Similarly, we can sum the amplitudes of all the reflection paths.

$$\begin{aligned} r_{31} &= r_{21} + t'_{21}r_{32}[1 + r'_{21}r_{32} + (r'_{21}r_{32})^2 + \cdots]t_{21} \\ &= r_{21} + t'_{21}r_{32}(1 - r'_{21}r_{32})^{-1}t_{21} \end{aligned}$$

t'_{31} and r'_{31} can also be obtained in a similar manner by starting with a wave incident from the right rather than from the left.

1.4 *Tunneling:* Consider the structure in Fig. 1.7 but with an electron energy E that is less than E_{C2} ($E_{C1} < E < E_{C2}$). Assume $k_x = k_y = 0$. It is apparent from eq. (1.62) that k_2 is imaginary. Derive an expression for T_B.

Solution:

$$k_1 = \sqrt{\frac{2m^*(E - E_{C1})}{\hbar^2}}$$

$$k_2 = i\gamma$$

$$\gamma = \sqrt{\frac{2m^*(E_{C2} - E)}{\hbar^2}}$$

From eq. (1.66), using eqs. (1.43), (1.48), and (1.49),

$$\begin{aligned} t_B &= \frac{i4k_1\gamma e^{-\gamma d}\, e^{-ikd}}{(k_1 + i\gamma)^2 - (k_1 - i\gamma)^2 e^{-2\gamma d}} \\ &= \frac{i2k_1\gamma}{(k_1^2 - \gamma^2)\sinh \gamma d + i2k_1\gamma \cosh \gamma d} \end{aligned} \tag{1.77}$$

Hence,

$$\begin{aligned} T_B = |t_B|^2 &= \frac{4k_1^2\gamma^2}{4k_1^2\gamma^2 + (k_1^2 + \gamma^2)^2 \sinh^2\gamma d} \\ &= \frac{1}{1 + \dfrac{(E_{C2} - E_{C1})^2 \sinh^2\gamma d}{4(E - E_{C1})(E_{C2} - E)}} \end{aligned} \tag{1.78a}$$

$$\simeq \frac{4(E - E_{C1})(E_{C2} - E)}{(E_{C2} - E_{C1})^2} e^{-2\gamma d}, \qquad \text{if } \gamma d \gg 1 \tag{1.78b}$$

It is straightforward to calculate the reflection coefficient r_B from eq. (1.67), using eqs. (1.43), (1.48), and (1.49).

$$r_B = \frac{(k_1^2 + \gamma^2)(1 - e^{-2\gamma d})}{(k_1 + i\gamma)^2 - (k_1 - i\gamma)^2 e^{-2\gamma d}}$$

$$= \frac{(k_1^2 + \gamma^2)\sinh \gamma d}{(k_1^2 - \gamma^2)\sinh \gamma d + i2k_1\gamma \cosh \gamma d} \tag{1.79}$$

Hence,

$$R_B = |r_B|^2 = \frac{(k_1^2 + \gamma^2)^2 \sinh^2 \gamma d}{4k_1^2\gamma^2 + (k_1^2 + \gamma^2)^2 \sinh^2 \gamma d} \tag{1.80}$$

It is easy to check that $R_B + T_B = 1$. Also, it can be shown that $t'_B = t_B$ and $r'_B = r_B$; this is expected from the symmetry of the structure.

It is apparent from eq. (1.78) that there is a non-zero probability for an electron to transmit through a barrier if it is thin enough, that is, if γd is small enough. This is a quantum mechanical phenomenon known as *tunneling,* not predicted by Newton's law. From a particle point of view, $T_B = 0$ regardless of the barrier thickness "d" as long as $E < E_{C2}$. But from a wave point of view, when $E < E_{C2}$ the wave becomes decaying (or evanescent) rather than propagating within the barrier. If the barrier ends before the wave has attenuated sufficiently—that is, if $\gamma d \lesssim 1$—then it can transmit significantly through the barrier. This phenomenon of tunneling has found extensive applications in both basic and applied sciences (Ref. [1.3]). A recent application is the *Scanning Tunneling Microscope* that makes it possible to get very high-resolution images of surfaces by exploiting the exponential dependence of the tunneling current on the barrier thickness "d".

