

Unsteady Aerodynamics

I. Field Description

- Lagrangian method --- particle point of view, position of any fluid particle P
 $x = x_p(x_0, y_0, z_0, t)$, ...
 → abundance of information
 not necessary nor useful
- Eulerian method --- field point of view, spatial distribution of flow variables at each instant
 $u = u(x, y, z, t)$, ...

II. Governing Eqr's

i. Integral form

① Mass conservation

$$\frac{d \text{m.c.v.}}{dt} = \frac{\partial}{\partial t} \int_{\text{c.v.}} \rho dV + \int_{\text{c.s.}} \rho (\vec{f} \cdot \vec{n}) dS = 0 \quad \dots (1)$$

↑
 change in the mass
 within the c.v.
 ↓
 rate of mass leaving across and
 normal to the surface

② Momentum conservation

$$\frac{d (\rho \vec{q})_{\text{c.v.}}}{dt} = \frac{\partial}{\partial t} \int_{\text{c.v.}} \rho \vec{q} dV + \int_{\text{c.s.}} \rho \vec{q} (\vec{f} \cdot \vec{n}) dS = \sum \vec{F}$$

↑
 accumulation of momentum
 ρq within the c.v.
 ↓
 charge of momentum
 across the c.s. boundary

$$(\sum \vec{F})_i = \int_{\text{c.v.}} \rho f_i dV + \int_{\text{c.s.}} \eta_j T_{ij} dS$$

↑
 body forces
 ↓
 surface forces

ii. Differential form

① Mass conservation

- divergence theorem

$$\int_{\text{c.s.}} \eta_j g_j dS = \int_{\text{c.v.}} \frac{\partial g_j}{\partial x_j} dV$$

$$(1) \rightarrow \int_{\text{c.v.}} \underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} \right)}_{\nabla \cdot \rho \vec{q}} dV = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho}_{D\rho/Dt} + \rho \vec{q} \cdot \nabla \vec{q} = 0$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{q} \cdot \nabla \quad \begin{array}{l} \text{"material derivative"} \\ \text{"rate of change following a} \\ \text{fluid particle"} \end{array} \quad (1)$$

$$\Rightarrow \frac{DP}{Dt} + \rho \nabla \cdot \vec{g} = 0$$

incompressible fluid \rightarrow constant density

$$\Rightarrow \nabla \cdot \vec{g} = 0$$

Q Momentum Eqn.

From the divergence theorem,

$$\int_{C.S.} \rho g_i (\vec{g} \cdot \vec{n}) dS = \int_{C.V.} \nabla \cdot \rho g_i \vec{g} dV$$

$$\int_{C.S.} \eta_j T_{ij} dS = \int_{C.V.} \frac{\partial T_{ij}}{\partial x_j} dV$$

$$\Rightarrow \int_{C.V.} \left[\frac{\partial}{\partial t} (\rho g_i) + \nabla \cdot (\rho g_i \vec{g}) - \rho f_i - \frac{\partial T_{ij}}{\partial x_j} \right] dV = 0$$

$$= 0$$

$$\frac{\partial}{\partial t} (\rho g_i) + \nabla \cdot (\rho g_i \vec{g}) = \rho \frac{Dg_i}{Dt}$$

$$\Rightarrow \rho \frac{Dg_i}{Dt} = \rho f_i + \frac{\partial T_{ij}}{\partial x_j}$$

For a Newtonian fluid

$$\Rightarrow \rho \left(\frac{\partial g_i}{\partial t} + \vec{g} \cdot \nabla g_i \right) = \rho f_i - \frac{\partial}{\partial x_i} \left(\rho + \frac{2}{3} \mu \nabla \cdot \vec{g} \right) + \frac{\partial}{\partial x_j} \mu \left(\frac{\partial g_i}{\partial x_j} + \frac{\partial g_j}{\partial x_i} \right)$$

incompressible fluid $\rightarrow \nabla \cdot \vec{g} = 0$

$$\rho \left(\frac{\partial \vec{g}}{\partial t} + \vec{g} \cdot \nabla \vec{g} \right) = \rho \vec{f} - \nabla p + \mu \nabla^2 \vec{g}$$

