$f : \mathbb{R}^n \to \mathbb{R}$ with $\text{dom} f$,
$f^*(y) = \sup_{x \in \text{dom} f} \{y^T x - f(x)\}$,
where $\text{dom} f^* = \{y | y^T x - f(x) < +\infty, \forall x \in \text{dom} f\}$ is called the conjugate function of f.

$\forall x \in \text{dom} f, \forall y \in \text{dom} f^*, f^*(y) \geqslant y^T x - f(x) \iff y^T x \leqslant f(x) + f^*(y)$

e.g. $f(x) = \frac{1}{2} x^T Q x \Rightarrow f^*(y) = \frac{1}{2} y^T Q^{-1} y$, where $Q \in \mathbb{S}^n_{++}$
$\forall x, y, y^T x \leqslant \frac{1}{2} x^T Q x + \frac{1}{2} y^T Q^{-1} y$

Observation. For $f^*(y) = \frac{1}{2} y^T Q^{-1} y$, $f^{**}(z) = \frac{1}{2} z^T (Q^{-1})^{-1} z = \frac{1}{2} z^T Q z$.
Then, is the statement, $f^{**} = f$, true in general?
The answer is yes if f is a convex and closed function.

Note. f is a closed function if $\text{epi} f$ is a closed set.
When f is closed, f can be represented as the following:
$f(x) = \sup\{h(x) | h$ are affine functions such that $h(x) \leqslant f(x), \forall x \in \text{dom} f\}$

Proposition. $f^{**} = f$ if f is a convex and closed function.
Proof.
1. $f^{**} \geqslant f$
Any affine function minorizing f of x can be represented as $y^T x + c$, where c is a constant.
$\Rightarrow y^T x + c \leqslant f(x), \forall x \in \text{dom} f$
$\Rightarrow y^T x - f(x) \leqslant -c, \forall x \in \text{dom} f$
$\Rightarrow y^T x - f(x)$ is bounded above over $\text{dom} f$
$\Rightarrow$ the supremum of $y^T x - f(x)$ exists, and $y \in \text{dom} f^*$
$\Rightarrow y \in \text{dom} f^*, f^*(y) \leqslant -c$
$$\Rightarrow f(x) = \sup_{\substack{y^T x + c \leqslant f(x) \\ \forall x \in \text{dom} f}} \{y^T x + c\} = \sup_{\substack{y \in \text{dom} f^* \\ f^*(y) \leqslant -c}} \{y^T x + c\} \leqslant \sup_{y \in \text{dom} f^*} \{y^T x - f*(y)\} = f^{**}(x)$$

2. $f^{**} \leqslant f$
Since $f^*(y) = \sup_{x \in \text{dom} f} \{y^T x - f(x)\}$, $x^T y - f^*(y) \leqslant f(x), \forall x \in \text{dom} f, \forall y \in \text{dom} f^*$.
$\Rightarrow \sup_{y \in \text{dom} f^*} \{x^T y - f^*(y)\} \leqslant f(x), \forall x \in \text{dom} f$
The left-hand-side of the inequality above can be defined as $f^{**}(x)$ if $\text{dom} f^{**} = \text{dom} f$.
Since $x^T y - f^*(y) < +\infty, \forall x \in \text{dom} f, \forall y \in \text{dom} f^*$, $\text{dom} f^{**} \supseteq \text{dom} f$.
If there exists $x^o \notin \text{dom} f$ but $x^o \in \text{dom} f^{**}$, then $f^{**}(x) \leqslant f(x) = +\infty$.
It leads to contradiction, so $\text{dom} f^{**} \subseteq \text{dom} f$.
Hence, $\text{dom} f^{**} = \text{dom} f$, and $\sup_{y \in \text{dom} f^*} \{x^T y - f^*(y)\} = f^{**}(x)$.
Therefore, $f^{**}(x) \leqslant f(x), \forall x \in \text{dom} f = \text{dom} f^{**}$

Legendre transformation
Let f be a convex and differentiable function with $\mathrm{dom} f = \mathbb{R}^n$.
If $y^T x - f(x)$ be maximized at x^*, then $y = \bigtriangledown f(x^*) \Rightarrow f^*(y) = \nabla f(x^*)^T x^* - f(x^*)$

Quasiconvex functions
$f : \mathbb{R}^n \to \mathbb{R}$ with $\mathrm{dom} f$ is quasiconvex(unimodal) function
if its every sublevel set is convex, that is, $S_\alpha = \{x | f(x) \leqslant \alpha\}$ is convex, $\forall \alpha \in \mathbb{R}$.
f is quasiconcave function if $-f$ is quasiconvex.
f is quasilinear function if f is both quasiconvex and quasiconvex.