Electromagnetic Theory : Maxwell Equations

Light Waves

(Electrons in Solids, 3rd Ed., R. H. Bube)

Light Waves

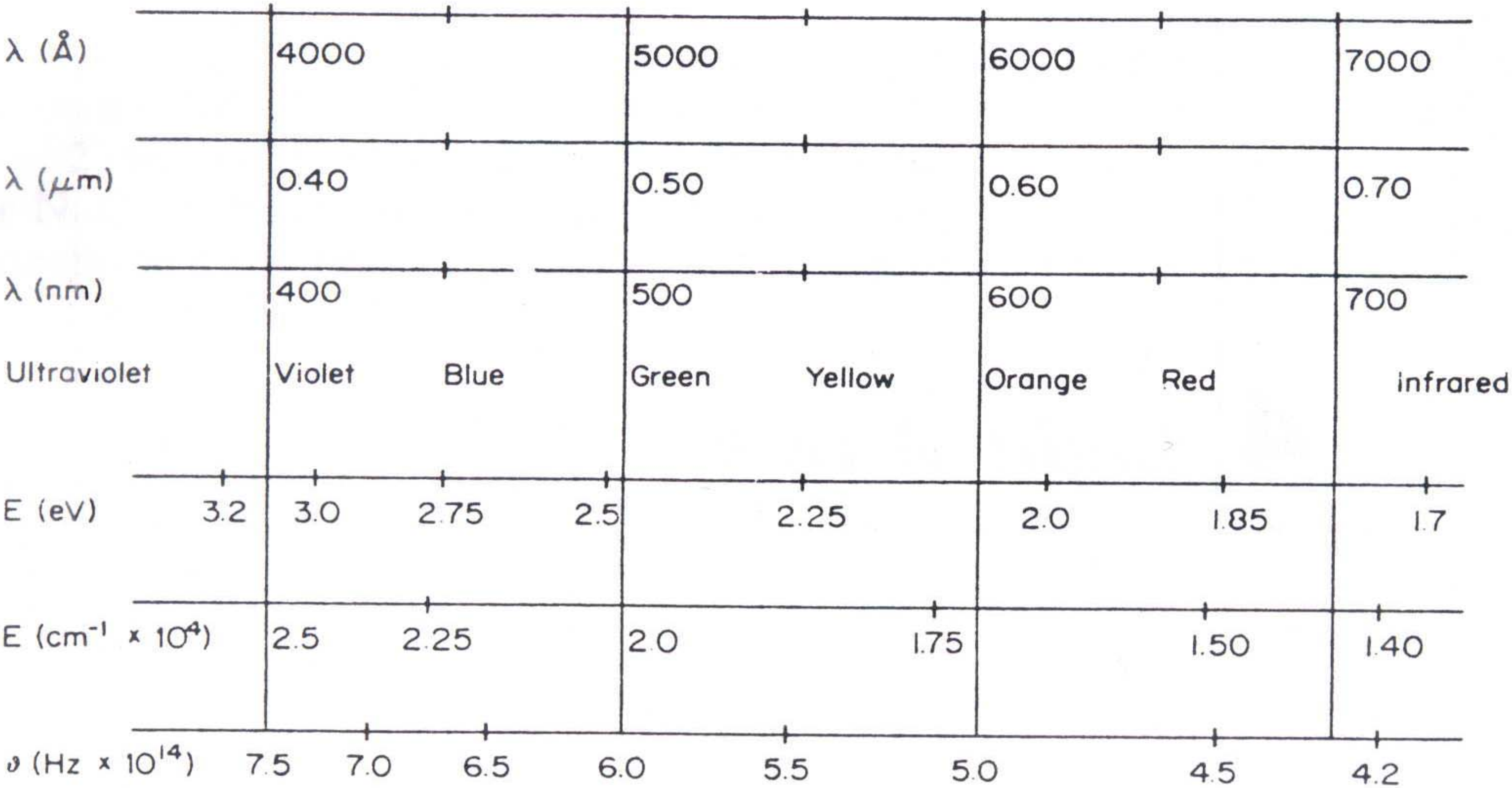

FIG. 4.1 The near-visible region of the electromagnetic spectrum with a variety of wavelenth, energy, wave-number, and frequency scales.

