Design of Piezoelectric Active Structures

Lecture 3:

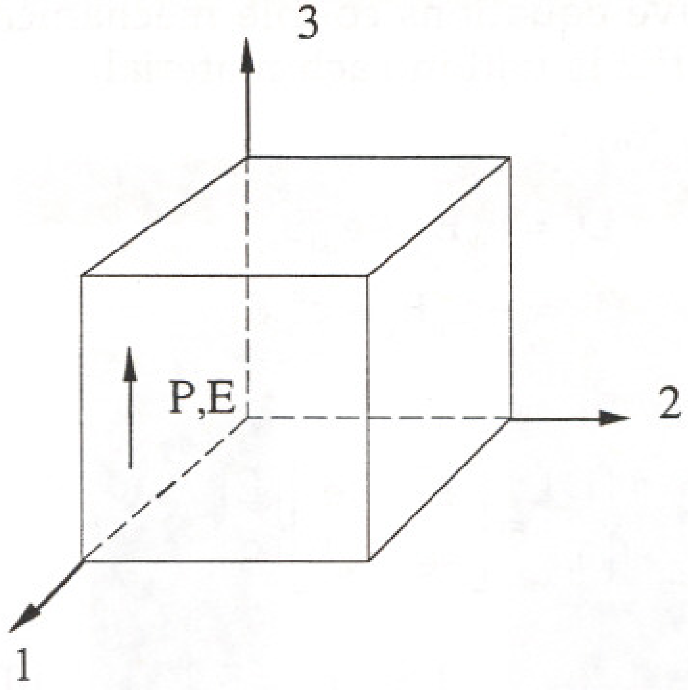
Linear Piezoelectric Properties

Outline

- Material Conventions
- Constitutive Properties for Common Materials
- Single Axis Cases and Coupling Coefficient
- Property Rotation
- Reduction to Plane Strain/Stress

Material Conventions

- IEEE Standard (Std-176-1978) defines properties relative to the poling axis:



- Material poled with high electric field to align dipoles in 3 direction

  Application of electric field in same polarity produces extension (primary actuation) in 3 direction

  Transverse piezoelectric effect produces shortening in 1 and 2 directions

- Poling produces anisotropy

  3 direction exhibits higher compliance, dielectric, and piezoelectricity

  Isotropy exists in plane defined by 1 and 2 directions

Linear Piezoelectricity

- Assumptions of small deformations allows linear forms of constitutive equations to be used
- Constitutive equations couple mechanical and electrical fields within each material

In tensor form:

$$\mathbf{D}_i = \varepsilon^S_{ik}\mathbf{E}_k + \mathbf{e}_{ikl}\mathbf{S}_{kl}$$

$$\mathbf{T}_{ij} = -\mathbf{e}_{kij}\mathbf{E}_k + \mathbf{c}^E_{ijkl}\mathbf{S}_{kl}$$

In matrix form:

$$\begin{Bmatrix} D \\ T \end{Bmatrix} = \begin{bmatrix} \varepsilon^S & e \\ -e_t & c^E \end{bmatrix} \begin{Bmatrix} E \\ S \end{Bmatrix}$$

$$\mathbf{D} = \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} \quad \mathbf{E} = \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix} \quad \mathbf{S} = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{13} \\ 2S_{12} \end{Bmatrix} = \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} \quad \mathbf{T} = \begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{Bmatrix} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix}$$

ε^S dielectric at constant strain condition (clamped)

c^E stiffness at constant electric field (shorted)

Superscript denotes property found at constant field for independent variable

Alternate Forms I

- Four field variables allows four alternate forms of constitutive equations, each with different pairs of independent and dependent fields:

- Form 1:

  In tensor form:

$$\mathbf{D}_i = \varepsilon^S_{ik}\mathbf{E}_k + \mathbf{e}_{ikl}\mathbf{S}_{kl}$$

$$\mathbf{T}_{ij} = -\mathbf{e}_{kij}\mathbf{E}_k + \mathbf{c}^E_{ijkl}\mathbf{S}_{kl}$$