Inviscid fluid

$$\frac{\partial \vec{g}}{\partial t} + \vec{g} \cdot \nabla \vec{g} = \vec{f} - \frac{\nabla p}{\rho} \quad \dots \text{Euler eqn.}$$

III. Inviscid, Incompressible Flow

high Reynolds No. flow \rightarrow effects of viscosity confined to thin boundary layers and thin wakes.

flow outside this regions \rightarrow inviscid, incompressible

i) Vorticity and circulation

$$\vec{\omega} = \nabla \times \vec{g} : \text{vorticity}$$

From Stokes theorem

$$\int_S \nabla \times \vec{g} \cdot \vec{n} dS = \int_S \vec{w} \cdot \vec{n} dS = \underbrace{\int_C \vec{g} \cdot d\vec{l}}_{\equiv \Gamma : \text{circulation}}$$

irrotational flow - fluid will not rotate by the shear force of the neighboring fluid elements $\rightarrow \nabla \times \vec{g} = 0$

(ii) Velocity Potential

irrotational flow $\rightarrow \vec{g} = \nabla \Phi$: velocity potential

continuity eqn. $\nabla \cdot \vec{g} = \nabla \cdot \nabla \Phi = \nabla^2 \Phi = 0$: Laplace's eqn.

Since viscosity is neglected, only one boundary condition can be enforced

- normal component of relative velocity between the fluid and the solid surface is zero

$$\vec{n} \cdot (\vec{g} - \vec{g}_s) = 0$$

velocity of solid surface

(iii) Bernoulli's eqn. for the pressure

- Vector identity $\vec{g} \cdot \nabla \vec{g} = \nabla \frac{g^2}{2} - \vec{g} \times \vec{\sigma}$

- incompressible Euler eqn. $\rightarrow \frac{\partial \vec{g}}{\partial t} - \vec{g} \times \vec{\sigma} + \nabla \frac{g^2}{2} = \vec{f} - \nabla \frac{p}{\rho}$

- irrotational flow $\rightarrow \frac{\partial \vec{g}}{\partial t} = \frac{\partial}{\partial t} \nabla \Phi = \nabla \left(\frac{\partial \Phi}{\partial t} \right)$

- Assumption --- body force is conservative with a potential E

$$\vec{f} = -\nabla E$$

$\Rightarrow \nabla (E + \frac{p}{\rho} + \frac{g^2}{2} + \frac{\partial \Phi}{\partial t}) = 0$ --- Euler eqn. for incompressible, irrotational flow

$$E + \frac{p}{\rho} + \frac{g^2}{2} + \frac{\partial \Phi}{\partial t} = C(t) \quad \text{--- Bernoulli's eqn.}$$

$$[E + \frac{p}{\rho} + \frac{g^2}{2} + \frac{\partial \Phi}{\partial t}] = [E + \frac{p}{\rho} + \frac{g^2}{2} + \frac{\partial \Phi}{\partial t}]|_\infty$$

$$\frac{p_\infty - p}{\rho} = \frac{\partial \Phi}{\partial t} + E + \frac{g^2}{2}$$

iv) The Biot - Savart Law

- velocity field due to a known vorticity distribution

$$\vec{v} = \frac{1}{4\pi} \int_V \nabla \times \frac{\vec{\omega}}{|\vec{r}_0 - \vec{r}_1|} dV \quad \vec{r}_0 = \text{position vector of the point}$$

\vec{r}_1 of the vorticity

$$= \frac{1}{4\pi} \int_V \frac{d\vec{\omega} \times (\vec{r}_0 - \vec{r}_1)}{|\vec{r}_0 - \vec{r}_1|^3} dV$$

$$= \frac{1}{4\pi} \int_V \frac{\vec{\omega} \times (\vec{r}_0 - \vec{r}_1)}{|\vec{r}_0 - \vec{r}_1|^3} dV \quad \dots \text{Biot - Savart Law}$$

