Design of Piezoelectric Active Structures

Lecture 2:

Fundamentals of Elasticity and Electromagnetism

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Outline

- Review of Elasticity
- Review of Electrostatics
- Review of Magnetostatics

Mechanical Fields in Matter

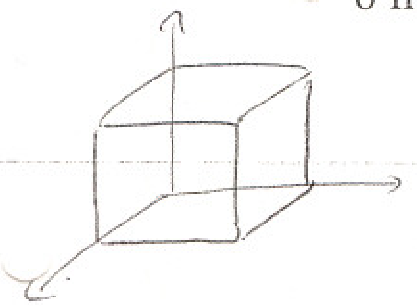
- 2 principle fields: S, strain field (tensor)

  T, stress field

- Both S and T are second order tensors

  $S_{ij} \quad \underbrace{T_{ij}}_{\text{face} \; \text{direction}} \quad i, j = 1, 2, 3$

  - 6 independent in each. (Voight Notation)

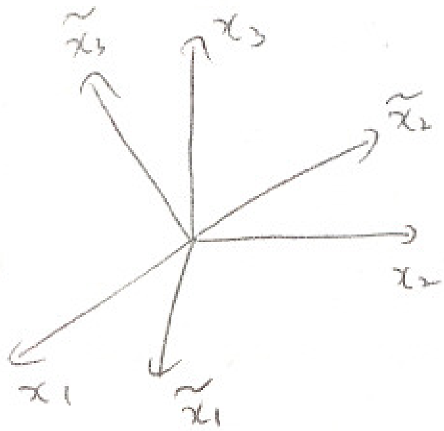


$$\underbrace{\vec{S} = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{31} \\ 2S_{21} \end{bmatrix}}_{\text{engineering strain}} \qquad \vec{T} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{21} \end{bmatrix}$$

- Forms of the equations are parallel in mechanical electrical and magnetic systems

  Newton, Coulomb, Ampere

· Tensor transformation



$= u_1 l_{\tilde{m}1} + u_2 l_{\tilde{m}2} + u_3 l_{\tilde{m}3}$

· $\tilde{u}_m = u_r l_{\tilde{m}r}$ (summation rule)

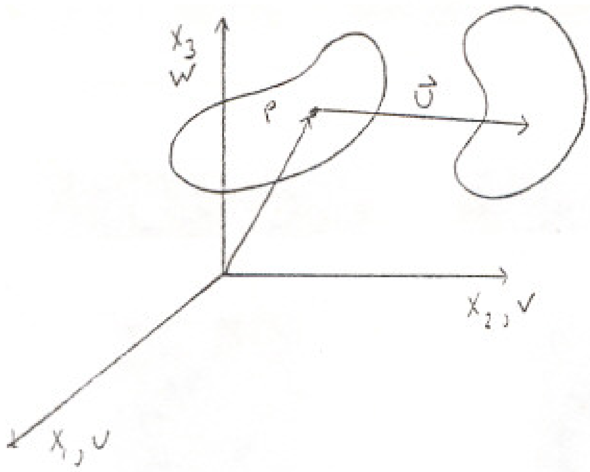
$l_{\tilde{m}r}$ = cosine of angle between r & $\tilde{m}$.

· $\tilde{S}_{ij} = S_{rs} l_{r\tilde{i}} l_{s\tilde{j}}$

or $\underset{6\times1}{\vec{\tilde{S}}} = \underset{6\times6}{[\quad]} \underset{6\times1}{\vec{S}}$

Displacement Field

- The displacement of a point, p, of a body is represented by $\vec{u}\,(x_1, x_2, x_3, t)$



- $\vec{u}$ is the single valued continuous function over the body with components.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