  In matrix form:

$$\begin{Bmatrix} D \\ T \end{Bmatrix} = \begin{bmatrix} \varepsilon^S & e \\ -e_t & c^E \end{bmatrix} \begin{Bmatrix} E \\ S \end{Bmatrix}$$

- Form 2:

  In tensor form:

$$\mathbf{D}_i = \varepsilon^T_{ik}\mathbf{E}_k + d_{ikl}T_{kl}$$

$$S_{ij} = d_{kij}\mathbf{E}_k + s^E_{ijkl}T_{kl}$$

  In matrix form:

$$\begin{Bmatrix} D \\ S \end{Bmatrix} = \begin{bmatrix} \varepsilon^T & d \\ d_t & s^E \end{bmatrix} \begin{Bmatrix} E \\ T \end{Bmatrix}$$

Alternate Forms II

- Form 3:

  In tensor form:

$$E_i = \beta^T_{ik} D_k - g_{ikl} T_{kl}$$
$$S_{ij} = g_{kij} D_k + s^D_{ijkl} T_{kl}$$

  In matrix form:

$$\begin{Bmatrix} E \\ S \end{Bmatrix} = \begin{bmatrix} \beta^T & -g \\ g_t & s^D \end{bmatrix} \begin{Bmatrix} D \\ T \end{Bmatrix}$$

open circuit

- Form 4:

  In tensor form:

$$E_i = \beta^S_{ik} D_k - h_{ikl} S_{kl}$$
$$T_{ij} = -h_{kij} D_k + c^D_{ijkl} S_{kl}$$

  In matrix form:

$$\begin{Bmatrix} E \\ T \end{Bmatrix} = \begin{bmatrix} \beta^S & -h \\ -h_t & c^D \end{bmatrix} \begin{Bmatrix} D \\ S \end{Bmatrix}$$

open circuit

Description of Symbols

ε^{T}	free dielectric constant
ε^{S}	clamped dielectric constant
k_{33}	longitudinal coupling constant
k_{31}	transverse coupling constant
k_{15}	shear coupling constant
d_{ij}	piezoelectric constant (strain/field)
e_{ij}	piezoelectric constant (stress/field)
s^{E}	compliance at constant field (short circuit)
s^{D}	compliance at constant electrical displacement (open circuit)

Typical Properties

Type*	PZT-4	PZT-5A	PZT-5H
$\varepsilon^T_{33}/\varepsilon_o$	1300	1700	3400
$\varepsilon^S_{33}/\varepsilon_o$	635	830	1470
d_{33}	289	374	593
d_{31}	-123	-171	-274
d_{15}	496	584	741
s^E_{33}	15.5	18.8	20.7
s^D_{33}	7.90	9.46	8.99
s^E_{11} ($\times 10^{-12}$ m^2/N)	12.3	16.4	16.5
s^D_{11}	10.9	14.4	14.1
s^E_{12}	-4.05	-5.74	-4.78
s^D_{12}	-5.42	-7.71	-7.27
k_{33}	.700	.705	.752
k_{31}	-.334	-.344	-.388
k_{15}	.710	.685	.675

* Designations of Morgan-Matroc Piezoceramics

PZT 4 - NAVY I PZT 5A - NAVY II PZT 5H - NAVY V

Form Relationships

- Simple matrix algebra provides relationships between various material properties

<u>example 1</u>:

From form 2,

$$T = (s^E)^{-1}S - \overbrace{(s^E)^{-1}d_t}^{=e}E$$

Substitute into equation for D:

$$D = d(s^E)^{-1}S - d(s^E)^{-1}d_tE + \varepsilon^TE$$

Compare to form 1:

$$\underline{c^E = (s^E)^{-1}} \quad \overset{3\times6}{e} = d\overset{6\times6}{c^E} \quad \boxed{\varepsilon^S = \varepsilon^T - dc^Ed_t}$$