IV. General Solution of the Incompressible, Potential Flow Eqs.

i) Problem statement

- incompressible, irrotational - continuity eqn. reduces to

$$\nabla^2 \phi = 0$$

- velocity normal to the body's surface and solid boundaries must be zero

$$\nabla \phi \cdot \vec{n} = 0$$

- disturbance created by the motion should decay far from the body

$$\lim_{r \rightarrow \infty} (\nabla \phi - \vec{v}) = 0$$

relative velocity between the undisturbed fluid and
the body

ii) Methodology of Solution

- solution is obtained by distributing elementary solutions on the problem boundaries

→ singular solutions

- potential specified on the boundaries → Dirichlet problem

Zero normal flow boundary condition → Neumann problem

- Additional considerations are required (Kutta condition)

iii) Separation of Thickness and Lifting Problems in Wing

- complete solution for the cambered wing with nonzero thickness at a certain angle of attack

= symmetric wing with nonzero thick at zero a.o.a.
(thickness effect) +

zero-thickness, uncambered wing at angle of attack
(effect of a.o.a.)

+ zero-thickness, cambered wing at zero a.o.a. (effect of camber)

viv. Zero-thickness cambered wing at Angle of Attack -- Lifting surfaces

- boundary condition requiring no flow across the surface

$$\frac{\partial \bar{P}}{\partial z}(x, y, 0\pm) = Q_{\infty} \left(\frac{\partial \eta}{\partial x} - \alpha \right)$$

- can be solved by a doublet distribution or a vortex distribution

For a vortex line distribution, vortex elements cannot be terminated at the wing and must be shed into the flow

vi) Vortex distribution for lifting surface

- velocity due to a vortex element $d\vec{l}$ with a strength $\Delta\Gamma$

$$\Delta\vec{v} = \frac{-1}{4\pi} \frac{\Delta\Gamma \vec{r}_x d\vec{l}}{r^3} \quad (\text{Biot-Savart Law})$$

- downwash induced by γ_y (over wing) and γ_x (in the wake)

$$w(x, y, z) = \frac{-1}{4\pi} \int_{\text{wing+wake}} \frac{\gamma_y(x-x_0) - \gamma_x(y-y_0)}{r^3} dx_0 dy_0$$

- Helmholtz vortex theorem -- vortex strength is const. along a vortex line

$$\left| \frac{\partial \gamma_x}{\partial x} \right| = \left| \frac{\partial \gamma_y}{\partial y} \right| \rightarrow \text{unknowns are one}$$

vii) Vortex wake

- Kutta condition -- flow leaves the sharp T.E. smoothly and the velocity is finite.

$$\gamma_{TE} = 0$$

V. Thin Airfoil Theory

i) zero-thickness airfoil at angle of attack

- small-disturbance flow over thin airfoils

-- divided into a thickness problem and a lifting problem

- Lifting problem --- thin, cambered airfoil, a.o.a. α , inviscid, incompressible, irrotational, continuity eqn.

$$\nabla^2 \Phi = 0$$

camberline of the airfoil is given by a known function $\eta_c(x)$

- boundary condition requiring no flow across the surface

$$\frac{\partial \Phi}{\partial z}(x, 0^\pm) = Q_\infty \left(\frac{d\eta_c}{dx} \cos\alpha - \sin\alpha \right) \approx Q_\infty \left(\frac{d\eta_c}{dx} - \alpha \right)$$

$$\Rightarrow \frac{-1}{2\pi} \int_0^c \eta(x_0) \frac{dx_0}{x-x_0} = Q_\infty \left(\frac{d\eta_c}{dx} - \alpha \right), \quad 0 < x < c$$

- Kutta condition

$$\nabla \Phi \leq \infty \text{ (at T.E.)} \rightarrow \gamma(x=c) = 0$$

ii) Classical solution of the Lifting problem

- Glauert's approach --- approximate $\eta(x)$ by a trigonometric expansion

$$\text{transformation } x = \frac{c}{2} (1 - \cos\theta), \quad \text{l.e. } x=0 \ (\theta=0) \\ \text{t.e. } x=c \ (\theta=\pi)$$

- No flow across the surface

$$\frac{-1}{2\pi} \int_0^\pi \eta(\theta_0) \frac{\sin\theta_0 d\theta_0}{\cos\theta_0 - \cos\theta} = Q_\infty \left[\frac{d\eta_c(\theta)}{dx} - \alpha \right], \quad 0 < \theta < \pi$$