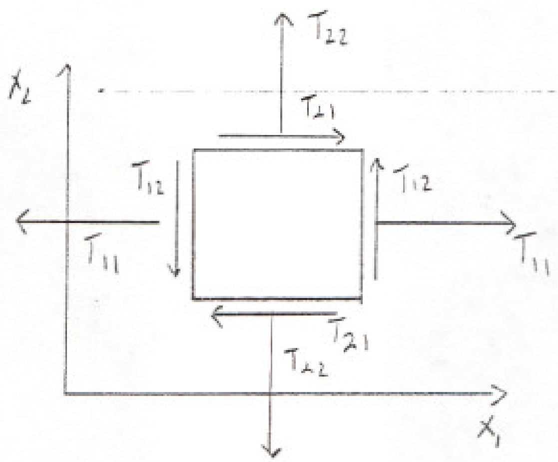
Newton's Law

- Forces on a particle or body (counting inertia) must sum to zero

$$\sum F = ma$$

- Within a body, the forces acting on the faces of a differential element are given by the stress tensor, T_{ij}

$$T_{ij} = \frac{\text{Force}}{\text{Area}} \text{ on ith face in jth direction}$$



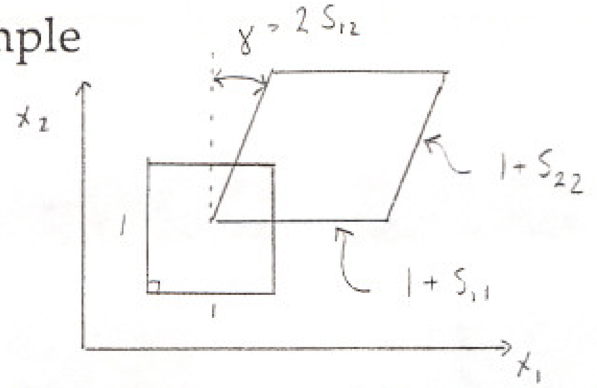
- Newton's Law can be applied to a differential element to yield the equilibrium equations for a body in differential form.

- ✓ Note T not necessarily symmetric if there are body moments.

Strain Field

- The relative deformation of a cell within the body is represented by the strain tensor.

  2-D example

  S_{ij} = strain tensor

- <u>Shear Strains</u> - represent 1/2 angle of deviation from 90°

  Engineering shear strains $\gamma_{ij} = 2S_{ij}$

- <u>Normal Strains</u>: relative side length change

- <u>Strain Displacement Relations</u>:

$$S_{ij} = \frac{1}{2}\left[\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_i}\right] = S_{ji}$$

  Symetric

- <u>Compatibility</u>: 3 displacements - 6 independent strains. Given S_{ij}, there are conditions which S_{ij} must satisfy for existence of single valued U_j

Equilibrium

- Differential Form: (Equations of Motion)

$$\boxed{\frac{\delta T_{ij}}{\delta X_j} + f_i = \rho a_i \quad \text{(3 equns)}}$$

  - applies to each particle in a body

- Integral Form - principle of virtual work (PVW)
  - ignoring $\vec{a}$ for now (If $\vec{a}$ is NOT ignored, → Hamilton's principle)

$$\int_v \left\{ \left[\frac{\delta T_{ij}}{\delta X_j} + f_i \right] \cdot \delta \vec{u} \right\} dv = 0$$

  - using calculus of variations can get to

$$\underbrace{\int_v T\delta S \; dv}_{du} = \int_V \underbrace{(\vec{f} \, . \, \delta u)\, dv + \int_s (\vec{f}_s \, . \, \delta u)\, ds}_{dw}$$

Principle of Minimum Potential Energy.

$$\text{where } T = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{13} \\ T_{12} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix}, \; S = \begin{bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ 2S_{23} \\ 2S_{13} \\ 2S_{12} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} \text{ Voight Notation}$$

Constitutive Relations

- To solve for the bodies deformation you need the relationship between stress and strain in the body. (in matrix notation)

compliance, 6×6

$$T = cS \qquad S = sT$$

c stiffness, 6×6

- Actually, the Elasticity Tensor relates second order strain tensor to 2nd order stress tensor

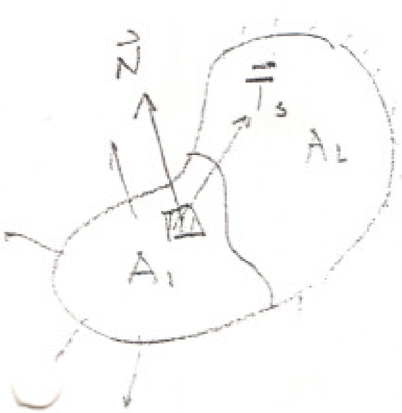
compliance tensor

$$T_{ij} = E_{ijmn}\, S_{mn} \qquad S_{ij} = S_{ijmp}\, T_{mp}$$

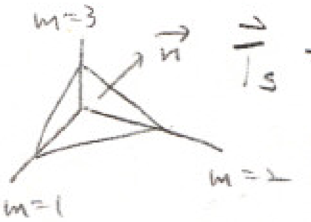
- Materials classified by the number of independent constants (forms of C & S matrices) needed to classify them.