(Thus, clamped dielectric is smaller than free dielectric)

<u>example 2</u>:

From form 2 again,

$$E = (\varepsilon^T)^{-1}D - (\varepsilon^T)^{-1}dT$$

Substitute into equation for S:

$$S = s^ET + d_t(\varepsilon^T)^{-1}D - d_t(\varepsilon^T)^{-1}dT$$

Compare to form 3:

$$\beta^T = (\varepsilon^T)^{-1} \quad g = (\varepsilon^T)^{-1}d \quad \boxed{s^D = s^E - d_t(\varepsilon^T)^{-1}d}$$

(Thus, material is less compliant in open circuit)

Active Material Types I

- Various crystal system types are available as active materials:

Poled ferroelectrics: Hexagonal Crystal - Class 6mm

$$\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix} = \begin{bmatrix} s_{11}^E & s_{12}^E & s_{13}^E & 0 & 0 & 0 \\ s_{12}^E & s_{11}^E & s_{13}^E & 0 & 0 & 0 \\ s_{13}^E & s_{13}^E & s_{33}^E & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44}^E & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44}^E & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{66}^E \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} + \begin{bmatrix} 0 & 0 & d_{31} \\ 0 & 0 & d_{31} \\ 0 & 0 & d_{33} \\ 0 & d_{15} & 0 \\ d_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}$$

$$\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} + \begin{bmatrix} \varepsilon_{11}^T & 0 & 0 \\ 0 & \varepsilon_{11}^T & 0 \\ 0 & 0 & \varepsilon_{33}^T \end{bmatrix} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \end{Bmatrix}$$

Active Material Types II

Rochelle Salt: Orthorhombic Crystal - Class 2

$$
\begin{bmatrix}
s_{11}^E & s_{12}^E & s_{13}^E & 0 & s_{15}^E & 0 \\
s_{12}^E & s_{22}^E & s_{23}^E & 0 & s_{25}^E & 0 \\
s_{13}^E & s_{23}^E & s_{33}^E & 0 & s_{35}^E & 0 \\
0 & 0 & 0 & s_{44}^E & 0 & s_{46}^E \\
s_{15}^E & s_{25}^E & s_{35}^E & 0 & s_{55}^E & 0 \\
0 & 0 & 0 & s_{46}^E & 0 & s_{66}^E
\end{bmatrix}
\begin{bmatrix}
0 & d_{21} & 0 \\
0 & d_{22} & 0 \\
0 & d_{23} & 0 \\
d_{14} & 0 & d_{34} \\
0 & d_{25} & 0 \\
d_{16} & 0 & d_{36}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11}^T & 0 & \varepsilon_{13}^T \\
0 & \varepsilon_{22}^T & 0 \\
\varepsilon_{13}^T & 0 & \varepsilon_{33}^T
\end{bmatrix}
$$

Quartz: Trigonal Crystal - Class 3m

$$
\begin{bmatrix}
s_{11}^E & s_{12}^E & s_{13}^E & s_{14}^E & 0 & 0 \\
s_{12}^E & s_{11}^E & s_{13}^E & -s_{14}^E & 0 & 0 \\
s_{13}^E & s_{13}^E & s_{33}^E & 0 & 0 & 0 \\
s_{14}^E & -s_{14}^E & 0 & s_{44}^E & 0 & 0 \\
0 & 0 & 0 & 0 & s_{44}^E & 2s_{14}^E \\
0 & 0 & 0 & 0 & 2s_{14}^E & s_{66}^E
\end{bmatrix}
\begin{bmatrix}
d_{11} & 0 & 0 \\
-d_{11} & 0 & 0 \\
0 & 0 & 0 \\
d_{14} & 0 & 0 \\
0 & -d_{14} & 0 \\
0 & -2d_{11} & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
\varepsilon_{11}^T & 0 & 0 \\
0 & \varepsilon_{11}^T & 0 \\
0 & 0 & \varepsilon_{33}^T
\end{bmatrix}
$$