Kutta condition $\gamma(\pi) = 0$ --- satisfied automatically

- Proposed function for the vortex distribution

$$\eta(\theta) = 2Q_\infty \left[A_0 \frac{1 + \cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin(n\theta) \right]$$

--- a large suction peak at the leading edge

$$-\cotangent \text{ function } A_0 \cot \frac{\theta}{2} = A_0 \frac{1 + \cos\theta}{\sin\theta}$$

- After inserting this expression into "no flow across the surface" condition and using Glauert's integral

$$-A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) = \frac{d\eta_c(\theta)}{dx} - \alpha \quad \text{--- Fourier expansion}$$

$$A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \frac{d\eta_c(\theta)}{dx} d\theta, \quad n=0$$

$$A_m = \frac{2}{\pi} \int_0^\pi \frac{d\eta_c(\theta)}{dx} \cos m\theta d\theta, \quad m=1, 2, 3, \dots$$

- downwash distribution

$$\frac{w}{\rho_\infty} = -A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta)$$

VII. Unsteady Incompressible Potential Flow

(i) 2-D Thin Airfoil

- modeling of the vortex wake's shape and strength -- discretized vortex wake model.
- inertial frame X, Z ; at $t > 0$, airfoil moves along a curved path S
the coordinates x, z are selected such that $\left. \begin{array}{l} \text{the origin is placed on } S, \\ x \text{ axis is always tangent to } S \end{array} \right\}$

camberline $y(x, t)$

path radius of curvature ρ ... $c/\rho = \dot{\theta}c/V(t) \ll 1$

- continuity eqn. in the moving frame of reference x, z system

$$\nabla^2 \Phi = 0$$

- time-dependent version of the boundary condition (no flow across the surface)

$$(\nabla \Phi - \vec{V}_0 - \vec{v}_{rel} - \vec{S} \times \vec{r}) \cdot \vec{n} = 0$$

$$\vec{n} = \frac{(-\partial y / \partial x, 0, 1)}{\sqrt{(\partial y / \partial x)^2 + 1}}, \quad \vec{V}_0 = [-U(t), 0, 0], \quad \vec{S} = [0, \dot{\theta}(t), 0],$$

$$\vec{v}_{rel} = (0, 0, \frac{\partial y}{\partial t})$$

- velocity potential ... $\Phi = \Phi_B + \Phi_w$

Φ_w wake potential, assumed to be known
 Φ_B airfoil potential, to be determined

$$\frac{\partial \Phi_B}{\partial z} = \left(\frac{\partial \Phi_B}{\partial x} + \frac{\partial \Phi_w}{\partial x} + U - \dot{\theta}z \right) \frac{\partial y}{\partial x} - \frac{\partial \Phi_w}{\partial z} - \dot{\theta}x + \frac{\partial y}{\partial t} \equiv w(x, t)$$

- equivalent steady-state flow problem at each time step,
by exchange the local downwash $w(x, t)$, the method developed
in the steady-state can be applied.

iii) wake modeling

- continuous vortex sheet shed from T.E. \rightarrow discrete vortex model of strength $\Gamma_{w,i}$ of each

$$\Gamma_{w,i} = \int_{t-\Delta t}^t \gamma_w(t) U(t) dt$$

- location of discrete vortex element --- place the lastest vortex closer to T.E. (within $0.2 \sim 0.3 U(t)\Delta t$)

- strength of the lastest vortex element --- Kelvin's theorem

$$\frac{d\Gamma}{dt} = \frac{d\Gamma(t)}{dt} + \frac{d\Gamma_w}{dt} = 0$$

At the i -th time step,

$$\Gamma_{w,i} = -[\Gamma(t_i) - \Gamma(t_{i-1})] = -[\Gamma(t_i) + \sum_{k=1}^{i-1} \Gamma_{w,k}]$$

- Helmholtz theorem --- no vortex decay, vortex strength will be conserved (good approximation for high Reynolds No. flows)

iii) Solution by the Time-Stepping Method

- downwash induced by the airfoil bound circulation $\pi(x,t)$

$$\frac{\partial \Phi_B}{\partial z}(x,t)|_{z=0} = \frac{-1}{2\pi} \int_0^c \pi(x,t) \frac{dx}{x-x_0}$$