  anisotropic - most general - no symmetries
  orthotropic - 3 orthogonal eyes – 9 independent const
  transversely isotropic - 1 plane of symmetry
  isotropic - 2 constants from E, ν, G

Boundary Conditions



- on A_1 - prescribed stress



$$\vec{T}_s = T_{sn}\hat{i}_n = \left[T_{mn}\cos(nm)\right]\hat{i}_n$$
$$= \hat{i}_1(T_{11}\cos N1 + T_{21}\cos N2 + \dots)$$
$$+ \hat{i}_2(+ \dots)$$

- on A_2 - prescribed displacement

$$u_n\hat{i}_n = U_n\hat{i}_n$$

Electric Fields in Matter

- Will investigate the relevant characteristics of electric fields in matter
- 2 principle fields, E, electric field and D, electrical displacement
- Both E and D are first order tensors, (vector fields).

$$E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \; D = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}$$

- Forms of equations are parallel to mechanical systems and magnetic systems.

Newton, Coulomb, Ampere

The Electric Field - I

- The Electric field is defined in terms of a force produced on a test charge

$$E = \lim_{q \to 0} \frac{F}{q} \qquad F\ (N),\ q\ (\text{Coulombs})$$

$$= \frac{\text{volts}}{\text{m}}$$

- Coulomb's Law (experimentally derived)
  - force between 2 point charges

$$F_2 = \frac{q_1 q_2 \vec{r}}{\underbrace{4\pi\varepsilon_0 r^3}_{\nabla\frac{1}{r}}} \qquad \varepsilon_0 = \frac{10^7}{4\pi C^2} = \frac{\text{farads}}{\text{meter}}$$

$$\frac{\vec{r}}{|\vec{r}|^3} = \nabla\left(\frac{1}{|r|}\right)$$

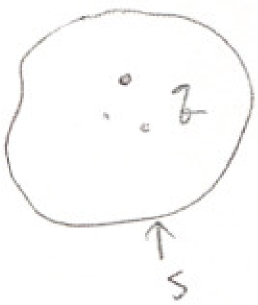
- Field of a point charge

$$\vec{E} = \frac{-q}{4\pi\varepsilon_0} \nabla\left(\frac{1}{r}\right) \qquad \vec{E} = \frac{-q_1}{4\pi\varepsilon_0} \nabla\left(\frac{1}{r}\right)$$

  - all other electrostatic properties can be derived from supperposition.

- Gauss's Flux Theorem (Integral Form)

$$\boxed{\int_S \vec{E} \cdot ds = \frac{q}{\varepsilon_0}}$$



The Electric Field - II

- Differential Forms:

divergence theorem. charge density.

$$\int_s \vec{E} \cdot ds = \int_v \nabla \cdot \vec{E}\, dv = \frac{q}{\varepsilon_0} = \int_v \frac{\rho}{\varepsilon_0} dv$$

$$\nabla \cdot \vec{E} = \frac{dE_1}{dx_1} + \frac{dE_2}{dx_2} + \frac{dE_3}{dx_3}$$

therefore

(*)

$$\boxed{\begin{aligned} \nabla \cdot E &= \frac{\rho}{\varepsilon_0} \\ \nabla \times E &= 0 \end{aligned}}$$

← from vector properties, Irrotational

- Irrotation implies

$\because E = \square\, \nabla(\frac{1}{r}) \to \nabla \times E = 0$

$$\oint \vec{E} \cdot d\vec{\ell} = 0$$

- no net work on charge traveling around a closed path.

conservative field.