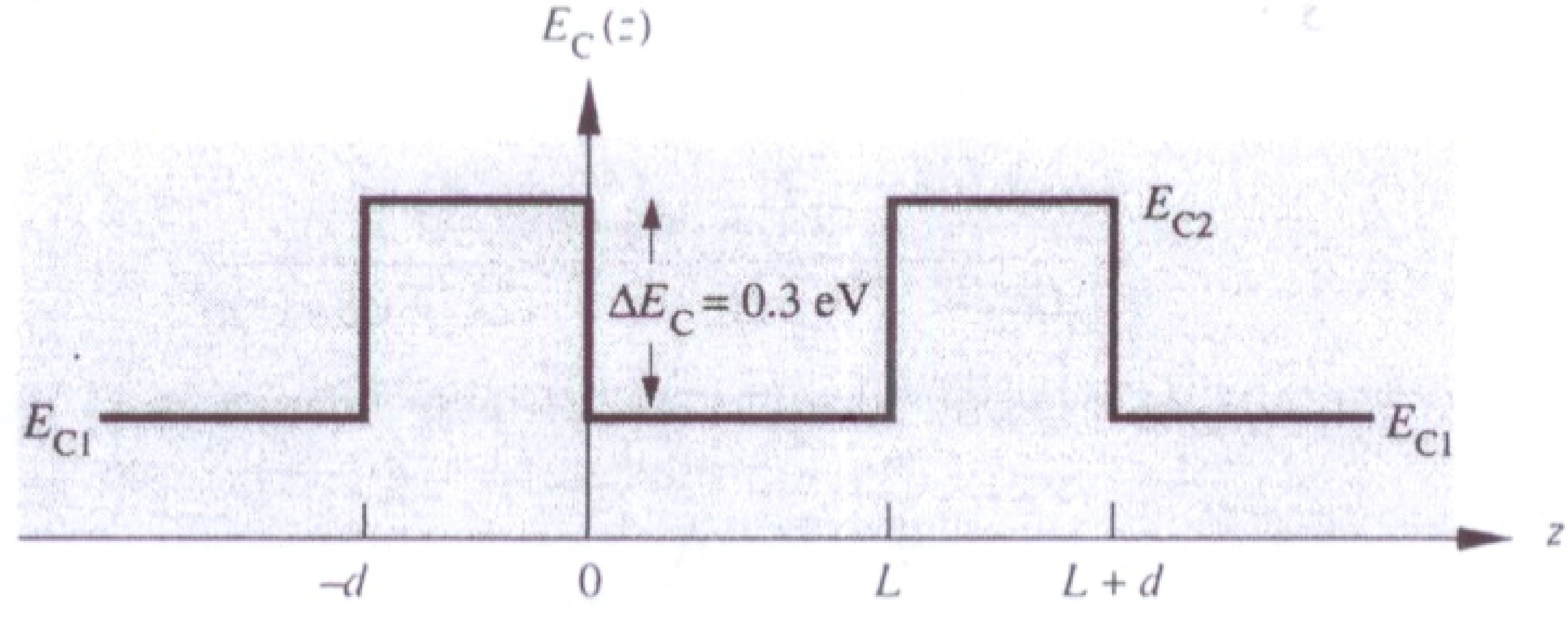
1.5 *Resonant Tunneling:* Consider the double barrier structure shown in Fig. 1.11a. We have already calculated the transmission coefficient t_B and the reflection coefficient r_B for a single barrier (Exercise 1.4). Assuming that the two barriers are identical, show that the overall transmission coefficient t_B through the double barrier structure is given by

