Aeroelasticity

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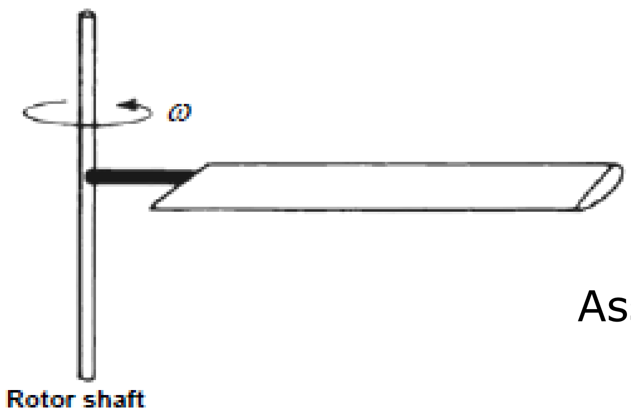
Structural Dynamics Overview

- ❖ Modeling
- ❖ Continuous and Discrete Systems
- ❖ Modal Methods
  - Eigenmodes
  - Rayleigh-Ritz
  - Galerkin
- ❖ Discrete Point Methods
  - Finite Difference
  - Finite Element
- ❖ Solution of Dynamic Problems
  - Mass Condensation – Guyan Reduction
  - Component Mode Synthesis

Modeling Levels

❖ Real structural dynamics system (structures)



Real structures, in 3-D space, comprised of different material, and subject to external excitation

Assumption : - material (linear elastic)
- geometry
- loads

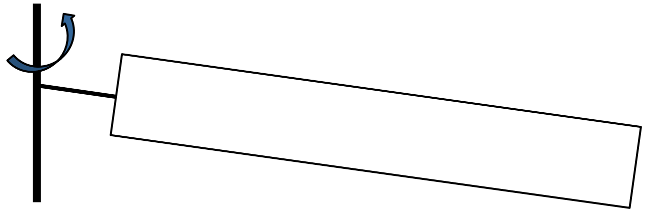
❖ Continuous representation of the structure

More assumptions

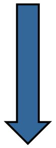
❖ Discrete representation of the structure

Modeling Levels

- ❖ Continuous representation of the structure
  - Idealized model (infinite d.o.f)

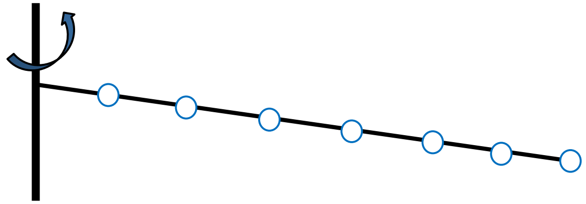


1-D (continuous beam)
representation of the blade



More assumptions, for example: low frequency behavior

- ❖ Discrete representation of the structure
  - Idealized model (finite d.o.f.)



1-D finite element
representation of the blade

Structural System Representation

❖ Methods for describing structural systems

- Continuous system : infinite D.O.F. → exact solution only available for special cases (e.g., vibration of uniform linear beams)

- Approximate solution : finite D.O.F. → two basic approaches
  1) Modal methods
  2) Discrete point methods

Discrete System

- Systems represented by finite number of degrees of freedom from the outset

- Properties described at certain locations can be obtained from (mass, stiffness) influence coefficient functions, or simply lumping techniques

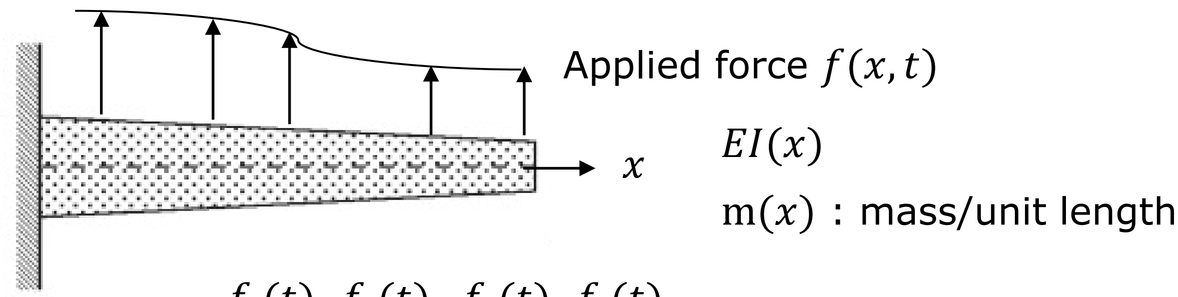
- General mass-spring system represented by