Electric Potential

- Since $\vec{E}$ s irrotational it has a potential function

$$\vec{E} = -\nabla\varphi$$

gradient operator

scaler

$$E_i = \frac{-\delta\varphi}{\delta x_i}$$

- From this we get

$$\nabla^2\varphi = \frac{-\rho}{\varepsilon_0} \quad \text{Poisson's Eq.}$$

- <u>Potential of a Charge Distribution</u>

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho(r')}{|r-r'|}\,dv'$$

for a point charge;

$$\varphi(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

- <u>General property of vector field</u>

$$\nabla\cdot\vec{v} = s, \quad \nabla\times\vec{v} = \vec{c}$$

→ $\vec{v}$ is uniquely defined by

$$\vec{v} = -\nabla\varphi + \nabla\times\vec{A}$$

$\varphi(\vec{r})$ = scalar potential

$$= \frac{1}{4\pi}\int_v \frac{s(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dv'$$

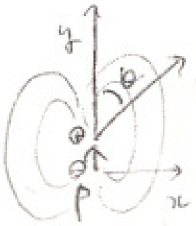
$\vec{A}(\vec{r})$ = vector potential

$$= \frac{1}{4\pi}\int_v \frac{\vec{c}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,dv'$$

Polarization Field

opposite

- Two charges separated by a small distance make a dipole which has a special electric field.

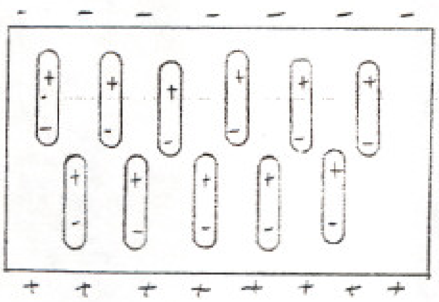


$\varphi = \frac{P\cos\theta}{r^2}$ $+q$ dx $-q$ $\vec{p}$ = dipole moment $= q\,d\vec{x}$

$E_x = \frac{3P\sin\theta\cos\theta}{r^3}$ $\varphi = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$

- Volume distribution of dipole moments is called the polarization field: $\vec{P} = \vec{p}/\text{volume}$

$E_y = \frac{P(3\cos^2\theta - 1)}{r^3}$



- The electric field due to the volume distribution can be interpreted as coming from free charges on the surface and bound charges in the volume.

$$\rho = \text{total change density} = \rho^f + \rho^B$$

$\rho^B = -\nabla\cdot\vec{P}$ see note

where $\rho^b = -\nabla\cdot\vec{P}$

ρ^f = free net charges
ρ^B = bound zero net polarization charge which on a homogeneous scale cannot be considered individually.

Electrical Displacement

- We have

$$\nabla^2 \varphi = -\nabla\cdot\vec{E} = \frac{-\rho^{tot}}{\varepsilon_0} = -\left(\frac{\rho^f + \rho^B}{\varepsilon_0}\right)$$

- Let $\rho^B = -\nabla\cdot\vec{P}$ a result of volume polarization

$$\nabla\cdot\left(\vec{E} + \frac{\vec{P}}{\varepsilon_0}\right) = \frac{\rho^f}{\varepsilon_0}$$

- It is convenient to describe a new vector

$$\vec{D} = \varepsilon_0 \cdot \vec{E} + \vec{P}$$

biggest term

then

$$\boxed{\nabla\cdot\vec{D} = \rho^f \qquad \int_s D\cdot ds = q^f}$$

- D represents field whose sources are only the free charges. Has units of charge/area

Electrical Constitutive Relations

- The polarization of a body is usually dependent on the electric field

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

Electric susceptibility, polarization coef.

more generally

$$\underset{3\times 1}{\vec{P}} = \varepsilon_0 \underset{3\times 3}{X} \overset{3\times 1}{\vec{E}}$$

matrix form

$P_i = \varepsilon_0 X_{ij} E_j$

- In terms of Electrical Displacement, $\vec{D}$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 (1 + \chi) \vec{E}$$

$$\vec{D} = \kappa \varepsilon_0 \vec{E} = \varepsilon \vec{E}$$

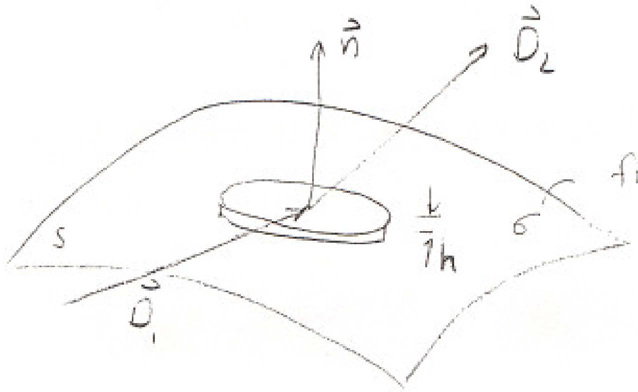
dielectric tensor (matrix)