Modes of Operation I

- Various one dimensional modes of operation are possible depending on:

  Electric field direction
  Poling direction
  Application of loads

- Three most common modes of operation are:

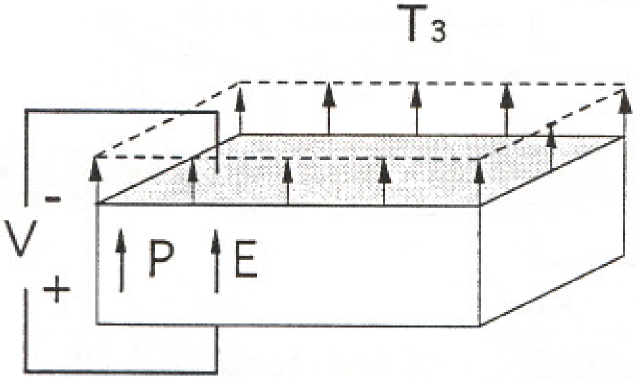
  Longitudinal Mode:
  Load and electric field applied along direction of poling

  Transverse Mode:
  Load applied transverse to poling direction, electric field applied along poling direction

  Shear Mode:
  Shear load and poling direction in plane of structure, electric field applied perpendicular to plane

Longitudinal Mode

- Longitudinal mode of operation utilizes primary actuation direction



- Mode of operation characterized by:
  - Electric field, E_3, applied along poling direction
  - Normal load T_3 also applied along poling direction

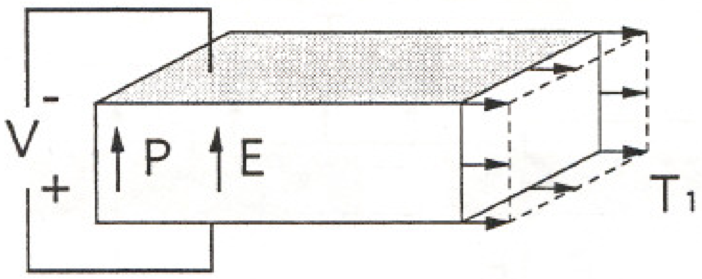
- With all other stresses zero, constitutive equations can be written as follows:

$$\begin{Bmatrix} S_3 \\ D_3 \end{Bmatrix} = \begin{bmatrix} s_{33}^E & d_{33} \\ d_{33} & \varepsilon_{33}^T \end{bmatrix} \begin{Bmatrix} T_3 \\ E_3 \end{Bmatrix} \qquad d_{33}^{PZT5H} \approx 593e^{-12} m/V$$

- Typical applications: extension motors, used in stacks for accurate positioning control

Transverse Mode

- Transverse mode utilizes transverse piezoelectric effects:



- Mode of operation characterized by:

  Electric field, E_3, applied along poling direction
  Normal load T_1 applied perpendicular to poling direction

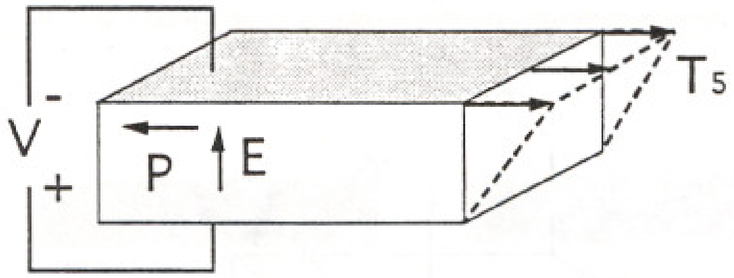
- With all other stresses zero, constitutive equations become:

$$\begin{Bmatrix} S_1 \\ D_3 \end{Bmatrix} = \begin{bmatrix} s_{11}^E & d_{31} \\ d_{31} & \varepsilon_{33}^T \end{bmatrix} \begin{Bmatrix} T_1 \\ E_3 \end{Bmatrix} \qquad d_{31}^{PZT5H} \approx -276e^{-12} m/V$$