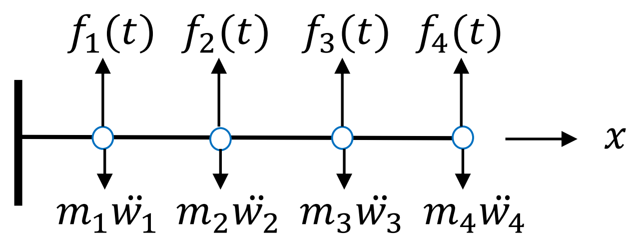
$$[\mathrm{M}]\{\ddot{u}\} + [K]\{u\} = \{F\}$$

Mass matrix Stiffness matrix Forcing vector

Discrete System

[Example] Lumped parameter formulation for a beam





Total force : $F_{i,Tot} = f_i - m_i\ddot{w}_i$ (D'Alembert's principle)

Applied force Inertial force

$$\{F_i\}_{tot} = \{f_i\} - \underbrace{\begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & m_3 & \\ & & & m_4 \end{bmatrix}}_{[\mathrm{M}]} \begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \\ \ddot{w}_3 \\ \ddot{w}_4 \end{Bmatrix}$$

Discrete System

Deflection w_i is

flexibility influence coefficient, Deflection @ i due to a unit load @ j

$$\{w_i\} = \{c_{ij}\}\{F_j\}_{tot}$$
$$= \{c_{ij}\}[\{f_j\} - [M]\{\ddot{w}_i\}]$$

c_{ij}

deflection load

Repose $\boxed{[\mathrm{M}]\{\ddot{w}\} + [K]\{w\} = \{f\}}$

This can also be extended to a full 2-D, 3-D structures

$$[\mathrm{M}]\begin{Bmatrix}\ddot{u}\\ \vdots\\ \ddot{v}\\ \vdots\\ \ddot{w}\\ \vdots\end{Bmatrix} + [K]\begin{Bmatrix}u\\ \vdots\\ v\\ \vdots\\ w\\ \vdots\end{Bmatrix} = \begin{Bmatrix}F_u\\ \vdots\\ F_v\\ \vdots\\ F_w\\ \vdots\end{Bmatrix}$$

Note : Generally both [M] and [K] have coupled structures (off-diagonal components), but still symmetric

Discrete System

$$[\mathrm{M}]\ddot{w} + [K]w = F$$

Set of simultaneous, coupled DE subject to IC's @ t=0

$$\left.\begin{aligned} w_i &= w_i^o \\ \dot{w}_i &= \dot{w}_i^o \end{aligned}\right\} @\,\mathrm{t} = 0$$

- First solve homogeneous equations for the lowest (few) eigenvalues (ω) and eigenvectors ([∅]: mode shape matrix)

$$[\mathrm{M}]\ddot{w} + [K]w = 0$$

Set $\mathrm{w} = \bar{w}e^{i\omega t}$

$$\underbrace{\left[-\omega[\mathrm{M}] + [\mathrm{K}]\right]}_{\text{characteristic eqn.}}\tilde{w}\underline{e}^{i\omega t} = 0 \qquad \cdots(*)$$

eigenvector

Discrete System

4 eigenvalues $\lambda_i = \omega_i^2$, natural frequency $f_i = \dfrac{\omega_i}{2\pi}$

Eigenvectors are obtained by placing any root into (*)

$$\begin{bmatrix} k_{11} - m_{11}\omega_1^2 & k_{12} - m_{12}\omega_2^2 & \cdots \\ & \ddots & \end{bmatrix} \phi^{(i)} = 0$$

Need to set at least one value of $\phi^{(i)}$

$\phi^{(1)}$

ω_1

Mode 1

$\phi^{(2)}$

ω_2 $\cdots$

Mode 2

A N-D.O.F system has N natural frequencies and N mode shapes associated to these natural frequencies.

Discrete System

- Orthogonality Relations

$\omega_j, \phi_i^{(j)}$ set of free vibration mode shapes

Each satisfies $-\omega^2[M]\phi + [K]\phi = 0$

$$-\omega_r^2[M]\phi^{(r)} = [K]\phi^{(r)} \cdots (1)$$

$$-\omega_s^2[M]\phi^{(s)} = [K]\phi^{(s)} \cdots (2)$$

Multiply (1) by $\phi^{(s)T}$ and (2) by $\phi^{(r)T}$

$$\omega_r^2\phi^{(s)T}[M]\phi^{(r)} = \phi^{(s)T}[K]\phi^{(r)}$$

$$\omega_s^2\phi^{(r)T}[M]\phi^{(s)} = \phi^{(r)T}[K]\phi^{(s)} \cdots (3)$$

Take transpose of both sides

$$\omega_r^2\phi^{(r)T}[M]^T\phi^{(s)} = \phi^{(r)T}[K]^T\phi^{(s)}$$

$$\omega_r^2\phi^{(r)T}[M]\phi^{(s)} = \phi^{(r)T}[K]\phi^{(s)} \cdots (4)$$

[M], [K] symmetric

Discrete System

Subtract (4) from (3)

