Design of Piezoelectric Active Structures

Lecture 4:

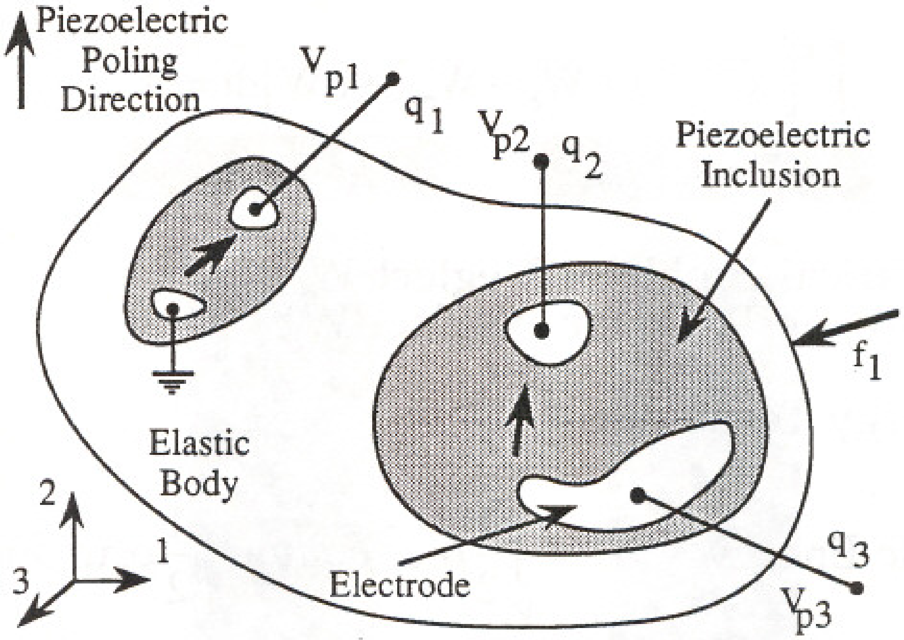
Coupled Equations of Motion of Active Structures

Outline

- Energy Methods for Coupled Electromechanical Systems
- Exact Modelling: Differential Equations
- Ritz-Galerkin Approximate Modelling
- Finite Elements Approximate Modelling

Problem Statement: Electroelastic Continuum

- General electroelastic body:



- Actuation Problem: Find Structure Response to Applied Voltage or Charge
- Sensing Problem: Find Electrode Charge or Voltage resulting form Structural Motion
- Will consider first a general framework for determining the equations of motion.
- Allow arbitrary piezoelectric electroding as well as poling directions which vary within the piezoelectric.

Generalized Hamilton's Principle

- Generalized Hamilton's Principle for coupled electromechanical systems

  Ref. Tiersten's (Electrical Enthalpy) or Crandal

$$\int_{t_1}^{t_2}\left[\partial(\mathrm{T}-\mathrm{U}+\mathrm{W_e}-\mathrm{W_m})+\partial\mathrm{W}\right]dt=0$$

- Electrical Problem: Neglect W_m
  Magnetic Problem: Neglect W_e

- Energy Terms are:

Kinetic Energy: $\delta \mathrm{T}=\int_{V_s}\frac{1}{2}\rho_s\dot{\mathbf{u}}^T\delta\dot{\mathbf{u}}\,dv+\int_{V_p}\frac{1}{2}\rho_p\dot{\mathbf{u}}^T\delta\dot{\mathbf{u}}\,dv$

Strain Energy: $\delta \mathrm{U}=\int_{V_s}\frac{1}{2}\delta\mathbf{S}^T\mathbf{T}\,dv+\int_{V_p}\frac{1}{2}\delta\mathbf{S}^T\mathbf{T}\,dv$

Electrical Enegy: $\delta \mathrm{W_e}=\int_{V}\frac{1}{2}\delta\mathbf{E}^T\mathbf{D}\,dv$

Magntic Energy: $\delta \mathrm{W_m}=\int_{V}\frac{1}{2}\delta\mathbf{B}^T\mathbf{H}\,dv$

S = Strain Vector, T = Stress Vector, etc.

Work Terms

- The work terms for point and distributed forces and applied external charges (ignoring magnetic terms):

$$\delta W = \sum_{i=1}^{nf} \partial \mathbf{u}(\mathbf{x}_i) \cdot \mathbf{f}(\mathbf{x}_i)$$

$$+\int_s \delta \mathbf{u} \cdot \mathbf{f}^s \, ds$$

$$+\int_v \delta \mathbf{u} \cdot \mathbf{f}^v \, dv$$

$$-\sum_{j=1}^{nq} \partial \varphi_j \cdot q_j$$

- Note that we have considered φ and u as the variable fields. There are four other possible variational principles using different choices.

