- $N(X+Y) = N(X) \cap N(Y)$

  1. $N(X+Y) \subseteq N(X) \cap N(Y)$
     $\forall v \in N(X+Y), (X+Y)v = 0$, and let the spectral decomposition of X and Y be the following:

     $$X = \sum_{i=1}^{n} \lambda_i^x x_i x_i^T, Y = \sum_{j=1}^{n} \lambda_j^y y_j y_j^T$$

     Then, since $(X+Y)v = 0$,

     $$\begin{aligned} v^T(X+Y)v = 0 &= v^T(\sum \lambda_i^x x_i x_i^T)v + v^T(\sum \lambda_j^y y_j y_j^T)v \\ &= \sum \lambda_i^x (x_i^T v)^T (x_i^T v) + \sum \lambda_j^y (y_j^T v)^T (y_j^T v) \end{aligned}$$

     Thus, $i, j = 1, ..., n, \|x_i^T v\|_2 = \|y_j^T v\|_2 = 0 \in \mathbb{R} \Rightarrow x_i^T v = y_j^T v = 0 \in \mathbb{R}^n$,
     and we have $Xv = Yv = 0$, $\forall v$ such that $(X+Y)v = 0$.

  2. $N(X) \cap N(Y) \subseteq N(X+Y)$: It is obvious.

- $rank(X+Y) \geqslant min\{rank(X), rank(Y)\}$

  $$\begin{aligned} N(X+Y) = N(X) \cap N(Y) &\Rightarrow nullity(X+Y) \leqslant min\{nullity(X), nullity(Y)\} \\ &\Rightarrow n - nullity(X+Y) \geqslant max\{n - nullity(X), n - nullity(Y)\} \\ &\Rightarrow rank(X+Y) \geqslant max\{rank(X), rank(Y)\} \\ &\Rightarrow rank(X+Y) \geqslant min\{rank(X), rank(Y)\} \end{aligned}$$