$$(\omega_s^2 - \omega_r^2)\, \phi^{(r)T}[M]^T\phi^{(s)} = 0$$

If $\;\mathrm{r} \neq \mathrm{s} \longrightarrow \phi^{(r)T}[M]\phi^{(s)} = 0$

$\mathrm{r} = \mathrm{s} \longrightarrow \phi^{(r)T}[M]\phi^{(s)} = M_r^*$ (some value : modal mass)

$$\phi^{(r)T}[M]\phi^{(s)} = \delta_{rs} M_r^*$$

Kronecker delta $\delta_{rs} = \begin{cases} 0 \;:\; r \neq s \\ 1 \;:\; r = s \end{cases}$

Also note that

$$\phi^{(r)T}[K]\phi^{(s)} = \omega_r^2 M_r^* \delta_{rs}$$ (modal stiffness)

Discrete System

- Complete solution

$$[\mathrm{M}]\ddot{w} + [K]w = F$$

let $\quad w_i(t) = \sum_{i=1}^{4} \phi_i^{(r)} \eta_i(t)$

Generalized coordinate

$$[\mathrm{M}]\phi\ddot{\eta} + [K]\phi\eta = F$$

Pre-multiply by ϕ^T

$$\phi^T[\mathrm{M}]\phi\ddot{\eta} + \phi^T[K]\phi\eta = \phi^T F$$

Orthogonality $\longrightarrow$ Decoupled equations

$$M_1^*\ddot{\eta} + M_1^*\omega_1^2\eta_1 = Q_1, \qquad Q_1 = \phi^{(1)}F$$

$$\vdots \qquad\qquad \vdots$$

Generalized or normalized coordinate

$$M_n^*\ddot{\eta}_n + \underbrace{M_n^*\omega_n^2}\eta_n = Q_n$$

Generalized mass

Generalized stiffness

Generalized force

Discrete System

- Initial conditions

@ t=0, given $w(0), \dot{w}(0)$

$$\phi\eta(0) = \begin{Bmatrix} w_1(0) \\ w_2(0) \\ w_3(0) \\ w_4(0) \end{Bmatrix} \quad \text{and} \quad \phi\dot{\eta}(0) = \dot{w}(0)$$

If all the modes are retained in solution, that is, $\mathrm{w} = \sum_{i=1}^{n} \phi^{(i)} \eta_i(t)$

$$\underset{\text{n x 1}}{\eta(0)} = \underset{\text{m x n}}{\phi^{-1}} \underset{\text{n x 1}}{w(0)}$$

Discrete System

- Truncation

Problem can be truncated by using only a few selected number of modes

$$w(x,t) = \sum_{i=1}^{m} \phi^{(i)}(x)\eta_i(t)$$

where m<<n

But now calculation of initial condition on η is not straightforward.

$$\underset{m\mathrm{x}1}{\eta(0)} = \underset{m\mathrm{x}n}{\phi^{-1}} \underset{n\mathrm{x}1}{w(0)}$$

not invertible!

$$\underset{n\mathrm{x}m}{\phi} \underset{m\mathrm{x}1}{\eta(0)} = \underset{n\mathrm{x}1}{w(0)}$$

Discrete System

Premultiply by $\phi^T[M]$,

$$\underbrace{\underset{\text{mxn}}{\phi^T}\,\underset{\text{nxn}}{[M]}\,\underset{\text{nxm}}{\phi}}_{M^*_{m\times m}\,:\,diagonal}\,\underset{\text{mx1}}{\eta(0)} = \underset{\text{mxn}}{\phi^T}\,\underset{\text{nxm}}{[M]}\,\underset{\text{nx1}}{w(0)}$$

$$M^*\eta(0) = \phi^T[M]w(0)$$

$$\eta_i(0) = \frac{1}{M_i^*}\left[\phi_1^i \cdots \phi_n^i\right][M]\begin{Bmatrix} w_1(0) \\ w_2(0) \\ \vdots \\ w_n(0) \end{Bmatrix}$$

→ Solve for $\eta(t)$ subject to $\eta(0)$ and $\dot{\eta}(0)$

and find w from $\mathrm{w(x,t)} = \sum_{i=1}^{m} \phi^{(i)}(x)\eta_i(t)$

[Note] The normal equations of motion are uncoupled on the left-hand side due to the modal matrix composed of eigenvectors.

Coupling, however, may come from motion-dependent forces, including damping.