$$t_{2B} = \frac{t_B^2 e^{ik_1L}}{1 - r_B^2 e^{i2k_1L}}$$

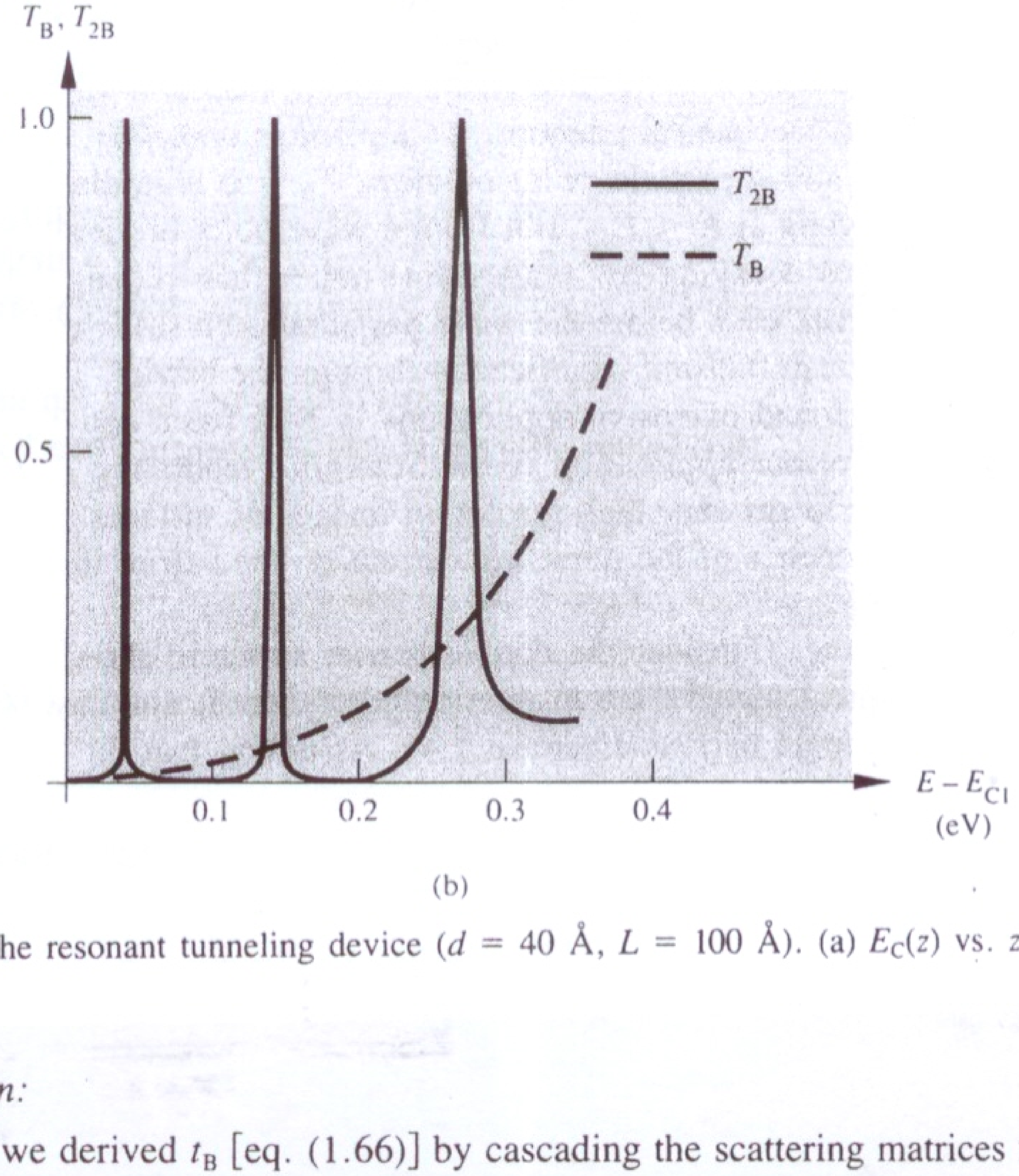
so that

$$T_{2B} = |t_{2B}|^2 = \left[1 + \frac{4R_B}{T_B^2}\sin^2(k_1L - \theta)\right]^{-1} \tag{1.81}$$