3x3 matrix

piezoelectrics $\kappa \sim$ 1,000 - 3,000
electrostrictors $\kappa \sim$ 10,000 - 30,000

Boundary Conditions for Electric Fields in Matter

- consider interface between 2 dielectrics



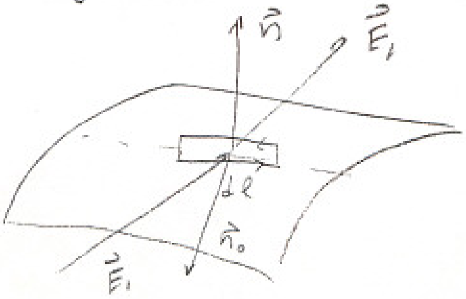
free charge at interface

- since $\oint \mathbf{D} \cdot d\mathbf{s} = q_f$ in limit $h \to 0$ you have

$$\boxed{\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma}$$

  - no charge at interface ($\sigma = 0$) $\Rightarrow$ normal comp. of D remains constant
  - note: normal component of E doesn't

- consider line integral at interface



since $\oint \mathbf{E} \cdot d\mathbf{s} = 0 \Rightarrow n \times (E_2 - E_1) = 0$

or $\boxed{\varphi_1 = \varphi_2 \text{ along boundary}}$

$E_n = \dfrac{\sigma}{\varepsilon_0}$ for conductor

Energy Relations:

Electrical Equilibrium

- Differential Form:

$$\nabla \cdot \vec{D} = \sigma \text{ applied free charge}$$

- Integral Form:
  - premultiply by allowable variation in φ, $\delta\varphi$

$$\int_v [\nabla \cdot \vec{D}]\, \delta\varphi \, dv = \int_v \sigma \, \delta\varphi \, dv$$

allowable $\delta\varphi$ (potential)
$\delta\varphi = 0$ on fixed conductors (applied voltage)
$\delta\varphi$ = constant along conductors
$\delta E = -\nabla\delta\varphi$

from calculus of variations,

$$\underbrace{\int_v \vec{D} \cdot \delta\vec{E} \, dv}_{\delta U} = \underbrace{\sum_i q_i \, \delta\varphi_i}_{\delta W}$$

- Also have a complementary principle with δD instead of δE.

- To solve you need $\vec{D}(\vec{E})$ from Constitutive Rel.

Magnetic Fields in Matter

- Will investigate the relevant characteristics of magnetic fields in matter.
- 2 principle fields: B, Magnetic field and $(\sim \vec{E})$

  H, Coercive field $(\sim \vec{D})$
- Both B and H are first order tensors

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}$$

- Forms of the equations are parallel to mechanical, electrical and magnetic systems.

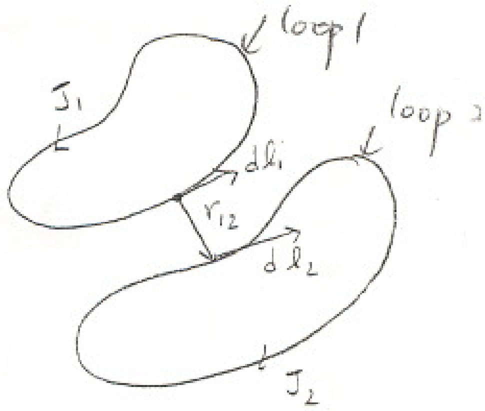
Newton, Coulomb, Ampere

The Magnetic Field - I

- Ampere's Law: represents the interaction between currents, gives force between two current carrying elements.