Discrete System

- Motion Dependent Forces

Forces F_i may be dependent on position, velocity, acceleration after structure @ its nodes i, as well as time

$$\Rightarrow \quad F_i = F_i(w_1, w_2, \cdots \dot{w}_1, \dot{w}_2, \cdots \ddot{w}_1, \ddot{w}_2 \cdots, t)$$

Consider a general case

$$F_i = \sum_{k=1}^{N} (a_{ik} w_k + c_{ik} \dot{w}_k + e_{ik} \ddot{w}_k) + F_i(t)$$

Consider an N degree of freedom system

$$[\mathrm{M}]\{\ddot{w}\} + [K]w = [a]\{w\} + [c]\{\dot{w}\} + [e]\{\ddot{w}\} + \{F_i(t)\}$$

Discrete System

Let $w_i = \sum_{j}^{n=3} \phi_i^{(j)} \eta_j(t)$

$$[M^*]\ddot{\eta} + [\omega^2 M^*]\eta = \underbrace{\underbrace{\phi^T[a]\phi}_{[A]}\eta + \underbrace{\phi^T[c]\phi}_{[C]}\dot{\eta} + \underbrace{\phi^T[e]\phi}_{[E]}}_{\text{fully populated (in general)}}\ddot{\eta} + Q$$

Can also write it as

$$M_r^*\ddot{\eta}_r + \omega_r^2 M_r^* \eta_r = \sum_{s=1}^{m} (A_{rs}\eta_s + C_{rs}\dot{\eta}_s + E_{rs}\ddot{\eta}_s) + Q_r$$

not necessarily positive definite

The terms on the summation on the right-hand side couple (in general) the equations of motion. This is typical in aeroelastic problem.

Discrete System

- For proportional damping,

$$[C] = \alpha[K] + \beta[M]$$ … damping matrix is proportional to a linear combination of the mass and stiffness matrices

any value, constants

Then, due to orthogonality on [K] and [M]

⇒ $C_{rs} = 0 \quad \text{when } r \neq s$

⇒ No coupling ⇒ $Set\ \ C_{rr} = 2\varsigma_r \omega_r M_r^*$

Critical damping ratio: obtained from experiments or guess

m set of uncoupled equations

$$\begin{cases} M_r^*(\ddot{\eta}_r + 2\varsigma_r \omega_r \eta_r + \omega_r^2 \eta_r) = Q_r(t) \\ \vdots \end{cases}$$

Continuous System

- At this point, a distinction between two main classes of approaches for approximating the solution of structural systems needs to be made.

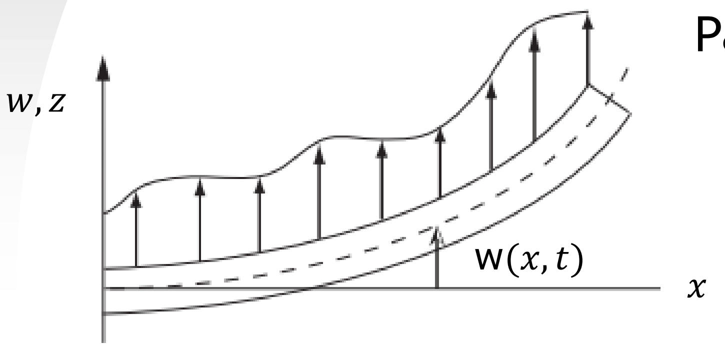
- The two basic approaches are

  1) modal methods: represent displacements by overall motion of the structure

  2) discrete point methods: represent displacement by motion at many discrete points distributed along the structures

Continuous System

- Consider a basic high-aspect ratio wing modeled as a cantilever beam for symmetric response



Partial differential equation for continuous beam

$$\mathrm{m}\ddot{w} - (Tw')' + (EIw'')'' = f_z$$

- Pinned end  $w = 0$, $M = EIw'' = 0$
- Fixed end  $w = 0$, $w' = 0$
- Free end  $M = EIw'' = 0$, $S = (EIw'')' = 0$
- Vertical spring  $M = EIw'' = 0$, $S = (EIw'')' = k_v w$

$m(x) : mass/unit \ \ length\,(kg/m)$

$w(x,t): vertical\ deflection\ (m)$

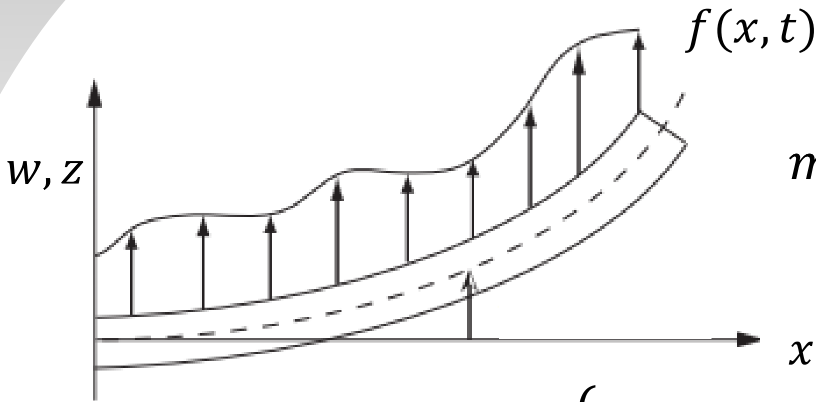
$T\ : axial\ force\ (N)$

$EI(x): bending\ stiffness\ (N \cdot m^2)$

$f_z\ : vertical\ applied\ force\ (N/m)$

$f_x\ : horizontal\ applied\ force\ (N/m)$

Response of a Uniform Cantilevered Beam



$$m = EI = const.$$
$$T = 0$$

B.C. $\quad @x = 0 \begin{cases} w = 0 \\ w' = 0 \end{cases} \longrightarrow$ geometric B.C.

