}^{(s)} &= \underline{\phi}^{(r)T} [K]^T \underline{\phi}^{(s)} \\ \omega_s^2 \underline{\phi}^{(r)T} [M] \underline{\phi}^{(s)} &= \underline{\phi}^{(r)T} [K] \underline{\phi}^{(s)} \quad \dots (4) \end{aligned}$$

$[M], [K]$ symmetric

Subtract (4) from (3)

$$(\omega_s^2 - \omega_r^2) \underline{\phi}^{(r)T} [M] \underline{\phi}^{(s)} = 0$$

$$\text{if } r \neq s \rightarrow \underline{\phi}^{(r)T} [M] \underline{\phi}^{(s)} = 0$$

$$r = s \rightarrow \underline{\phi}^{(r)T} [M] \underline{\phi}^{(s)} = M_r^* \text{ (same value : modal stiffness)}$$

$$\underline{\phi}^{(r)T} [M] \underline{\phi}^{(s)} = \delta_{rs} M_r^*$$

$$\hookrightarrow \text{Kronecker delta } \delta_{rs} = \begin{cases} 0 & : r \neq s \\ 1 & : r = s \end{cases}$$

Also note that

$$\underline{\phi}^{(r)T} [K] \underline{\phi}^{(s)} = \omega_r^2 M_r^* \delta_{rs} \text{ (modal stiffness)}$$

- complete solution

$$[M] \ddot{\underline{w}} + [K] \underline{w} = \underline{F}$$

$$\text{let } w_i(t) = \sum_{i=1}^4 \phi_i^{(r)} \underbrace{\eta_i(t)}_{\text{Generalized coordinate}}$$

Generalized coordinate

$$[M] \underline{\phi} \ddot{\eta} + [K] \underline{\phi} \eta = \underline{F}$$

Pre-multiply by $\underline{\phi}^T$

$$\underline{\phi}^T [M] \underline{\phi} \ddot{\eta} + \underline{\phi}^T [K] \underline{\phi} \eta = \underline{\phi}^T \underline{F}$$

Orthogonality \rightarrow Decoupled equations

$$M_1^* \ddot{\eta}_1 + M_1^* \omega_1^2 \eta_1 = Q_1, \quad Q_1 = \underline{\phi}^{(1)T} \underline{F}$$

$$\begin{matrix} \text{Generalized} \\ \text{mass} \end{matrix} \downarrow \quad \begin{matrix} \vdots \\ M_n^* \ddot{\eta}_n \\ \vdots \end{matrix} + \begin{matrix} \text{Generalized or normalized coordinate} \\ \text{Generalized stiffness} \end{matrix} \downarrow \quad \begin{matrix} \vdots \\ M_n^* \omega_n^2 \eta_n \\ \vdots \end{matrix} = \begin{matrix} \text{Generalized force} \\ Q_n \end{matrix}$$

- Initial conditions

@ $t=0$, given $w(0), \dot{w}(0)$

$$\underline{\phi} \underline{\eta}(0) = \begin{Bmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \\ w_4(0) \end{Bmatrix} \quad \text{and} \quad \underline{\phi} \dot{\eta} = \dot{w}(0)$$

If all modes are retained in solution, that is, $w = \sum_{i=1}^n \phi^{(i)} \eta_i(t)$

$$\begin{matrix} \eta(0) \\ n \times 1 \end{matrix} = \begin{matrix} \phi^{-1} \\ n \times n \end{matrix} \begin{matrix} w(0) \\ n \times 1 \end{matrix}$$

- truncation

Problem can be truncated by using only a few selected number of modes

$$w(x,t) = \sum_{i=1}^m \phi^{(i)}(x) \eta_i(t)$$

where $m \ll n$

But now calculation of initial condition on η is not straightforward.

$$\begin{matrix} \eta(0) \\ m \times 1 \end{matrix} = \begin{matrix} \phi^{-1} \\ m \times m \end{matrix} \begin{matrix} w(0) \\ m \times 1 \end{matrix}$$

\uparrow not invertible!