- Typical applications: bonded on (or embedded in) structures for damping/vibration control

Shear Mode

- Shear mode of operation utilizes in-plane shear load and perpendicular dipole alignment



- Mode of operation characterized by:

  Electric field, E_3, applied perpendicular to poling direction

  Shear load T_5 applied perpendicular to poling direction

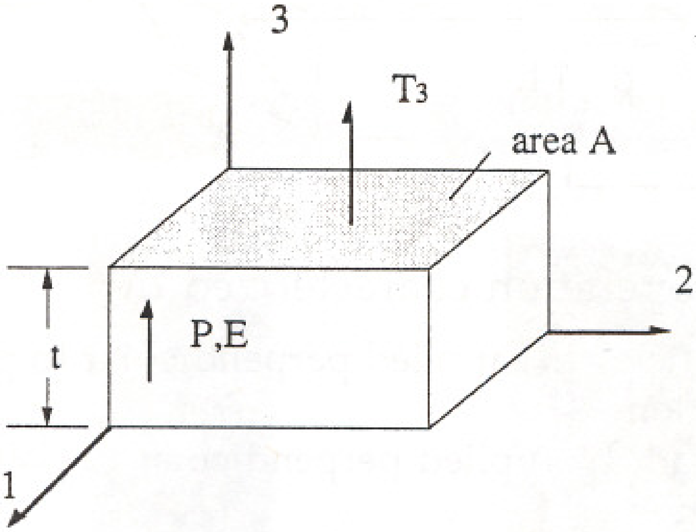
- With all other stresses zero, constitutive equations become:

$$\begin{Bmatrix} S_5 \\ D_1 \end{Bmatrix} = \begin{bmatrix} s_{44}^E & d_{15} \\ d_{15} & \varepsilon_{11}^T \end{bmatrix} \begin{Bmatrix} T_5 \\ E_1 \end{Bmatrix} \quad d_{15}^{PZT5H} \approx 741e^{-12} m/V$$

- Typical applications: shear mode motors, shear type accelerometers

Coupling Coefficient I

- Definition: ratio of electrical work which may be done, to the total energy stored from a mechanical source



- Consider following steps for longitudinal case:

  1. Under short circuit conditions (E=0), apply stress T_3. Element is free to expand in other directions so that $T_1 = T_2 = 0$. Then, mechanical energy stored per unit volume is:

$$U_m^E = \int T_3 \delta S_3 = \frac{1}{2} s_{33}^E T_3^2$$

Coupling Coefficient II

2. Open circuit element (D=0) and remove stress. For this process, electric field can be related to stress as:

$$E_3 = -\frac{T_3 d_{33}}{\varepsilon_{33}^T}$$ ← form II.

3. Apply electric field to return strain to original state. Energy delivered to the electrical load is:

$$U_e^T = \int D_3 \delta E_3 = \frac{1}{2}\varepsilon_{33}^T E_3^2$$

4. Finally, comparing electrical energy done to mechanical energy stored, replacing E_3 by the above relation gives:

$$\frac{U_e^T}{U_m^E} = k_{33}^2 = \frac{d_{33}^2}{\varepsilon_{33}^T s_{33}^E}$$

- Similar arguments may be used to develop coupling coefficients for other modes of operation:

transverse mode: $$k_{31}^2 = \frac{d_{31}^2}{\varepsilon_{33}^T s_{11}^E}$$

shear mode: $$k_{15}^2 = \frac{e_{15}^2}{\varepsilon_{11}^S c_{55}^D}$$ (?)