$\quad @x = l \begin{cases} M = EIw'' = 0 \\ S = EIw''' = 0 \end{cases} \longrightarrow$ natural B.C.

I.C. $\quad @t = 0 \begin{cases} w = 0 \\ \dot{w} = 0 \end{cases}$ (Rest I.C.'s)

Same solution procedure as before

i) find solution to homogeneous equation

ii) then determine complete solution as expansion of homogeneous solution

Response of a Uniform Cantilevered Beam

$$\mathrm{EI}w'''' + \mathrm{m}\ddot{w} = 0 \qquad \cdots(1)$$

let $\mathrm{w(x,t)} = \bar{w}(x)e^{i\omega t}$

$$\longrightarrow \quad \left(\mathrm{EI}\bar{w}'''' - \mathrm{m}\omega^2\bar{w}\right)e^{i\omega t} = 0 \quad \cdots(2)$$

$$\longrightarrow \quad \bar{w}'''' - \frac{\mathrm{m}\omega^2}{EI}\bar{w} = 0 \qquad \cdots(3)$$

To solve, let $\bar{w} = e^{px}$ $(\rightarrow sin, cos, sinh, cosh)$

$$\longrightarrow \quad p^4 e^{px} - \frac{m\omega^2}{EI}e^{px} = 0$$

nontrivial solution $p^4 = \dfrac{m\omega^2}{EI}$

$4\ roots$ $\quad \mathrm{p} = \lambda, -\lambda, \mathrm{i}\lambda, -\mathrm{i}\lambda \qquad$ where $\lambda^2 = \omega\sqrt{\dfrac{m}{EI}}$

Response of a Uniform Cantilevered Beam

$$\bar{w}(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x} + C_3 e^{i\lambda x} + C_4 e^{-i\lambda x}$$

or $\bar{w}(x) = A sinh\lambda x + B\cosh\lambda x + C\sin\lambda x + D\cos\lambda x$

Determine A, B, C, D from B.C.'s in matrix form

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ sinh\lambda l & \cosh\lambda l & -\sin\lambda l & -\cos\lambda l \\ cosh\lambda l & \sinh\lambda l & -\cos\lambda l & \sin\lambda l \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0.$$

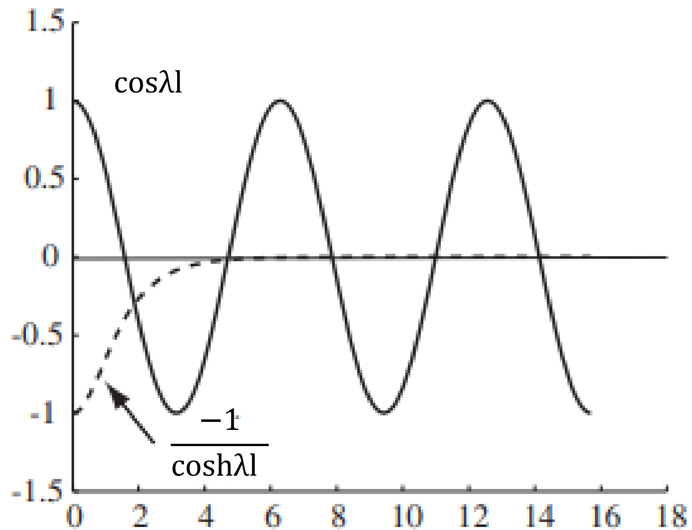
$\longleftarrow$ Transcendental equation