- due to N_w discrete vortices of the wake

$$\frac{\partial \Phi_w}{\partial z}(x,t)|_{z=0} = \sum_{k=1}^{N_w} \frac{-\Gamma_k}{2\pi} \frac{x-x_k}{(x-x_k)^2 + (z-z_k)^2}, \quad k: \text{counter of the wake vortices}$$

- boundary conditions (no flow across the surface)

$$\frac{-1}{2\pi} \int_0^c \pi(x,t) \frac{dx}{x-x_0} = U(t) \frac{\partial \pi(x,t)}{\partial x} - \frac{\partial \Phi_w}{\partial z}(x,t) - \dot{\theta}(t)x + \frac{\partial \pi(x,t)}{\partial t}, \quad 0 < x < c$$

- Kutta condition $\pi(c,t) = 0$

r.h.s. is known except the lastest vortex influence \rightarrow assume that it is known

- Glauert transformation

$$x = \frac{c}{2} (1 - \cos \theta)$$

$$\frac{w(x,t)}{U(t)} = A_0(t) + \sum_{n=1}^{\infty} A_n(t) \cos(n\theta)$$

$$A_0(t) = -\frac{1}{\pi} \int_0^\pi \frac{w(x,t)}{U(t)} d\theta, \quad n=0$$

$$A_n(t) = \frac{2}{\pi} \int_0^\pi \frac{w(x,t)}{U(t)} \cos n\theta d\theta, \quad n=1, 2, 3, \dots$$

-- if momentary chordwise downwardwash $w(x, t)$ is known, then the momentary circulation distribution is known, too.

- Determining the strength of the latest vortex element --- Kelvin's condition

$$f(\Gamma) = \Gamma(t) + \Gamma_{wi} + \sum_{k=1}^{\infty} \Gamma_{wk} \quad \{ = 0 \text{ for the converged sol.}\}$$

↑ ↑ ↑
 airfoil latest circulation of all
 vortex wake the other wake vortices (known from the previous
 wake time step)

- Newton-Raphson iteration scheme

$$(\Gamma_{wi})_{j+1} = (\Gamma_{wi})_j - \frac{f(\Gamma_{wi})_j}{f'(\Gamma_{wi})_j}, \quad f'(\Gamma)_j = \frac{[f(\Gamma)_j - f(\Gamma)_{j-1}]}{(\Gamma_w)_j - (\Gamma_w)_{j-1}}$$

- Iterative procedure

① at a given time step t_i , $w(x, t)$ is calculated by

$$w(x, t) \approx U \frac{\partial \bar{\Phi}}{\partial x} - \frac{\partial \bar{\Phi}_w}{\partial x} - \dot{\theta}x + \frac{\partial \bar{\eta}}{\partial t}$$

② Assuming Γ_{wi} for the most recently shed T.E. vortex,

can calculate wake influence by (1)

③ Now $w(x, t)$ can be calculated at any point along the chord.

→ allows numerical computation of $A_{\Gamma}(t)$ and $f(\Gamma)$.

④ Using (2), next value of the latest wake vortex is obtained

iv) Fluid Dynamic Loads

- unsteady Bernoulli's eqn.

$$\frac{p_\infty - p}{\rho} = \frac{1}{2} \left[\left(\frac{\partial \bar{\Phi}}{\partial x} \right)^2 + \left(\frac{\partial \bar{\Phi}}{\partial y} \right)^2 + \left(\frac{\partial \bar{\Phi}}{\partial z} \right)^2 \right] - (\vec{V}_0 + \vec{\Omega} \times \vec{r}) \cdot \nabla \bar{\Phi} + \frac{\partial \bar{\Phi}}{\partial t}$$

$$\approx U(t) \frac{\partial \bar{\Phi}}{\partial x} + \dot{\theta}(t)x \frac{\partial \bar{\Phi}}{\partial z} + \frac{\partial \bar{\Phi}}{\partial t} = U(t) \frac{\partial \bar{\Phi}}{\partial x} + \frac{\partial \bar{\Phi}}{\partial t}$$