Properties of electric and magnetic fields

Wave equations for EM waves:

Maxwell's equations

TABLE 4.1 Maxwell's Equations

	Gaussian Units	SI Units
1.	$\nabla \cdot \mathbf{D} = 4\pi\rho$	$\nabla \cdot \mathbf{D} = \rho$
2.	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$
3.	$\nabla \times \mathcal{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$\nabla \times \mathcal{E} = -\frac{\partial \mathbf{B}}{\partial t}$
4.	$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} + \frac{4\pi \mathbf{J}}{c}$	$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

D—electric displacement

$\mathbf{D} = \varepsilon_r \mathcal{E}$ $\qquad$ $\mathbf{D} = \varepsilon_r \varepsilon_o \mathcal{E}$

ε_r—dielectric constant; $\varepsilon_o = (36\pi \times 10^9)^{-1}$ F/m—permittivity of free space

$\mathcal{E}$—electric field

ρ—charge density

B—magnetic induction

$\mathbf{B} = \mu_r \mathbf{H}$ $\qquad$ $\mathbf{B} = \mu_r \mu_o \mathbf{H}$

μ_r—permeability; $\mu_0 = 4\pi \times 10^{-7}$ H/m—permeability of free space

H—magnetic field

$\mathbf{J} = \sigma\mathcal{E}$—current density; σ—electrical conductivity

Properties of electric and magnetic fields

The First Maxwell Equation : $\nabla \bullet \mathbf{D} = \rho$ **in SI unit**

From coulomb's Law: the force on a charge *q* due to another charge *q'* separated by distance *r*

$$F = \frac{qq'}{4\pi\varepsilon_0 r^2}$$

If we define the electric field generated from *q* is E_q, the force on a charge *q'* is given by *F* = *q'* E_q

$$E_q = \frac{q}{4\pi\varepsilon_0 r^2}$$

Electric field due to charge *q* in the dielectric material with dielectric constant ε_r

$$E_q = \frac{q}{4\pi\varepsilon_r\varepsilon_0 r^2}$$

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Properties of electric and magnetic fields

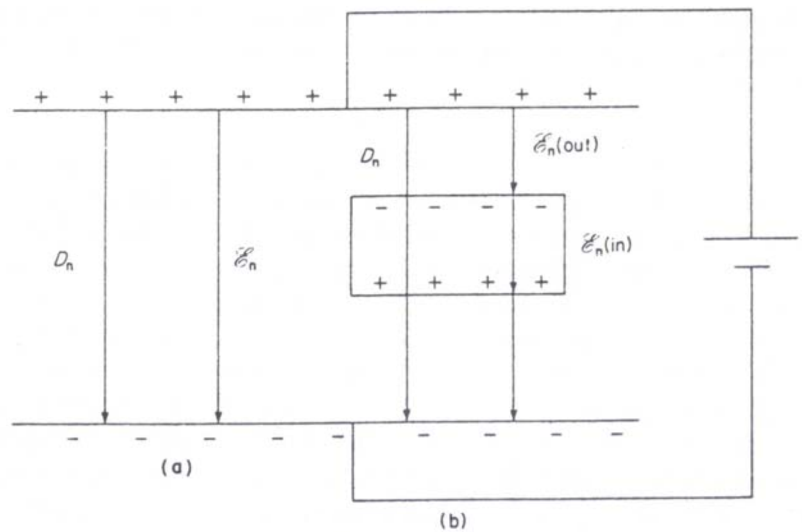

FIG. 4.2 Illustration of relationship between displacement **D** and electric field $\mathcal{E}$ with the simple case of a material between two condenser plates. (a) With no material between the plates the normal components are related $D_n = \varepsilon_o \mathcal{E}_n$. (b) With polarizable material between the plates $D_n = \varepsilon_r \varepsilon_o \mathcal{E}_n$ is the same both in free space and in the material, since it is conserved upon crossing the boundary. Because of the polarizability of the material, however, the electric field in the material is reduced so that $\mathcal{E}_n(\text{in}) = \mathcal{E}_n(\text{out})/\varepsilon_r$. Note that in this simple example, the tangential component of $\mathcal{E}$ that is conserved on crossing the boundary between the two materials is zero in all cases.