where $r_B^2 = R_B e^{-i2\theta}$, $|t_B|^2 = T_B$. Plot T_B and T_{2B} as a function of electron energy E in the range $E_{C1} < E < E_{C1} + 0.3$ eV assuming that $E_{C2} = E_{C1} + 0.3$ eV, $d =$ 40 Å, $L =$ 100 Å, $m^* = 0.07\ m_0$.



(a)



(b)

Fig. 1.11 The resonant tunneling device (d = 40 Å, L = 100 Å). (a) $E_C(z)$ vs. z; (b) $T_B(E)$, $T_{2B}(E)$.

Solution:

Earlier we derived t_B [eq. (1.66)] by cascading the scattering matrices for two potential steps and a free propagation region in between. t_{2B} is derived by cascading the scattering matrices for two barriers and a free propagation region in the same manner.

Figure 1.11b shows plots of T_B and T_{2B} as a function of the electron energy E. It is interesting to note how an electron with energy $E \simeq 0.035$ eV has a very low probability of getting through a single barrier ($T_B \simeq 0$) but can go clean through two barriers ($T_{2B} \simeq 1$). This behavior is very hard to rationalize with a particle-like picture. This phenomenon is known as resonant tunneling and has received considerable attention lately as a high-frequency negative-differential resistance device (Ref. [1.4]).

1.6 What is the probability current density J_z for a single electron with a wavefunction given by

$$\Psi(z,t) = (e^{-\gamma z} + ae^{+\gamma z})e^{-iEt/\hbar}$$

where $\gamma = \kappa - ik$, κ and k being real numbers.

Solution:

$$\left(\frac{\partial \Psi}{\partial z}\right)^* = (-\gamma^* e^{-\gamma^* z} + \gamma^* a^* e^{+\gamma^* z})e^{iEt/\hbar}$$

Hence,

$$\left(\frac{\partial \Psi}{\partial z}\right)^* \Psi = -\gamma^* e^{-(\gamma+\gamma^*)z} + \gamma^*|a|^2 e^{(\gamma+\gamma^*)z} + \gamma^* a^* e^{-(\gamma-\gamma^*)z} - \gamma^* a e^{(\gamma-\gamma^*)z}$$

From eq. (1.10),

$$\begin{aligned}
J_z &= -\frac{iq\hbar}{2m^*}\left[\left(\frac{\partial \Psi}{\partial z}\right)^* \Psi - \text{complex conjugate}\right] \\
&= \frac{-q\hbar k}{m^*}(e^{-2\kappa z} - |a|^2 e^{2\kappa z}) + \frac{iq\hbar\kappa}{m^*}(ae^{-i2kz} - a^* e^{i2kz}) \\
&= \frac{q\hbar k}{m^*}(1 - |a|^2) \quad \text{if } \kappa = 0 \\
&\quad \frac{iq\hbar\kappa}{m^*}(a - a^*) \quad \text{if } k = 0
\end{aligned}$$

When $\kappa = 0$, the net current is simply the difference between that carried by the forward wave ($\sim e^{ikz}$) and that carried by the reverse wave ($\sim e^{-ikz}$). It is interesting to note that this is not true if $\kappa \neq 0$. The current carried by a decaying wavefunction ($\sim e^{-\kappa z}$) or a growing wavefunction ($\sim e^{+\kappa z}$) is individually zero. But a linear combination of the two (as in a tunnel barrier, Exercise 1.4) carries a current proportional to Im(a).