Mathematical Background in Aircraft Structural Mechanics

CHAPTER 2. Beam Theory

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❖ One of its dimensions much large than the other two

- ➢ Civil engineering structure – assembly on grid of beams with cross-sections having shapes such as T's on I's
- ➢ Machine parts – beam-like structures lever arms, shafts, etc.
- ➢ Aeronautic structures – wings, fuselages → can be treated as thin-walled beams

❖ Beam theory

– important role, simple tool to analyze numerous structures
valuable insight at a pre-design stage

❖ Euler-Bernoulli beam theory – simplest, must be useful

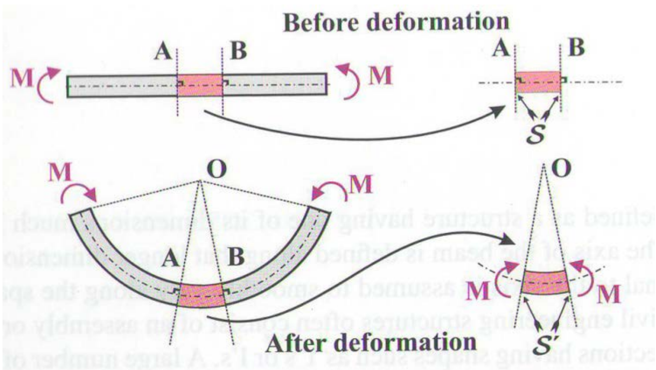
- ➢ Assumption
  - ① Cross-section of the beam is infinitely rigid in its own plane
    → in-plane displacement field → $\begin{cases}\text{2 rigid body translations}\\ \text{1 rigid body rotation}\end{cases}$
  - ② The cross-section is assumed to remain plane
  - ③ The cross-section is assumed to normal to the deformed axis

5.1 The Euler-Bernoulli assumptions

❖ **Fig. 5.1**

"pure bending" beam deforms into a curve of constant curvature

→ a circle with center O, symmetric w.r.t. any plane perpendicular to its deformed axis



❖ **Kinematic assumptions "Euler-Bernoulli"**

① Cross-section is infinitely rigid in its own plane

② Cross-section remains plane after deformation

③ Cross-section remains normal to the deformed axis of the beam

→ valid for long, slender beams made of isotropic materials with solid cross-sections

5.2 Implication of the E-B assumption

❖ $\begin{Bmatrix} u_1(x_1,x_2,x_3) \\ u_2(x_1,x_2,x_3) \\ u_3(x_1,x_2,x_3) \end{Bmatrix}$ **displacement of an arbitrary point of the beam**

➢ E-B assumption

① Displacement field in the plane of cross-section consists solely of 2 rigid body translations $\bar{u}_2(x_1), \bar{u}_3(x_1)$

$$u_2(x_1,x_2,x_3)=\bar{u}_2(x_1)\ ,\ \ u_3(x_1,x_2,x_3)=\bar{u}_3(x_1) \qquad (5.1)$$

② Axial displacement field consists of $\left\{ \begin{array}{l} \text{rigid body translation } \bar{u}_1(x_1) \\ \text{2 rigid body rotation } \Phi_2(x_1), \Phi_3(x_1) \end{array} \right.$

$$u_1(x_1,x_2,x_3)=\bar{u}_1(x_1)+x_3\Phi_2(x_1)-x_2\Phi_3(x_1) \qquad (5.2)$$

③ Equality of $\left\{ \begin{array}{l} \text{the slope of the beam} \\ \text{the rotation of the section} \end{array} \right.$

$$\Phi_3=\frac{d\bar{u}_2}{dx_1} \qquad \Phi_2=-\frac{d\bar{u}_3}{dx_1} \qquad (5.3)$$

consequence of the sign convention

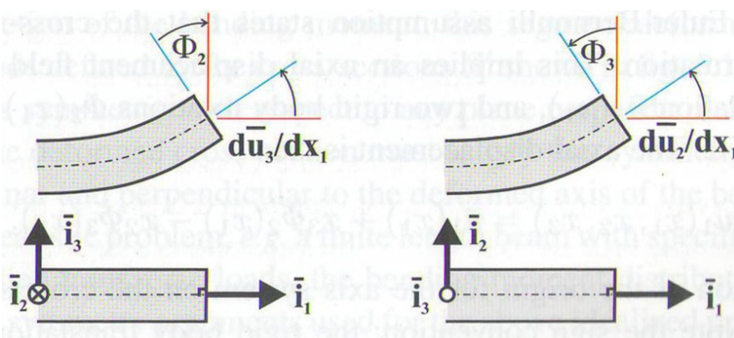


Fig. 5.4. Beam slope and cross-sectional rotation.