For a nontrivial solution, $|\Delta| = 0$

$$|\Delta| = 2cosh\lambda l\cos\lambda l + \underbrace{(sin^2\lambda l + cos^2\lambda l)}_{=1} + \underbrace{(cosh^2\lambda l - sinh^2\lambda l)}_{=1} = 0$$

$$\longrightarrow \boxed{\cos\lambda l = \frac{-1}{cosh\lambda l}}$$

Response of a Uniform Cantilevered Beam



many solutions possible

$$\lambda l = \underset{\underset{0.5\pi}{\wr}}{0.597\pi},\ \underset{\underset{\frac{3}{2}\pi}{\wr}}{1.49\pi},\ \frac{5}{2}\pi,\ \frac{7}{2}\pi$$

$$\omega_r = (\lambda l)^2 \sqrt{\frac{EI}{ml^4}}$$

For eigenvectors (mode shapes), place λl into first three equations

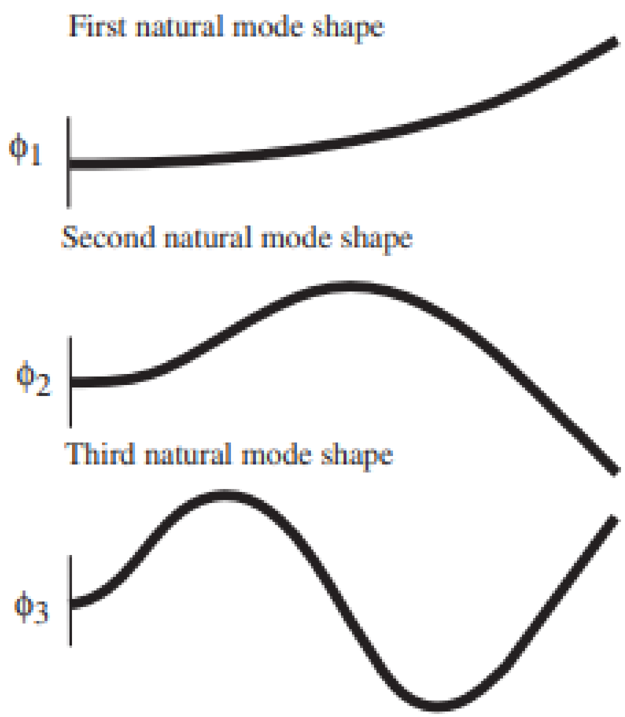
$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ sinh\lambda l & \cosh\lambda l & -\sin\lambda l & -\cos\lambda l \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = 0$$

$$\bar{w}_r(x) = (cosh\lambda_r x - cos\lambda_r x) - (\frac{cosh\lambda_r l + cos\lambda_r l}{sinh\lambda_r l + sin\lambda_r l})(sinh\lambda_r x - sin\lambda_r x\Big)$$

Ref. : Blevins "Formulas for Natural Frequency and Mode Shapes"

Response of a Uniform Cantilevered Beam

$$\mathrm{w(x,t)} = \sum_{r=1}^{\infty} \phi_r(x) e^{i\omega_r t}$$



$$\omega_1 = 3.52\sqrt{\frac{EI}{ml^4}} \ \ (rad/s)$$

$$\omega_2 = 22\sqrt{\frac{EI}{ml^4}}$$

$$\omega_3 = 61.7\sqrt{\frac{EI}{ml^4}}$$

Orthogonality

Since each solution satisfies $\mathrm{w}(\mathrm{x},\mathrm{t}) = \phi_r(x)e^{i\omega_r t}$

$$\mathrm{m}\ddot{w} + (EIw'')'' = 0$$

$$-\mathrm{m}\omega_r^2\phi_r + (EI\phi_r'')'' = 0 \quad \cdots(1)$$

$$-\mathrm{m}\omega_s^2\phi_s + (EI\phi_s'')'' = 0 \quad \cdots(2)$$

Multiply (1) by ϕ_s and integrate

$$\omega_r^2\int_0^l \phi_s m\phi_r dx = \int_0^l \phi_s(EI\phi_r'')''dx \quad \cdots(3)$$

and (2) by ϕ_r and integrate

$$\omega_s^2\int_0^l \phi_r m\phi_s dx = \int_0^l \phi_r(EI\phi_s'')''dx \quad \cdots(4)$$

Orthogonality

Subtract (4) from (3), and integrate by parts

$$(\omega_r^2-\omega_s^2)\int_0^l \phi_r m\phi_s dx = \phi_s(EI\phi_r'')'|_0^l - \phi_s' EI\phi_r''|_0^l + \cancel{\int_0^l \phi_s EI\phi_r'' dx}$$

$$-\underbrace{\phi_r}_{\text{defection}}\underbrace{(EI\phi_s'')'}_{\text{shear}}|_0^l + \underbrace{\phi_r'}_{\text{slope}}\underbrace{EI\phi_s''}_{\text{moment}}|_0^l - \cancel{\int_0^l \phi_r'' EI\phi_s'' dx}$$

Note that all the constant terms on RHS=0 because of BC's

for example :

- pinned $\rightarrow$ $w=0 \Rightarrow \phi=0$
  $w''=0 \Rightarrow \phi''=0$
- fixed $\rightarrow$ $w=0 \Rightarrow \phi=0$
  $w'=0 \Rightarrow \phi'=0$
- free $\rightarrow$ $\underset{\text{M=0}}{\phi''=0}$ and $\underset{\text{S=0}}{(EI\phi'')'=0}$