Polarization, P: dielectric dipole moment per unit volume induced by the electric field

$$\mathbf{P} = N\mathbf{p} = Nq^{*}\mathbf{d}$$

Where N is the volume density of dipoles, p is the dipole moment, and q^{*} is the charge involved in the dipole moment, with positive and negative charges separated by distance d

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Properties of electric and magnetic fields

The polarization is proportional to electric field $\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$

Proportional constant, χ: dielectric susceptibility

Since the electric displacement, **D** is defined by

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

Thus $\varepsilon_r = 1 + \chi$

Since there are a number of possible mechanism contributing the dielectric susceptibility (depending on frequency of EM waves)

$$\varepsilon_r = 1 + \sum_i \chi_i$$

For the displacement of atoms in the lattice χ_L and that of electrons in an atom, χ_e $\varepsilon_{\mathrm{r(lo)}} = 1 + \chi_{\mathrm{L}} + \chi_{\mathrm{e}}$ $\varepsilon_{\mathrm{r(hi)}} = 1 + \chi_{\mathrm{e}}$

The following relation is a good indication of the relative degree of ionic binding for materials

$$\{[1/\varepsilon_{r(hi)}] - [1/\varepsilon_{r(lo)}]\} = \chi_L / [\varepsilon_{r(hi)} \varepsilon_{r(lo)}]$$

Properties of electric and magnetic fields

Consider charge *q* with spherical symmetry and $E_q(r)$ at a distance r

$$E_q = \frac{q}{4\pi\varepsilon_r\varepsilon_0 r^2}$$

Divergence theorem $\oint\!\!\oint_A \mathbf{F}\cdot dS = \iiint_V \nabla\cdot\mathbf{F}\,dV$

$$4\pi r^2 E_q = \frac{q}{\varepsilon_r\varepsilon_0} \longrightarrow \int \mathbf{E}\cdot dS = \frac{q}{\varepsilon_r\varepsilon_0} \longrightarrow \int \nabla\cdot\mathbf{E}\,dV = \int \frac{\rho}{\varepsilon_r\varepsilon_0}\,dV$$

By writing $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$, $\nabla\bullet\mathbf{D} = \rho$

The first Maxwell Equation for an isotropic and homogeneous material.

$\mathbf{E} = -\nabla\phi$ where Φ the is called electrostatic potential

$$\frac{\partial^2\phi}{\partial x^2} = -\frac{\rho}{\varepsilon_r\varepsilon_0}$$

Poisson's Equation

Properties of electric and magnetic fields

The Second Maxwell Equation $\nabla \bullet \mathbf{B} = 0$

"Isolated magnetic poles do not exist"

Only magnetic dipole exist. A line of force starting on a "North" pole is terminated on a "South pole". No divergence of magnetic field line.

When we apply a magnetic field to a material, the magnetization M

$$\mathbf{M} = \kappa \mathbf{H}$$

Where κ is the magnetic susceptibility

A quantity B is conserved at an interface even when magnetization is present

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} \quad \text{(in SI unit)} \qquad \mathbf{B} = \mathbf{H} + 4\pi \mathbf{M} \quad \text{(in cgs unit)}$$

$$\mu_r = 1 + \kappa \qquad\qquad \mu_r = 1 + 4\pi\kappa$$

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Properties of electric and magnetic fields

The Third Maxwell Equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

When a wire is moved into or out of the pole of magnet. The wire will be subjected to a changing magnetic flux $\partial \Phi / \partial t$, with $\Phi = \int \mathbf{B} \bullet \mathrm{d}\mathbf{S}$

A potential difference ϕ **has been induced in the wire with a value which can be given simply by**

$$\phi = -\frac{\partial \Phi}{\partial t}$$

ϕ**: a line integral of electric field E**

$$\int \mathbf{E} \cdot \mathbf{dl} = -\int \frac{\partial \mathrm{B}}{\partial t} \cdot \mathbf{dS} \longrightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

By Stoke's Theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{l} = \iint_A \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

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Properties of electric and magnetic fields

The Fourth Maxwell Equation $$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

" A continuous electrical current I or a displacement current ($\frac{\partial \mathbf{D}}{\partial t}$) gives rise to a magnetic field"

Consider the attempt to measure a magnetic field around a wire carrying a current. At distance *r* from a wire direct current I

$$\mathbf{H} = \frac{\mathbf{I}}{2\pi r}$$

$$\int \mathbf{H} \cdot \mathbf{dl} = \int \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{dS} + \mathbf{I}$$