- Pressure difference across the airfoil Δp

$$\Delta p = p_e - p_u = \rho e \left[U(t) \frac{\partial \bar{\Phi}}{\partial x} + \frac{\partial \bar{\Phi}}{\partial t} \right]_e = \rho \left[U(t) \frac{\partial \bar{\Phi}}{\partial x} + \frac{\partial \bar{\Phi}}{\partial t} \right]$$

$$\text{where } \Delta \bar{\Phi} = \bar{\Phi}(x, 0+, t) - \bar{\Phi}(x, 0-, t) = \int_0^x \gamma(x, t) dx = \Gamma(x, t)$$

$$\Delta p = \rho \left[U(t) \gamma(x, t) + \frac{\partial \bar{\Phi}}{\partial t} \int_0^x \gamma(x, t) dx \right]$$

- lift

$$\begin{aligned}
 L' &= F_z = \int_0^c \Delta p dx = \int_0^c \rho [U(t) \gamma(x, t) + \rho \frac{\partial}{\partial t} \Gamma(x, t)] dx \\
 &= \rho U(t) \Gamma(t) + \rho \int_0^c \frac{\partial}{\partial t} \Gamma(x, t) dx
 \end{aligned}$$

↑ instantaneous circulation
 (similar to steady-state circulatory term)
 ↓ contribution of time dependency

- Glauert transformation and integral:

$$\begin{aligned}
 \frac{\partial}{\partial t} \Delta \bar{\Phi}(x, t) &= \frac{\partial}{\partial t} \int_0^x \gamma(x_0, t) dx_0 = \frac{\partial}{\partial t} \int_0^\theta \gamma(\theta_0, +) \frac{c}{2} \sin \theta_0 d\theta_0 \\
 &= \frac{\partial}{\partial t} \left\{ 2U(t) \int_0^\theta \left[A_0(t) \frac{1 + \cos \theta_0}{\sin \theta_0} + \sum_{n=1}^{\infty} A_n(t) \sin(n\theta_0) \right] \frac{c}{2} \sin \theta_0 d\theta_0 \right\} \\
 &= 2 \left\{ B_0(0 + \sin \theta) + B_1 \left(\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right) + \sum_{n=2}^{\infty} B_n \left[\frac{\sin(n-1)\theta}{2(n-1)} - \frac{\sin(n+1)\theta}{2(n+1)} \right] \right\}
 \end{aligned}$$

where $B_m = \frac{c}{2} \frac{\partial}{\partial t} [A_m(t) U(t)]$, $m = 0, 1, 2, 3, \dots$

$$\begin{aligned}
 - L'(t) &= \rho c \left\{ \frac{3\pi}{2} B_0 + \frac{\pi}{2} B_1 + \frac{\pi}{4} B_2 + \pi U^2 A_0 + \frac{\pi}{2} U^2 A_1 \right\} \\
 &= \pi \rho c \left\{ \left[U^2 A_0 + \frac{3c}{2} \frac{\partial}{\partial t} (U A_0) \right] + \left[U^2 \frac{A_1}{2} + \frac{c}{4} \frac{\partial}{\partial t} (U A_1) + \frac{c}{2} \frac{\partial}{\partial t} (0 A_1) \right] \right\}
 \end{aligned}$$

- Pitching moment

$$\begin{aligned}
 M_o(t) &= - \int_0^c \Delta p x dx = - \int_0^c \rho [U(t) \frac{\partial}{\partial x} \Delta \bar{\Phi} + \frac{\partial}{\partial t} \Delta \bar{\Phi}] x dx \\
 &= - \rho c^2 \frac{\pi}{2} \left[\frac{U^2}{2} (A_0 + A_1 - \frac{A_2}{2}) + \frac{1}{4} B_0 + \frac{3}{4} B_1 + \frac{1}{4} B_2 - \frac{1}{16} B_3 \right] \\
 &= - \rho c^2 \frac{\pi}{2} \left[\frac{U^2}{2} A_0 + \frac{1}{8} \frac{\partial}{\partial t} (U A_0) + \frac{U^2}{2} A_1 + \frac{3}{8} \frac{\partial}{\partial t} (U A_1) - \frac{U^2}{4} A_2 + \frac{c}{32} \frac{\partial}{\partial t} (U A_2) - \frac{c}{32} \frac{\partial}{\partial t} (U A_3) \right]
 \end{aligned}$$

v) small-amplitude oscillation of a thin airfoil (Theodorsen)