Constitutive Relations

- The structural constitutive relations have no coupling between the electrical and mechanical terms.

$$\begin{bmatrix} D' \\ T' \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} E' \\ S' \end{bmatrix}$$

- The piezoelectric constitutive relations couple the electrical and mechanical terms.

$$\begin{bmatrix} D' \\ T' \end{bmatrix} = \begin{bmatrix} \varepsilon^{S} & e \\ -e_t & c^{E} \end{bmatrix} \begin{bmatrix} E' \\ S' \end{bmatrix}$$

$$\begin{bmatrix} D'_1 \\ D'_2 \\ D'_3 \\ T'_1 \\ T'_2 \\ T'_3 \\ T'_4 \\ T'_5 \\ T'_6 \end{bmatrix} = \left[\begin{array}{ccc:cccccc} \varepsilon_1^{S} & 0 & 0 & 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & \varepsilon_1^{S} & 0 & 0 & 0 & 0 & e_{15} & 0 & 0 \\ 0 & 0 & \varepsilon_3^{S} & e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \\ \hdashline 0 & 0 & -e_{31} & c_{11}^{E} & c_{12}^{E} & c_{13}^{E} & 0 & 0 & 0 \\ 0 & 0 & -e_{31} & c_{12}^{E} & c_{11}^{E} & c_{13}^{E} & 0 & 0 & 0 \\ 0 & 0 & -e_{33} & c_{13}^{E} & c_{13}^{E} & c_{33}^{E} & 0 & 0 & 0 \\ 0 & -e_{15} & 0 & 0 & 0 & 0 & c_{55}^{E} & 0 & 0 \\ -e_{15} & 0 & 0 & 0 & 0 & 0 & 0 & c_{55}^{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{66}^{E} \end{array}\right] \begin{bmatrix} E'_1 \\ E'_1 \\ E'_1 \\ S'_1 \\ S'_2 \\ S'_3 \\ S'_4 \\ S'_5 \\ S'_6 \end{bmatrix}$$

- Use this form because pof choice of independent fields.

Arbitrary Poling Directions

- The piezoelectric material properties are defined relative to the material local poling direction and must be rotated into the global coordinate frame.

$$\mathbf{S}' = \mathbf{R}_S(\mathbf{x},\mathbf{p})\mathbf{S} \quad and \quad \mathbf{E}' = \mathbf{R}_E(\mathbf{x},\mathbf{p})\mathbf{E}$$

$$\mathbf{T}' = \mathbf{R}_T(\mathbf{x},\mathbf{p})\mathbf{T}$$

- $\mathbf{R}_E$ is a matrix of direction cosines (orthonormal), $\mathbf{R}_S$ is the engineering strain rotation matrix and $\mathbf{R}_T$ is the engineering stress rotation matrix given from tensor relations.

$$\text{if} \quad R_T = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \text{ then } R_S = \begin{bmatrix} R_{11} & R_{12}/2 \\ 2R_{21} & R_{22} \end{bmatrix}$$

- Given these rotations the piezoelectric properties can be written:

$$\begin{bmatrix} \mathbf{D} \\ \mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{R}_E^T \varepsilon^S \mathbf{R}_E & \mathbf{R}_E^T \mathbf{e} \mathbf{R}_S \\ -\mathbf{R}_T^{-1} \mathbf{e}_t \mathbf{R}_E & \mathbf{R}_T^{-1} \mathbf{c}^E \mathbf{R}_S \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{S} \end{bmatrix}$$

- Usefull in 3-D electroelasticity applications. Rotation matrices can be functions of position.

Strain Displacement Relations

- The strain and electrical fields are derived from the displacement and potential with differential operators.

$$\mathbf{S} = \mathbf{L}_u \mathbf{u}(x) \quad \text{and} \quad \mathbf{L}_\varphi \varphi(x) = -\nabla \cdot \varphi(x)$$

- The mechanical differential operator can be standard or chosen for the particular problem. For the general 3-d case:

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Further Examples of Operators

- Sometimes only the displacements at a representative location (eg, centerline) are chosen to characerize structural motion.
- For a Bernoulli Euler Beam:

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & -y\frac{\partial}{\partial x^2} & -z\frac{\partial}{\partial x^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

For example, for Classical Laminated Plates:

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & -z\frac{\partial}{\partial x^2} \\ 0 & \frac{\partial}{\partial y} & -z\frac{\partial}{\partial y^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & -2z\frac{\partial}{\partial xy} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Options for Modelling

- Spatiallly Continuous Description
- Spatial Discrete Representations: Approximate Solution Techniques:

Rayleigh-Ritz - deformation shape assumed throughout domain.