$$\begin{matrix} \underline{\phi} \eta(0) \\ m \times m \end{matrix} = \begin{matrix} w(0) \\ m \times 1 \end{matrix}$$

Premultiply by $\underline{\phi}^T [M]$,

$$\underbrace{\phi^T [M] \phi}_{M_{m \times n}^+ : \text{diagonal}} \underline{\eta}(0) = \underbrace{\phi^T [M]}_{m \times n} \underbrace{w(0)}_{n \times 1}$$

$$M^+ \underline{\eta}(0) = \phi^T [M] \underline{w}(0)$$

$$\eta_i(0) = \frac{1}{M_i^+} [\phi_1^{(i)} \dots \phi_n^{(i)}] [M] \begin{Bmatrix} w_1(0) \\ w_2(0) \\ \vdots \\ w_n(0) \end{Bmatrix}$$

→ Solve for $\eta(t)$ subject to $\eta(0)$ and $\dot{\eta}(0)$

and find w from $w(x,t) = \sum_{i=1}^m \phi^{(i)}(x) \eta_i(t)$

Note: The Normal Equations of Motion are uncoupled on the left-hand side due to the modal matrix composed of eigenvectors.

Coupling, however, may come from motion-dependent forces, including damping.

- Motion Dependent Forces

Forces F_i may be dependent on position, velocity, acceleration of structure @ its nodes i , as well as time

$$\Rightarrow F_i = F_i(w_1, w_2, \dots, \dot{w}_1, \dot{w}_2, \dots, \ddot{w}_1, \ddot{w}_2, \dots, t)$$

Consider a general case

$$F_i = \sum_{k=1}^N (a_{ik} w_k + c_{ik} \dot{w}_k + e_{ik} \ddot{w}_k) + F_i(t)$$

consider an N degree of freedom system

$$[M] \{\ddot{w}\} + [K] \{w\} = [a] \{w\} + [c] \{\dot{w}\} + [e] \{\ddot{w}\} + \{F(t)\}$$

$$\text{Let } w_i = \sum_j^{n=3} \phi_i^{(j)} \eta_j(t)$$

o Modal and Discrete Methods

- At this point, a distinction between two main classes of approaches for approximating the solution of structural systems needs to be made.

- The two basic approaches are

- i) Modal methods: represent displacements by overall motion of the structure
- ii) Discrete Point Methods: represent displacement by motion at many discrete points distributed along the structures.

o Continuous System

Consider a basic high-aspect ratio wing modeled as a cantilever beam for symmetric response



Partial Differential Equation for Continuous Beam

$$m\ddot{w} - (Tw')' + (EIw'')'' = f_z$$

$$T' = -f_x$$

$m(x)$: mass/unit length (kg/m)

$w(x, t)$: vertical deflection (m)

T : axial force (N)

$EI(x)$: bending stiffness (N.m²)

f_z : vertical applied force (N/m)

f_x : horizontal applied force (N/m)

plus two boundary conditions at each end (Total 4)

- pinned end 

$$w = 0$$

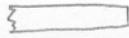
$$M = EIw'' = 0$$

- fixed end 

$$w = 0$$

$$w' = 0$$

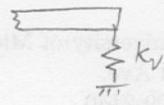
- Free end



$$M = EI w'' = 0$$

$$S = (EI w''')' = 0$$

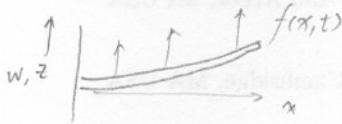
- vertical spring



$$M = EI w'' = 0$$

$$S = (EI w''')' = k_v w$$

• Response of uniform cantilevered beam



$$m = EI = \text{const.}$$

$$T = 0$$

$$\text{B.C. : @ } x = 0 \quad \begin{cases} w = 0 \\ w' = 0 \end{cases} \rightarrow \text{geometric B.C.}$$

$$\text{@ } x = l \quad \begin{cases} M = EI w'' = 0 \\ S = EI w''' = 0 \end{cases} \rightarrow \text{natural B.C.}$$