- $U(t) = U = \text{const}$, (x, z) frame does not rotate $\rightarrow \theta = \dot{\theta} = 0$

- time-dependent chordline position --- represented by $\left\{ \begin{array}{l} \text{vertical displacement } h(t) \\ \text{instantaneous a.o.a } \alpha(t) \end{array} \right.$

$$y = h - \alpha(x - a)$$

Assume that the pitching axis is at the origin ($a = 0$)

$$y = h - \alpha x, \quad \frac{\partial y}{\partial t} = h - \alpha x, \quad \frac{\partial y}{\partial x} = -\alpha$$

- downwash $w(x, t)$

$$w(x, t) = -U\dot{x} + h - \dot{\alpha}x - \frac{\partial \bar{w}}{\partial z}$$

- loads due to the motion only $\rightarrow w^*(x, t)$

$$w^*(x, t) = -U\dot{x} + h - \dot{\alpha}x = -U\dot{x} + h - \frac{c}{z}\dot{z} + \frac{c}{z}\alpha \cos \theta$$

$$A_0 = \frac{1}{V}(U\dot{x} - h + \frac{c}{z}\dot{z}), \quad A_1 = \frac{\partial c}{\partial V}, \quad A_2 = A_3 = \dots = A_N = 0$$

- circulation due to the downwash w^*

$$\Gamma^*(t) = \int_0^c \gamma(x, t) dx = \pi c V (A_0 + \frac{A_1}{z}) = \pi c (U\dot{x} - h + \frac{3}{4}c\dot{\alpha})$$

$$L^* = \rho U P + \pi \rho c^2 V \left(\frac{3}{4} \frac{\partial A_0}{\partial t} + \frac{1}{4} \frac{\partial A_1}{\partial t} \right)$$

$$= \pi \rho U c (U\dot{x} - h + \frac{3}{4}c\dot{\alpha}) + \pi \rho c^2 \left[\frac{3}{4} (U\dot{x} - h) + \frac{c}{z} \dot{\alpha} \right]$$

--- Kutta condition was satisfied, but the downwash of the unsteady wake is not included

- Theodorsen, von Karman, Sears --- for a small-amplitude oscillatory motion, the final result will include similar terms, and the effect of wake is to reduce the lift due to the first term by a factor of $C(k)$
lift deficiency factor

- harmonic heave and pitch oscillation

$$h = h_0 \sin \omega t, \quad \alpha = \alpha_0 \sin \omega t \quad \text{reduced frequency } k = \frac{\omega c}{2U}$$

$$L' = \pi \rho U c C(k) \left[U\dot{x} - h + \frac{3}{4}c\dot{\alpha} \right] + \pi \rho \frac{c^2}{4} \left[U\dot{x} - h + \frac{c}{z}\dot{z} \right]$$

in case pitch axis is moved to a location a

$$L' = \pi \rho U c C(k) \left[U\dot{x} - \dot{h} + \left(\frac{3}{4} - \frac{a}{c} \right) c\dot{\alpha} \right] + \pi \rho \frac{c^2}{4} \left[(U\dot{x} - \dot{h}) + c \left(\frac{1-a}{z} - \frac{a}{c} \right) \dot{\alpha} \right]$$

$$= L'_1 + L'_2$$

circulatory lift term in a
steady motion

lift due to acceleration
(added mass)

- Delaying effect --- $L'_1(t) = L'_1 \sin(\omega t - \bar{\omega})$

time shift effect of the wake

- Pitching moment

$$M_o = -\frac{\pi \rho c^3}{4} \left\{ -\frac{c}{z} \dot{h} + \frac{3U_c}{4} \dot{\alpha} + \frac{9}{32} c^2 \dot{\alpha}^2 + U C(k) [-\dot{h} + U\dot{x} + \frac{3c}{4} \dot{\alpha}] \right\}$$