$$F_2 = \frac{\mu_0}{4\pi} J_1 J_2 \int_1\int_2 \frac{d\vec{l}_2 \times (d\vec{l}_1 \times \vec{r}_{12})}{|\vec{r}_{12}|^3}$$

$$\cancel{F_2 = \frac{\mu_0}{4\pi} J_1 J_2 \left[\underbrace{\oint_1 \vec{r}_{12} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{|r_{12}|^3}}_{\text{geometry}}\right]}$$



compare Coulombs Law

$$F_2 = \frac{1}{4\pi\varepsilon_0} q_1 q_2 \frac{\vec{r}_{12}}{|r_{12}|^3}$$

geometry

- Represent the Force with the magnetic field

$$F_2 = J_2 \oint_2 d\ell_2 \times B_2$$

$$B_2 = \frac{\mu_0}{4\pi} J_1 \oint_1 \frac{d\ell_1 \times \vec{r}_{12}}{|r|^3}$$

- Generalizes to consider current densities, $\vec{J}$

$$F = \int_v (\vec{J} \times \vec{B})dv \qquad \vec{B} = \frac{\mu_0}{4\pi} \int_v \frac{\vec{J} \times \vec{r}\,dv'}{|r|^3}$$

$\vec{r}$ → vector from point of integration to point of field.

$$\vec{r} = \vec{r}_1 - \vec{r}\,'$$

The Magnetic Field - II

- Differential Relations

$$\boxed{\begin{aligned}\nabla \cdot B &= 0 \\ \nabla \times B &= \mu_0 J\end{aligned}} \qquad \begin{aligned}\nabla \cancel{\times} E &= 0 \\ \nabla \cdot E &= \frac{q}{\varepsilon_0}\end{aligned}$$

- Vector (not scalar) potential: of magnetic field.

$$\boxed{\begin{aligned}\vec{B} &= \nabla \times \vec{A} \\ \vec{A} &= \frac{\mu_0}{4\pi}\int_v \frac{\vec{J}\,dv'}{|r|}\end{aligned}} \qquad \begin{aligned}E &= -\nabla \cdot \varphi \\ \varphi &= \frac{1}{4\pi\varepsilon_0}\int_v \frac{\rho\, dv^1}{|r|}\end{aligned}$$

- Current density, $\vec{J}$, can be divided into

$$\vec{J}_{TOT} = \underbrace{J^F}_{\text{free}} + \underbrace{J^M}_{\text{"bound" magnatization currents}}$$

- Volume Polarization replaced by Volume Magnitization, $\vec{M}$

$$\boxed{J^M = \nabla \times \vec{M}} \qquad \rho_b = -\nabla \cdot \vec{P}$$

The Coercive Field $\vec{H}$

- We have

$$\nabla \times \vec{B} = \mu_0 J^{TOT} = \mu_0 \left(J^F + J^M\right)$$

- Let $J^M = \nabla \times \vec{M}$

$$\nabla \times \left(\vec{B} - \mu_0 \vec{M}\right) = \mu_0 J^F$$

- It is convenient to describe a new vector

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \qquad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

then

ignore

$$\boxed{\begin{aligned} \nabla \times \vec{H} &= J^F \\ \nabla \cdot \vec{H} &= 0 \end{aligned}} + \frac{\delta \vec{D}}{\delta t} \qquad \nabla \cdot \vec{D} = \rho^F$$

$$\boxed{\oint \vec{H} \cdot \vec{d}\ell = J^F}$$ integral form

- H represents the field whose source currents represent the free (extremally applied) current.

Magnetic Constitutive Relations

- The magnitization of a body is usually dependent on the coercive field, H.

$$\vec{M} = \chi_m \vec{H}$$

magnetic succeptability

more generally

$$\vec{M} = X \vec{H}$$

- In terms of Magnetic Field, B

$$\begin{aligned}\vec{B} &= \mu_0 (\vec{H} + \vec{M}) = \mu_0 (\chi_m + 1) \vec{H} \\ &= \kappa_m \mu_0 \vec{H} \\ &= \mu \vec{H}\end{aligned}$$

$\mu_0 = 4\pi \times 10^{-7} \frac{\text{Henry}}{m}$

permeability

Terfenol - $\kappa_m = 7.9$

Good magnet iron $\kappa_m = \sim 1000$