Orthogonality

For $r \neq s$, we have

$$\int_0^l \phi_r(x)m(x)\phi_s(x)dx = 0$$

$$\int_0^l \phi_r(x)m(x)\phi_s(x)dx = \delta_{rs}M_r^*$$

Also, $$\int_0^l \phi_s(EI\phi_r'')''dx = \delta_{rs}M_r^*w_r^2$$

$\Rightarrow$ can transform to normal coordinates

Complete solution

$$\mathrm{m}\ddot{w} + (EIw'')'' = f(x,t) \quad \cdots (5)$$

let $\mathrm{w(x,t)} = \sum_{r=1}^{\infty} \phi_r(x)\eta_r(t) \quad \cdots (6)$

Place (6) into (5) and integrate after multiplying with ϕ_s

$$\sum_{r=1}^{\infty} \ddot{\eta} \int_0^l m\phi_s\phi_r dx + \sum_{r=1}^{\infty} \eta_r \int_0^l \phi_s(EI\phi_r'')''dx = \int_0^l \phi_s f(x,t)dx$$

because of orthogonality

$$\begin{bmatrix} M_r\ddot{\eta}_r + M_r\omega_r^2\eta_r = Q_r \\ \vdots \end{bmatrix}$$

$$M_r = \int_0^l \phi_r^2(x)m(x)dx$$

$$Q_r = \int_0^l \phi_r(x)f(x,t)dx$$

Note : can also show orthogonality conditions hold if $-(Tw')'$ term is present

Complete solution

To find I.C.'s on η_r,

@ t = 0, $$w(x,0) = \sum_{r=1}^{\infty} \phi_r(x)\eta_r(0) = w_0(x)$$

and

$$\dot{w}(x,0) = \sum_{r=1}^{\infty} \phi_r(x)\dot{\eta}_r(0) = \dot{w}_0(x)$$

Multiply by $m\phi_s(x)$ and integrate

$$\int_0^l m\,\phi_s w_0 dx = \sum_{r=1}^{\infty} \eta_r(0) \int_0^l m\phi_s\phi_r dx = \eta_s(0)M_s^*$$

$$\Rightarrow \begin{cases} \eta_r(0) = \dfrac{1}{M_r^*}\displaystyle\int_0^l m\phi_r w_0(x)dx \\ \dot{\eta}_r(0) = \dfrac{1}{M_r^*}\displaystyle\int_0^l m\phi_r \dot{w}_0(x)dx \end{cases}$$

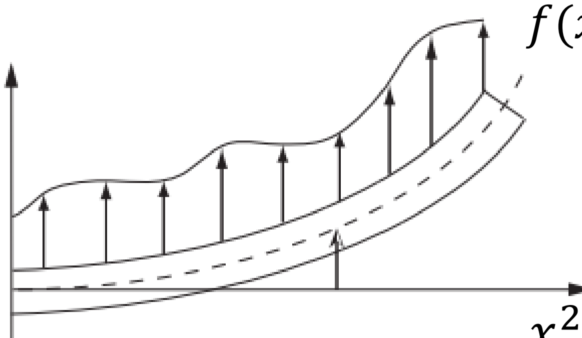
Rayleigh-Ritz Method

❖ Energy-based method

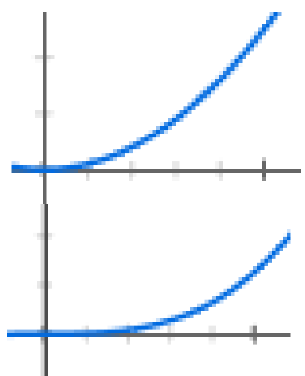
- Form of the solutions is assumed to be as :

$$\mathrm{w(x,t)} \approx \sum_{r=1}^{N} \gamma(x)\, q_r(t)$$

assumed modes need to satisfy at least geometrical boundary conditions



assume $\mathrm{w(x,t)} = \sum_{i=1}^{N} \gamma_i(x)\, q_i(t)$



$$\left.\begin{aligned} \gamma_1 &= \frac{x^2}{l^2} \\ \gamma_2 &= \frac{x^3}{l^3} \end{aligned}\right\} \text{satisfy} \quad \left.\begin{aligned} w &= 0 \\ w' &= 0 \end{aligned}\right\} @x = 0$$

Rayleigh-Ritz Method

$$\mathrm{T} = \frac{1}{2}\int_0^l m(x)\sum_{i=1}^{M}\gamma_i\dot{q}_i\sum_{j=1}^{M}\gamma_j\dot{q}_j\,dx = \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M}\underbrace{\int_0^l m\gamma_i(x)\gamma_j(x)dx}_{m_{ij}^*}\dot{q}_i\dot{q}_j$$

$$\mathrm{V} = \frac{1}{2}\int_0^l EI(w'')^2\,dx = \frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M}\underbrace{\int_0^l EI(x)\gamma_i''(x)\gamma_j''(x)dx}_{k_{ij}^*}q_iq_j$$

$$\delta\mathrm{W} = \int_0^l f\delta w dx = \sum_{i=1}^{M}\int_0^l f(x)\gamma_i dx\delta q_i$$

Plug into Lagrange’s equations,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

which gives

$$\sum_{i=1}^{M} m_{ij}^*\ddot{q}_j + \sum_{i=1}^{M} k_{ij}^*\ddot{q}_j = Q_i$$

coupled set of equations!