$$\text{I.C. : @ } t = 0 \quad \begin{cases} w = 0 \\ \dot{w} = 0 \end{cases} \quad (\text{Rest I.C.'s})$$

same solution procedure as before

i) find solution to homogeneous equation

ii) Then determine complete solution as expansion of homogeneous solution

$$EI w^{IV} + m \ddot{w} = 0 \quad \dots (1)$$

$$\text{let } w(x, t) = \bar{w}(x) e^{i\omega t}$$

$$\rightarrow (EI \bar{w}^{IV} - m\omega^2 \bar{w}) e^{i\omega t} = 0 \quad \dots (2)$$

$$\rightarrow \bar{w}^{IV} - \frac{m\omega^2}{EI} \bar{w} = 0 \quad \dots (3)$$

to solve, let $\bar{w} = e^{px}$ ($\rightarrow \sin, \cos, \sinh, \cosh$)

$$\rightarrow p^4 e^{px} - \frac{m\omega^2}{EI} e^{px} = 0$$

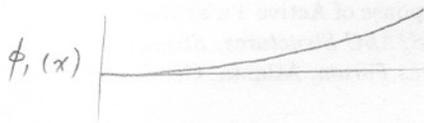
for eigenvectors (mode shapes)

place λl into first three equations

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \sinh \lambda l & \cosh \lambda l & -\sin \lambda l & -\cos \lambda l \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0$$

$$\bar{w}_r(x) = (\cosh \lambda_r x - \cos \lambda_r x) - \left(\frac{\cosh \lambda_r l + \cos \lambda_r l}{\sinh \lambda_r l + \sin \lambda_r l} \right) (\sinh \lambda_r x - \sin \lambda_r x)$$

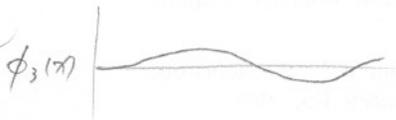
$$w(x, t) = \sum_{r=1}^{\infty} \phi_r(x) e^{i\omega_r t}$$



$$\omega_1 = 3.52 \sqrt{\frac{EI}{ml^4}} \quad (\text{rad/s})$$



$$\omega_2 = 22 \sqrt{\frac{EI}{ml^4}}$$



$$\omega_3 = 61.7 \sqrt{\frac{EI}{ml^4}}$$

• Orthogonality

Since each solution satisfies $w(x, t) = \phi_r(x) e^{i\omega_r t}$

$$m\ddot{w} + (EI w'')'' = 0$$

$$-m\omega_r^2 \phi_r + (EI \phi_r'')'' = 0 \quad \dots (1)$$

$$-m\omega_s^2 \phi_s + (EI \phi_s'')'' = 0 \quad \dots (2)$$

multiply (1) by ϕ_s and integrate

$$\omega_r^2 \int_0^l \phi_s m \phi_r dx = \int_0^l \phi_s (EI \phi_r'')'' dx \quad \dots (3)$$

and (2) by ϕ_r and integrate

$$\omega_s^2 \int_0^l \phi_r m \phi_s dx = \int_0^l \phi_r (EI \phi_s'')'' dx \quad \dots (4)$$

subtract (4) from (3), and integrate by parts

$$(\omega_r^2 - \omega_s^2) \int_0^l \phi_r m \phi_s dx = \phi_s (EI \phi_r''')' \Big|_0^l - \phi_s' EI \phi_r'' \Big|_0^l + \int_0^l \phi_s'' EI \phi_r'' dx$$

$$- \underbrace{\phi_r (EI \phi_s''')'}_{\text{deflection}} \Big|_0^l + \underbrace{\phi_r' EI \phi_s''}_{\text{shear}} \Big|_0^l - \int_0^l \phi_r'' EI \phi_s'' dx$$

slope moment

note that all constant terms on RHS = 0 because of BC's.