Finite Elements - deformation shape assumed within a finite element.

Rayleigh Ritz Modelling: Assumed Mechanical and Electrical Modes

- The displacement and potential mode shapes can be expressed in terms of generalized coordinates:

$$\mathbf{u}(\mathbf{x}, t) = \Psi_r(\mathbf{x})\mathbf{r}(t) = \begin{bmatrix} \psi_{r_1}(\mathbf{x}) & \cdots & \psi_{r_n}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{bmatrix}$$

$$\varphi(\mathbf{x}, t) = \Psi_v(\mathbf{x})v(t) = \begin{bmatrix} \psi_{v_1}(\mathbf{x}) & \cdots & \psi_{v_m}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} v_1(t) \\ \vdots \\ v_m(t) \end{bmatrix}$$

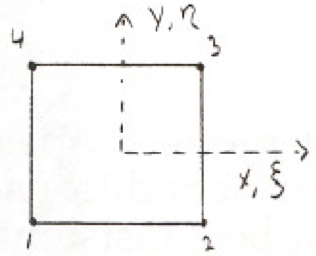
- The displacement mode shapes must obey the geometric boundary conditions. The potential distributions must be consistent with the voltage boundary conditions and equipotential at conductors.

Piezoelectric Finite Elements

- FE Analysis is a speicial form of the Rayleigh-Ritz Analysis
- For Finite Element applications, use nodal mechanical displacements and nodal values of the electric potential (volts) as the generalized coordinates.

  NOTE: pick up new nodal dof - volt (or potential, φ)
- Use standard interpolation functions as the assumed mechanical and electrical shapes.

  2-D example



Assuming linear interpolation

$$[u\,(x,y)\;\; v\,(x,y)\;\; \varphi\,(x,y)] = \frac{1}{4}\sum_{1=1}^{4}(1-\xi_1\xi)\,(1+\eta_1\eta)\,[u_i\;\; v_i\;\; \varphi_i]$$

which can be easily represented in standard form

$$\mathbf{u}\,(\mathbf{x},t) = \boldsymbol{\psi}_r(\mathbf{x})\,\mathbf{r}(t) \qquad \text{where} \qquad \mathbf{r} = \begin{bmatrix} u_1 \\ v_1 \\ \cdot \\ \cdot \\ u_4 \\ v_4 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} \varphi_1 \\ \cdot \\ \cdot \\ \varphi_4 \end{bmatrix}$$

$$\varphi\,(\mathbf{x},t) = \boldsymbol{\psi}_v\,(\mathbf{x})\,\mathbf{v}(t)$$

Combining Shape with Differential Operators

- Combining the shapes with the differential operators gives:

$$\mathbf{S}(x,t) = \mathbf{N}_r(x)\,\mathbf{r}(t) \qquad \mathbf{N}_r(x) = \mathbf{L}_u \mathbf{\Psi}_r(x)$$
$$\mathbf{E}(x,t) = \mathbf{N}_v(x)\,\mathbf{u}(t) \qquad \mathbf{N}_v(x) = \mathbf{L}_\varphi \mathbf{\Psi}_r(x)$$

Coupled Equations of Motion

- Taking the variation of Hamilton's Principle gives two coupled equations of motion.

$$(\mathbf{M}_s + \mathbf{M}_p)\mathbf{r} + (\mathbf{K}_s + \mathbf{K}_p)\,\mathbf{r} - \theta\mathbf{v} = \mathbf{B}_f\,\mathbf{f} \text{ Actuator Equation}$$

$$\theta^T\mathbf{r} + (\mathbf{C}_s + \mathbf{C}_p)\,\mathbf{v} = \mathbf{B}_q\,\mathbf{q} \text{ Sensor Equation}$$

- The mass and stiffness terms or the structure and piezoelectric:

$$\mathbf{M}_{s,p} = \int_{v_s, v_p} \psi_r^T \rho_{s,p}\, \psi_r\, dv \qquad \mathbf{K}_{s,p} = \int_{v_s, v_p} \mathbf{N}_r^T \mathbf{c}_{s,p}\, \mathbf{N}_r\, dv$$

- The electromechanical coupling matrix, Θ, and capacitance, C_p, are functions of the piezoelectric material properties and the assumed mode shapes.

$$\mathrm{C}_{s,p} = \int_{v_s, v_p} \mathrm{N}_v^T\, \varepsilon_{s,p}^S\, \mathrm{N}_v\, dv \qquad \theta = \int_{v_p} \mathrm{N}_r^T\, e_{s,p}^E \mathrm{N}_v\, dv$$

- The forcing terms due to applied external force or charge.