Rayleigh-Ritz Method

For a quick and "dirty" way to find the first natural frequency,

assume only one mode shape,

$$m_{11}^{*}\ddot{q}_1 + k_{11}^{*}q_1 = Q_1$$

Rayleigh quotient with $\mathrm{q} = \bar{q}e^{i\omega t}$

$$\omega^2 = \frac{\int_0^l EI(\gamma_1'')^2 dx}{\int_0^l m\gamma_1^2 dx}$$

… upper bound for the actual frequency

Clearly we can obtain higher modes by assuming more than one mode

$$\omega_r^2 = \frac{\{\gamma\}_r^T[K]\{\gamma\}_r}{\{\gamma\}_r^T[M]\{\gamma\}_r}$$

Galerkin's Method

- Galerkin's method applies to P.D.E. directly – residual method

$$\int_{\text{Domain}} \gamma_j [P.D.E.]dx = 0 \qquad for\ j = 1,2,\dots,N$$

- Assumed modes must satisfy all the boundary conditions (geometric and natural ones)

$$\mathrm{w(x,t)} = \sum_{i=1}^{N} \gamma_i(t)\, q_i(t)$$

Look at general beams

$$\mathrm{m}\ddot{w} + (EIw'')'' - (Tw')' = f(x,t)$$

for a pinned-pinned beam,

$$\gamma_j = \sin\left(\frac{j\pi x}{L}\right)$$

If γ_j is on exact mode shape, P.D.E. would be satisfied exactly
But if not $\rightarrow$ error

Galerkin's Method

$$\mathrm{E} = \mathrm{m}\ddot{w}_{\text{approx}} + \left[EIw''_{approx}\right]'' - \left[Tw'_{approx}\right]' - \mathrm{f}$$

Now set

$$\int_0^l h_i(x)E(x)dx = 0$$

: Average error in PDE with respect to some weighting function $h_i(x)$ that minimize the error in the interval, usually take $h_i(x) = \gamma_i(x)$

$$\sum_{j=1}^{M} \ddot{q}_j\left[\int_0^l \gamma_i(x)m(x)\gamma_j(x)dx\right] + \sum_{j=1}^{M}\left[\left[\int_0^l \gamma_i\left(EI\gamma_j''\right)''dx - \int_0^l \gamma_i\left(T\gamma_j'\right)'dx\right]\right.$$

Different from Rayleigh-Ritz

$$= \int_0^l \gamma_i\, f(x,t)dx$$

Galerkin's Method

For M different weighting function $\gamma_1,\ \gamma_2,\ \ldots\gamma_M$,

we have M equations to find M unknowns $q_1,\ q_2,\ \ldots q_M$

To find M unknowns $q_1,\ q_2,\ \ldots q_M$ in matrix form

$$[m_{ij}]\ddot{q}_j + [k_{ij}]q_j = Q_j$$

⋯ coupled set of DE's
(except when γ_j is natural mode shape)

Used standard technique, let $\mathrm{q} = \bar{q}e^{i\omega t}$

⟹ $$\left[I\omega^2 - [m]^{-1}[k]\right]\bar{q} = 0$$

⟹ Eigenvalues → approximate natural frequencies

Eigenvectors → approximate natural mode shapes

Galerkin's Method

Note :

i) more assumed modes → better approximation

$$\phi_1(x) = Acos\lambda_1 x + Bsin\lambda_1 x + Ccosh\lambda_1 x + Dsinh\lambda_1 x$$
$$= a_0 + a_1 x + a_1 x^2 + a_3 x^3 + \cdots$$

ii) more accurate assumed shapes → better approximation

iii) If $\gamma_j(x)$ is natural mode shapes, system will be uncoupled

iv) The closer $\gamma_j(x)$ is to $\phi(x)$, the less the coupling

Galerkin : very powerful, turn PDE's into ODE's
very general, can also be used in nonlinear problem !!

$$m\ddot{w} + (EIw'')'' + F(w^n) = f$$

v) If Rayleigh-Ritz assumed mode shapes satisfy both geometric and natural B.C.'s, two methods are identical

(can be shown by integration by parts)