for example: pinned $\rightarrow w = 0 \Rightarrow \phi = 0$

$$w'' = 0 \Rightarrow \phi'' = 0$$

fixed $\rightarrow w = 0 \Rightarrow \phi = 0$

$$w' = 0 \Rightarrow \phi' = 0$$

free $\rightarrow \phi'' = 0$ and $(EI \phi''')' = 0$

$$(M = 0) \quad (S = 0)$$

\Rightarrow for $r \neq s$, we have

$$\int_0^l \phi_r(x) m(x) \phi_s(x) dx = 0$$

$$\int_0^l m(x) \phi_r(x) \phi_s(x) dx = \delta_{rs} M_r^*$$

$$\text{Also, } \int_0^l \phi_s (EI \phi_r''')' dx = \delta_{rs} M_r^* \omega_r^2$$

\Rightarrow can transform to Normal Coordinates

• Complete solution

$$m \ddot{w} + (EI w'')'' = f(x, t) \quad \dots (5)$$

$$\text{let } w(x, t) = \sum_{r=1}^{\infty} \phi_r(x) \eta_r(t) \quad \dots (6)$$

place (6) into (5) and integrate after multiplying with ϕ_s

$$\sum_{r=1}^{\infty} \ddot{\eta}_r \int_0^l m \phi_s \phi_r dx + \sum_{r=1}^{\infty} \eta_r \int_0^l \phi_s (EI \phi_r''')' dx = \int_0^l \phi_s f(x, t) dx$$

because of orthogonality

$$\begin{bmatrix} M_r \ddot{\eta}_r + M_r \omega_r^2 \eta_r = Q_r \\ \vdots \\ \infty \end{bmatrix}$$

$$M_r = \int_0^l \phi_r^2(x) m(x) dx$$

$$Q_r = \int_0^l \phi_r(x) f(x, t) dx$$

Note: can also show orthogonality conditions hold if $-(T w)'$ term is present.

To find I.C.'s on η_r , substitute with

$$\textcircled{a} t=0, \quad w(x, 0) = \sum_{r=1}^{\infty} \phi_r(x) \eta_r(0) = w_0(x)$$

and

$$\dot{w}(x, 0) = \sum_{r=1}^{\infty} \phi_r(x) \dot{\eta}_r(0) = \dot{w}_0(x)$$

Multiply by $m \phi_s(x)$ and integrate

$$\int_0^l m \phi_s(x) w_0 dx = \sum_{r=1}^{\infty} \eta_r(0) \int_0^l m \phi_s \phi_r dx = \eta_s(0) M_s^*$$

$$\rightarrow \begin{cases} \eta_r(0) = \frac{1}{M_r^*} \int_0^l m \phi_r w_0(x) dx \\ \dot{\eta}_r(0) = \frac{1}{M_r^*} \int_0^l m \phi_r \dot{w}_0(x) dx \end{cases}$$

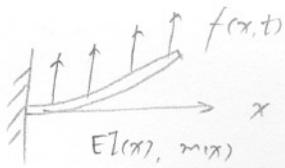
o Rayleigh - Ritz Method

- Energy-based method

- Form of the solutions is assumed to be as:

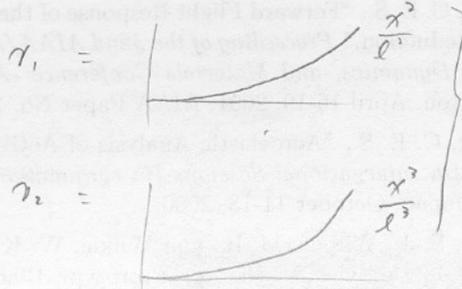
$$w(x, t) \approx \sum_{r=1}^N \phi_r(x) \eta_r(t)$$

assumed modes needs to satisfy at least geometrical boundary conditions



assume $w(x,t) = \sum_{i=1}^M r_i(x) q_i(t)$

for example,



satisfy $w=0$
 $w'=0$ } @ $x=0$

$$T = \frac{1}{2} \int_0^l m(x) \sum_{i=1}^M r_i q_i \sum_{j=1}^M r_j \dot{q}_j dx$$

$$= \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M \int_0^l m r_i(x) r_j(x) dx \dot{q}_i \dot{q}_j$$

m_{ij}^*

$$V = \frac{1}{2} \int_0^l EI (w'')^2 dx$$

$$= \frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M \int_0^l EI(x) r_i''(x) r_j''(x) dx q_i q_j$$

k_{ij}^*

$$\delta W = \int_0^l f \delta w dx$$

$$= \sum_{i=1}^M \int_0^l f(x) r_i dx \delta q_i$$

plug into Lagrange's Equations,

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

which gives

$$\sum_{i=1}^M m_{ij}^* \ddot{q}_j + \sum k_{ij}^* q_j = Q_i$$

coupled set of equations!