$$\mathrm{B}_f = \begin{bmatrix} \Psi_{r_1}^T\,(x_{f_i}) & \cdot & \cdot & \cdot & \Psi_{r_1}^T\,(x_{f_n}) \\ \Psi_{r_n}^T\,(x_{f_i}) & \cdot & \cdot & \cdot & \Psi_{r_n}^T\,(x_{f_n}) \end{bmatrix}$$

$$\mathrm{B}_q = \begin{bmatrix} \Psi_{v_1}\,(q_1) & \cdot & \cdot & \cdot & \Psi_{v_1}\,(q_m) \\ \Psi_{v_m}\,(q_1) & \cdot & \cdot & \cdot & \Psi_{v_m}\,(q_m) \end{bmatrix}$$

PiezoStructural Theory

$$\begin{Bmatrix} F \\ Q \end{Bmatrix} = \begin{bmatrix} K_{uu} & K_{uv} \\ -K_{uv}^T & K_{vv} \end{bmatrix} \begin{Bmatrix} U \\ V \end{Bmatrix}$$

where:

F = Applied Forces at Structural DOFs

Q = Applied Charges on Active Elements (Sensors & Actuators)

U = Displacements of Structural Model DOFs

V = Electric Potentials (Voltages) on Active Elements

K_{uu} = Elastic Stiffness Matrix,
Active Elements Shorted (V=0)

K_{vv} = Diagonal Matrix of Sensor Element Capacitances,
Blocked Structure (U=0)

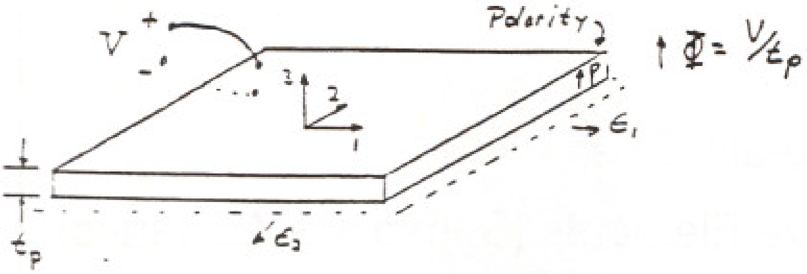
K_{uv} = Piezoelectric Load Matrix, i.e. Forces due to Unit Actuator Voltages, Blocked Structure (U=0)

$-K_{uv}^T$ = Piezoelectric Charge Matrix, i.e. Charge Migration on Sensors Generated by Displacements,
Active Elements Shorted (V=0)

Thermal Analogy

To implement PiezoStructural Analysis in a standard Finite Element code, employ a simple Thermal Analogy.

Equate the piezoelectric strain generated by an electric field Φ applied to a PZT wafer of thickness t_p, to the thermal strain generated by an equivalent temperature change, ΔT.



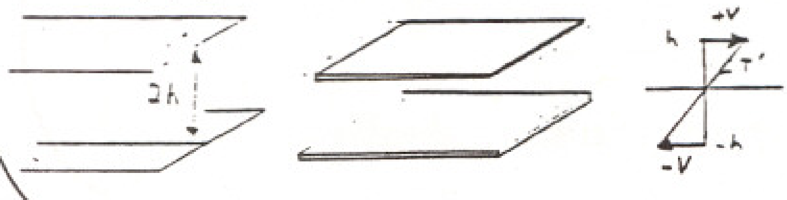
Piezoelectric Strain: $$\varepsilon_1^{piezo} = d_{31}\Phi = d_{31}\frac{V}{t_p}$$

Thermal Strain: $$\varepsilon_1^{thermal} = \alpha\,\Delta T$$

Equivalent CTE: $$\boxed{\alpha_{eq} \equiv \frac{d_{31}}{t_p}}$$

Equivalent Temperature: $$\boxed{\Delta T_{eq} \equiv V}$$

To Actute in Bending Employ the Correct PZT Inertia and Apply a Thermal Gradient



$$\boxed{\Delta T'_{eq} \equiv V/h}$$

Actuation

- To Actuate Apply Temperatures to the Actuator Elements and Recover Blocked Thermal Reaction Loads from the Finite Element Model

- The Resultant Matrix gives Piezoelectric Loads on the Model due to Applied Actuator Voltages V_a

$$\{F^{piezo}\} = [K_{uv}]\{V_{actuator}\}$$

Sensing

To generate the sensing matrix

- To Sense Apply Temperatures to the Sensor Elements and Recover Blocked Thermal Reaction Forces.
- Transpose the Resultant Matrix, and Premultiply by the Inverse of the Capacitance to Convert Sensed Charge to Sensed Voltage.
- Obtain Sensed Voltage in Terms of Structural Displacements.

$$\boxed{V_s = K_{vv}^{-1} K_{uv}^{T} U}$$