Let $A \in \mathbb{R}^{n\times n}$ be a real symmetric matrix.

1. All eigenvalues are real.

   Proof. Assume that there exists an complex eigenvalue, $\lambda + \mu i$, where $\lambda, \mu \in \mathbb{R}$ and $\mu \neq 0$.

   Then, the corresponding eigenvector must be a complex one, and let it $x + yi$, where $x, y \in \mathbb{R}^n$ and $y \neq 0$. Otherwise, $Ax \neq (\lambda + \mu i)x$ if $x \neq 0$.

   Hence, $A(x + yi) = (\lambda + \mu i)(x + yi)$, and we have $Ax = \lambda x - \lambda y, Ay = \lambda y + \mu x$.

$$\begin{array}{rcl} y^T Ax & = & \lambda y^T x - \mu y^T x \\ - \quad x^T Ay & = & \lambda x^T y + \mu x^T x \\ \hline 0 & = & -\mu(y^T y + x^T x) \end{array}$$

   Since $x^T x + y^T y \neq 0$, $\mu = 0$, which is contradiction to the assumption.

2. Any two eigenvectors obtained from two distinct eigenvalues are orthogonal.

   Proof. Let the eigenvalues and the corresponding eigenvectors be λ_i, λ_j and x_i, x_j. Then,

$$\begin{array}{rcl} x_j^T Ax_i & = & \lambda_i x_j^T x_i \\ - \quad x_i^T Ax_j & = & \lambda_j x_i^T x_j \\ \hline 0 & = & x_i^T x_j(\lambda_i - \lambda_j) \end{array}$$

   Since λ_i and λ_j are distinct, $x_i^T x_j = 0$, or x_i and x_j are orthogonal.

3. Let $X = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}$, where $\|x_i\|_2 = 1, i = 1, ..., n$. Then, $X^T = X^{-1}$ by 2.

4. $X^T AX = \begin{bmatrix} - & x_1^T & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}$

   $= \begin{bmatrix} - & x_1^T & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ Ax_1 & \cdots & Ax_n \\ | & & | \end{bmatrix} = \begin{bmatrix} - & x_1^T & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ | & & | \end{bmatrix}$

   $= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda.$

5. Spectral Decomposition: $A = \lambda_1 x_1 x_1^T + ... + \lambda_n x_n x_n^T$

   Proof. $X^T AX = \Lambda \Rightarrow A = X\Lambda X^T$ since $X^T = X^{-1}$.

   $A = X\Lambda X^T = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} - & x_1^T & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix}$

   $= \begin{bmatrix} | & & | \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & x_1^T & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix}$

   $= \lambda_1 x_1 x_1^T + ... + \lambda_n x_n x_n^T$