For a quick and "dirty" way to find first natural frequency,
assume only one mode shape,

$$m_{11} \ddot{z}_1 + k_{11} z_1 = Q_1$$

Rayleigh Quotient with $z = \bar{z} e^{i\omega t}$

$$\omega^2 = \frac{\int_0^L EI (z_1'')^2 dx}{\int_0^L m z_1^2 dx} \quad \dots \text{upper bound for the actual frequency}$$

Clearly we can get higher modes by assuming more than one mode

$$\omega_r^2 = \frac{\{r\}_r^T [K] \{r\}_r}{\{r\}_r^T [M] \{r\}_r}$$

• Galerkin's method

- Galerkin's method applied to P.D.E. directly - residual method

$$\int_{\text{Domain}} r_j [\text{P.D.E.}] dx = 0 \quad \text{for } j = 1, 2, \dots, N$$

- Assumed modes must satisfy all boundary conditions (geometric and natural ones)

$$w(x, t) = \sum_{i=1}^N r_i(x) g_i(t)$$

look at general beams

$$m \ddot{w} + (EI w'')'' - (T w')' = f(x, t)$$

for a pinned-pinned beam,

$$r_j = \sin\left(\frac{j\pi x}{L}\right)$$

If r_j is an exact mode shape, P.D.E. would be satisfied exactly.

But if not \Rightarrow Error

$$E = m \ddot{w}_{\text{approx}} + [EI w_{\text{approx}}'']'' - [T w_{\text{approx}}']' - f$$

Now set

$\int_0^l h_i(x) E(x) dx = 0$: Average error in PDE with respect to some weighting function $h_i(x)$ that minimize the error in the interval,

usually take $h_i(x) = \tau_i(x)$

$$\sum_{j=1}^M \ddot{\delta}_j \left[\int_0^l \tau_i(x) m(x) \tau_j(x) dx \right] + \sum_{j=1}^M \left[\int_0^l \tau_i (EI \tau_j'')'' dx \right] - \int_0^l \tau_i (T \tau_j')' dx \Big] = \int_0^l \tau_i f(x, t) dx$$

different from Rayleigh-Ritz

For M different weighting function τ_1, \dots, τ_M , we have M equations to find M unknowns $\delta_1, \dots, \delta_M$.

In matrix form,

$$[m_{ij}] \ddot{\delta}_j + [k_{ij}] \delta_j = Q_j \quad \dots \text{coupled set of DE's. (except when } \tau_j \text{ is natural mode shape)}$$

Used standard technique, let $\bar{z} = \bar{z} e^{i\omega t}$

$$\Rightarrow [L\omega^2 - [m]^{-1}[K]] \bar{z} = 0$$

\Rightarrow Eigenvalues \rightarrow approximate natural frequencies

Eigenvectors \rightarrow " natural mode shapes

Note: i) more assumed modes \rightarrow better approximation

$$\phi_1(x) = A \cos \lambda_1 x + B \sin \lambda_1 x + C \cosh \lambda_1 x + D \sinh \lambda_1 x$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

ii) more accurate assumed shapes \rightarrow better approximation

iii) if $\tau_j(x)$ is natural mode shapes, system will be uncoupled.

iv) The closer $\psi_j(x)$ is to $\phi_j(x)$, the less the coupling.

Galerkin is very powerful, turn PDE's into ODE's.

very general, can also be used in nonlinear problems!

$$m\ddot{w} + (EI w'')'' + F(w^m) = f$$

v) If Rayleigh-Ritz assumed mode shapes satisfy both geometric and natural B.C.'s, two methods are identical.

(can be shown by integration by parts)