$\frac{\partial \mathbf{D}}{\partial t}$ displacement current

Stoke's Theorem and $\mathbf{I} = \int \mathbf{J} \bullet d\mathbf{S}$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

where J is electrical current density

$$\mathbf{J} = \sigma \mathbf{E}$$

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Dielectric Relaxation Time

Suppose that a charge is placed on a neutral material. The length of time it takes for this charge to relax either to a uniform charge density if the material is electrically isolated, or to zero, restoring the neutral state, by the excess charge leaking off to ground: **dielectric relaxation time,** τ_{dr}

$$\nabla \bullet \nabla \times \mathbf{A} = 0$$

A is any vector: applied to the 4th Maxwell Equation

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

$$\Big\downarrow \qquad \nabla \bullet \nabla \times \mathbf{H} = 0, \quad \mathbf{J} = \sigma \mathbf{E} \qquad \mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E}$$

$$0 = \frac{\partial (\nabla \bullet \mathbf{D})}{\partial t} + \nabla \bullet \mathbf{J}$$

$$\Big\downarrow \ \text{1st Maxwell equation}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\sigma \rho}{\varepsilon_r \varepsilon_0} \xrightarrow{\text{solution}} \rho = \rho_0 \exp\left(-\frac{\mathrm{t}}{\tau_{\mathrm{dr}}}\right) \qquad \tau_{\mathrm{dr}} = \frac{\varepsilon_r \varepsilon_0}{\sigma}$$

Electromagnetic wave equation

Deviation of wave equation from Maxwell Equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ (3rd Maxwell equation)

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E} \text{ , } \mathbf{B} = \mu_0 \mu_r \mathbf{H} \text{ , } \nabla \bullet E = \frac{\rho}{\varepsilon_r \varepsilon_0}$$

$$-\mu_r \mu_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = \frac{\nabla \rho}{\varepsilon_r \varepsilon_0} - \nabla^2 \mathbf{E}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \text{ , } \mathbf{D} = \varepsilon_0 \varepsilon_r \mathbf{E} \text{ , } \mathbf{J} = \sigma \mathbf{E}$$

$$-\varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_r \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} = \frac{\nabla \rho}{\varepsilon_r \varepsilon_0} - \nabla^2 \mathbf{E}$$

Electromagnetic wave equation

(continued) we neglect the first term on the right of the equation since we are interested in the steady-state condition after the decay of any such space charge

$$\nabla^2 \mathbf{E} = \varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_r \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t}$$

If we calculate $\nabla \times \nabla \times \mathbf{H}$, rather than $\nabla \times \nabla \times \mathbf{E}$

We obtain the same form of equations for the magnetic field **H**

$$\nabla^2 \mathbf{H} = \varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2} + \mu_r \mu_0 \sigma \frac{\partial \mathbf{H}}{\partial t}$$

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Electromagnetic wave equation: the case of no absorption

In the absence of all absorption processes: a first time derivative term of the EM equation equal to zero: then EM equation → "Harmonic wave"

$$\nabla^2 \mathbf{E} = \varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2 \mathbf{H} = \varepsilon_r \varepsilon_0 \mu_r \mu_0 \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

Solution of these equations have the form

$$\mathbf{E} = \mathbf{E_0} \exp\left[i(\mathbf{k} \cdot \mathbf{r} - \omega t)\right] \qquad \mathbf{H} = \mathbf{H_0} \exp\left[i(\mathbf{k} \cdot \mathbf{r} - \omega t)\right]$$

In the form of EM wave equation, we can conclude immediately the phase velocity of the electromagnetic waves is given by

$$\upsilon = \frac{1}{\left(\varepsilon_r \varepsilon_0 \mu_r \mu_0\right)^{1/2}} = c, \quad \text{in vacuum} \quad c = \left(\varepsilon_0 \mu_0\right)^{-1/2}$$

Appendix: The Divergence Theorem and Stoke's Theorem

The Divergence Theorem: Because of the definition of the divergence, it can be shown that

$$\int_V \nabla \bullet \mathbf{A} dV = \int_V \mathbf{A} \bullet \mathbf{n} dS$$

Thus converting a volume integral of the divergence of A into surface integral of the scalar product of A with the vector n, the outward drawn unit vector normal to *dS*

Stoke's Theorem: It follows from the properties of the curl that

$$\int_S \mathbf{n} \bullet (\nabla \times \mathbf{F}) dS = \oint \mathbf{F} \bullet \mathrm{d}\mathbf{s}$$

thus converting a surface integral of the curl F to a line integral of F over a closed path on the surface