Modeling of Active Materials

- Active elements have been widely used as:

  Actuators: on adaptive structures such as beams, plates, and trusses
  Sensors: for modal and waveform control

- Applications require problem-specific manipulations to the material relations:

  Property Rotations:

  Material properties are all derived relative to the material poling direction, which may be different than the principal structural axes

  Rotations can be derived for transforming coupled material matrices to other axes

  Reduction to plane strain/stress problems:

  Geometry of certain problems allow simplification from three-dimensional states of stress and strain

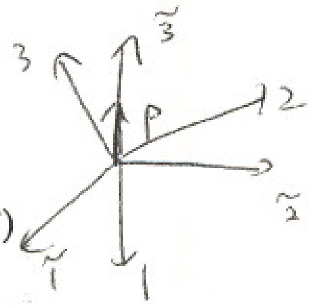
  Simplification requires choosing the correct forms of constitutive relations

Property Rotations I

- Assume fully anisotropic material in coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$:

$$\begin{Bmatrix} \tilde{D} \\ \tilde{T} \end{Bmatrix} = \begin{bmatrix} \tilde{\varepsilon}^S & \tilde{e} \\ -\tilde{e}_t & \tilde{c}^E \end{bmatrix} \begin{Bmatrix} \tilde{E} \\ \tilde{S} \end{Bmatrix}$$

- Goal is to find effective material properties in axes (x, y, z). Rotations may be found through tensor transformations: cosine between m & $\tilde{p}$



$$a_m = l_{m\tilde{p}} \tilde{a}_p \qquad \text{(First order, ex: D, E)}$$

$$a_{mn} = l_{m\tilde{p}} l_{n\tilde{q}} \tilde{a}_{pq} \qquad \text{(Second order, ex: S, T)}$$

- Transformations can also be represented in matrix form, such that rotations between variables can be written as:

$$\tilde{D} = \mathbf{F}D \quad \tilde{E} = \mathbf{F}E \quad \tilde{S} = \mathbf{B}S \quad \tilde{T} = AT$$

- Using these relations in the constitutive equations above yields:

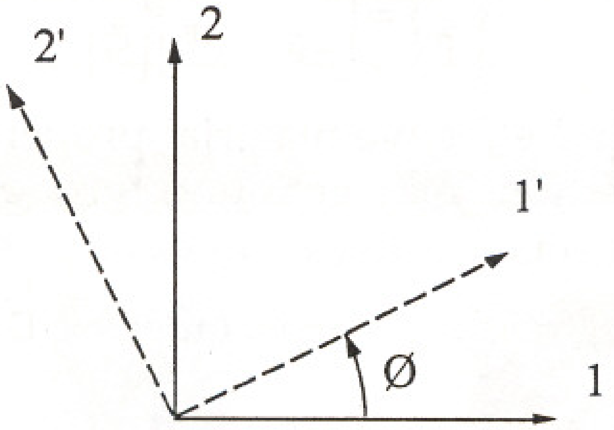
$$\begin{bmatrix} F & 0 \\ 0 & A \end{bmatrix} \begin{Bmatrix} D \\ T \end{Bmatrix} = \begin{bmatrix} \tilde{\varepsilon}^S & \tilde{e} \\ -\tilde{e}_t & \tilde{c}^E \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & B \end{bmatrix} \begin{Bmatrix} E \\ S \end{Bmatrix}$$

- Thus, the rotated properties are:

$$\begin{bmatrix} \varepsilon^S & e \\ -e_t & c^E \end{bmatrix} = \begin{bmatrix} F^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} \tilde{\varepsilon}^S & \tilde{e} \\ -\tilde{e}_t & \tilde{c}^E \end{bmatrix} \begin{bmatrix} F & 0 \\ 0 & B \end{bmatrix}$$

Property Rotations II

- For the specialized case of two dimensional rotation in-plane:



- Rotations of material properties in the plane about the 3 (z) direction can be achieved through the 1st and 2nd order transformation matrices:

$$\tilde{D} = \mathbf{R}_E D \quad \tilde{E} = \mathbf{R}_E E \quad \tilde{S} = \mathbf{R}_S S \quad \tilde{T} = (\mathbf{R}_{S_t})^{-1} T$$

transpose.

$$\mathbf{R}_S = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -2cs & 2cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix} \qquad \mathbf{R}_E = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

strain

where: $c = \cos\phi \quad s = \sin\phi$

Property Rotations III

- As before, rotate constitutive relations given in material axes:

$$\begin{Bmatrix}\tilde{D}\\ \tilde{T}\end{Bmatrix}=\begin{bmatrix}\tilde{\varepsilon}^S & \tilde{e}\\ -\tilde{e}_t & \tilde{c}^E\end{bmatrix}\begin{Bmatrix}\tilde{E}\\ \tilde{S}\end{Bmatrix}$$

- From first row equation:

$$\tilde{D}=\tilde{\varepsilon}^S\tilde{E}+\tilde{e}\tilde{S} \quad\rightarrow\quad \mathbf{R}_E D=\tilde{\varepsilon}^S\mathbf{R}_E E+\tilde{e}\mathbf{R}_S S$$

$$D=\mathbf{R}_{E_t}\tilde{\varepsilon}^S\mathbf{R}_E E+\mathbf{R}_{E_t}\tilde{e}\mathbf{R}_S S$$

- From second row equation:

$$\tilde{T}=-\tilde{e}_t\tilde{E}+\tilde{c}^E\tilde{S} \quad\rightarrow\quad (\mathbf{R}_{S_t})^{-1}T=-\tilde{e}_t\mathbf{R}_E E+\tilde{c}^E\mathbf{R}_S S$$

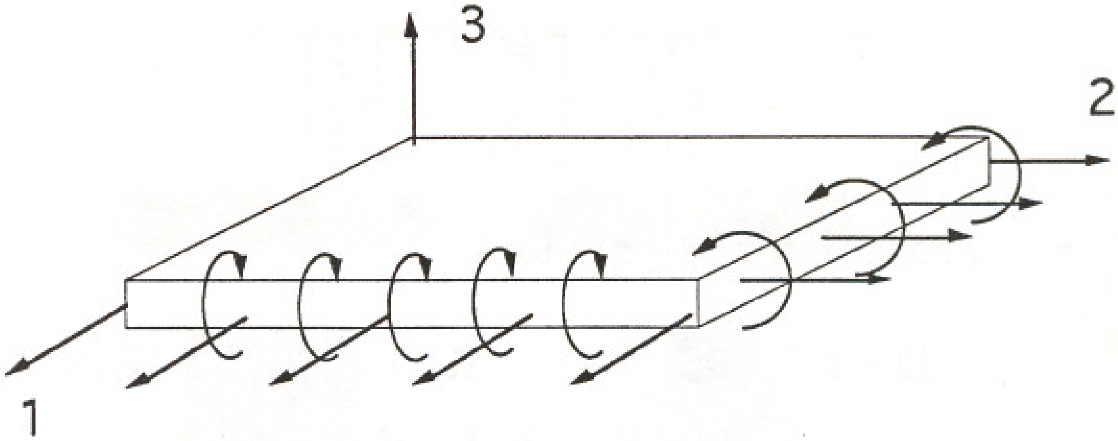
$$T=-\mathbf{R}_{S_t}\tilde{e}_t\mathbf{R}_E E+\mathbf{R}_{S_t}\tilde{c}^E\mathbf{R}_S S$$

- Finally, rotated properties are expressed in global axes:

$$\begin{Bmatrix}D\\ T\end{Bmatrix}=\begin{bmatrix}\mathbf{R}_{E_t}\tilde{\varepsilon}^S\mathbf{R}_E & \mathbf{R}_{E_t}\tilde{e}\mathbf{R}_S\\ -\mathbf{R}_{S_t}\tilde{e}_t\mathbf{R}_E & \mathbf{R}_{S_t}\tilde{c}^E\mathbf{R}_S\end{bmatrix}\begin{Bmatrix}E\\ S\end{Bmatrix}$$

Plane Stress Problems

- Plane stress assumptions may be made for many typical structures: plates, shells, beams



- Assumptions : $T_3 = T_4 = T_5 = 0$.

  Material poled in 3 direction
  Thickness of structure small
  Stresses through thickness small: $T_3 << T_1, T_2$
  Ignore shear stresses T_4, T_5; shear strains S_4, S_5 then must also be zero (shear problem uncoupled)

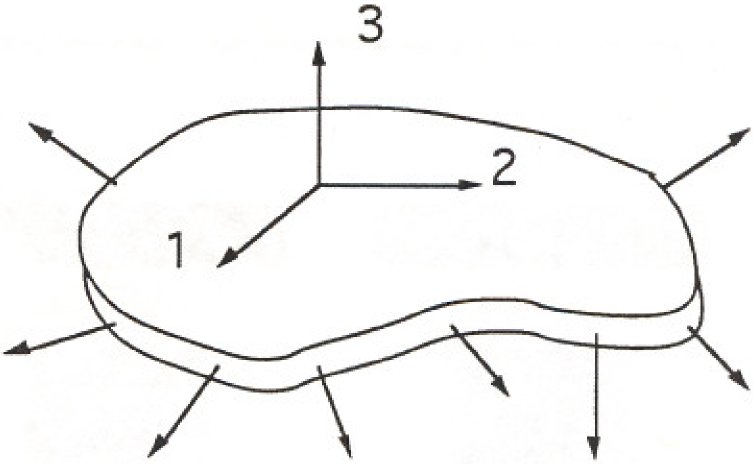
- Constitutive equations must be in form with stress as independent variable for reduction to plane problem:

$$\begin{Bmatrix} D_3 \\ S_1 \\ S_2 \\ S_6 \end{Bmatrix} = \begin{array}{c} \begin{array}{cccc} \phantom{\varepsilon^T_{33}} & \phantom{d_{31}} & d_{32} & \end{array} \\ \begin{bmatrix} \varepsilon^T_{33} & d_{31} & d_{31} & 0 \\ d_{31} & s^E_{11} & s^E_{12} & 0 \\ d_{31} & s^E_{12} & s^E_{11} & 0 \\ 0 & 0 & 0 & s^E_{66} \end{bmatrix} \end{array} \begin{Bmatrix} E_3 \\ T_1 \\ T_2 \\ T_6 \end{Bmatrix}$$

* Note that inverted properties will not have one-to-one correspondence to 3 dimensional properties

Plane Strain Problems

- Plane strain problems may be used to model rods of infinite length, or finite length with fixed ends



- Assumptions: $S_3 = S_4 = S_5 = 0$.

  Material poled along 3 direction
  Material infinite along 3 direction, with no variation of applied loads or properties along this direction
  Strains S_3, S_4, S_5 zero, thus corresponding stresses T_4, T_5 also zero (Stress T_3 may be solved for separately after initial problem solved)

- Constitutive equations must be in form with strain as independent variable for reduction to plane problem:

$$\begin{Bmatrix} D_3 \\ T_1 \\ T_2 \\ T_6 \end{Bmatrix} = \begin{bmatrix} \varepsilon_3^S & e_{31} & e_{31} & 0 \\ -e_{31} & c_{11}^E & c_{12}^E & 0 \\ -e_{31} & c_{12}^E & c_{11}^E & 0 \\ 0 & 0 & 0 & c_{66}^E \end{bmatrix} \begin{Bmatrix} E_3 \\ S_1 \\ S_2 \\ S_6 \end{Bmatrix}$$