5.2 Implication of the E-B assumption

- To eliminate the sectional rotation from the axial displacement field

$$u_1(x_1,x_2,x_3)=\bar{u}_1(x_1)-x_3\frac{d\bar{u}_3(x_1)}{dx_1}-x_2\frac{d\bar{u}_2(x_1)}{dx_1} \qquad (5.4.a)$$

→ Important simplification of E-B : unknown displacements are functions of the span-wise coord, x_1 ,alone

❖ Strain field

$$\varepsilon_2=0,\ \ \varepsilon_3=0,\ \ \gamma_{23}=0 \qquad (5.5.a) \leftarrow \text{E-B(1)}$$

$$\gamma_{12}=0,\ \ \gamma_{13}=0 \qquad (5.5.b) \leftarrow \text{E-B(2)}$$

$$\varepsilon_1=\frac{\partial u_1}{\partial x_1}=\frac{d\bar{u}_1(x_1)}{dx_1}-x_3\frac{d^2\bar{u}_3(x_1)}{dx_1^{\,2}}-x_2\frac{d^2\bar{u}_2(x_1)}{dx_1^{\,2}} \qquad (5.5.c)$$

$$\bar{\varepsilon}_1(x_1)=\frac{d\bar{u}_1(x_1)}{dx_1},\qquad \kappa_2(x_1)=-\frac{d^2\bar{u}_3(x_1)}{dx_1^{\,2}},\quad \kappa_3(x_1)=\frac{d^2\bar{u}_2(x_1)}{dx_1^{\,2}}. \qquad (5.6)$$

Sectional axial strain   Sectional curvature about $\bar{i}_2,\bar{i}_3$ axes

➡ $$\varepsilon_1(x_1,x_2,x_3)=\bar{\varepsilon}_1(x_1)+x_3\kappa_2(x_1)-x_2\kappa_3(x_1) \qquad (5.7) \leftarrow \text{E-B(2)}$$

- Assuming a strain field of the form Eqs (5.5.a),(5.5.b),(5.7)

  → Math. Expression of the E-B assumptions

5.3 Stress resultants

❖ **3-D stress field ⇨ described in terms of sectional stresses called "stress resultants"**

→ equipollent to specified components of the stress field

➢ 3 force resultants $\begin{cases} N_1(x_1) & \text{axial force} \\ V_2(x_1), V_3(x_1) & \text{transverse shearing forces} \end{cases}$

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.8)$$

$$V_2(x_1) = \int_A \tau_{12}(x_1, x_2, x_3)\, dA, \quad V_3(x_1) = \int_A \tau_{13}(x_1, x_2, x_3)\, dA \qquad (5.9)$$

➢ 2 moment resultants : $M_2(x_1), M_3(x_1)$ bending moments

$$M_2(x_1) = \int_A x_3 \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.10a)$$

$$M_3(x_1) = -\int_A x_2 \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.10b)$$

(+) equipollent bending moment about $\bar{i}_3$ (Fig 5.5)

bending moments computed about point $P(x_{2p}, x_{3p})$

$$M_2^{\,p}(x_1) = \int_A (x_3 - x_{3p}) \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.11a)$$

$$M_3^{\,p}(x_1) = \int_A (x_2 - x_{2p}) \sigma_1(x_1, x_2, x_3)\, dA \qquad (5.11b)$$

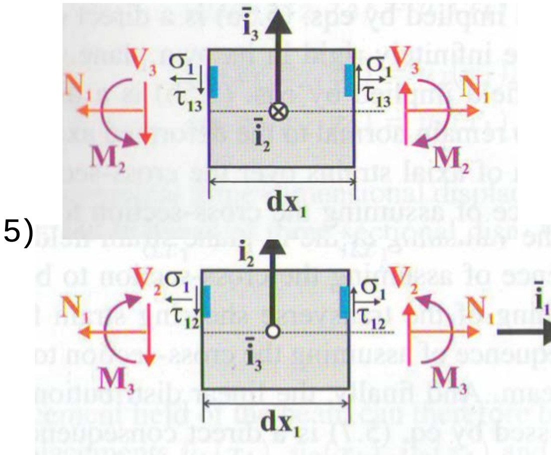


Fig. 5.5. Sign convention for the sectional stress resultants

5.4 Beams subjected to axial loads

❖ **Distributed axial load $p_1(x_1)$ [N/m], concentrated axial load P_1[N]**

→ axial displacement field $\bar{u}_1(x_1)$ ➡ 'bar' rather than 'beam'

5.4.1 Kinematic description

❖ **Axial loads causes only axial displacement of the section**

Eq.(5.4) → $u_1(x_1, x_2, x_3) = \bar{u}_1(x_1)$ (5.12a) → uniform over the x-s (Fig. 5.7)

$u_2(x_1, x_2, x_3) = 0$ (5.12b)

$u_3(x_1, x_2, x_3) = 0$ (5.12c)

Axial strain field $\varepsilon_1(x_1, x_2, x_3) = \bar{\varepsilon}_1(x_1)$ (5.13)

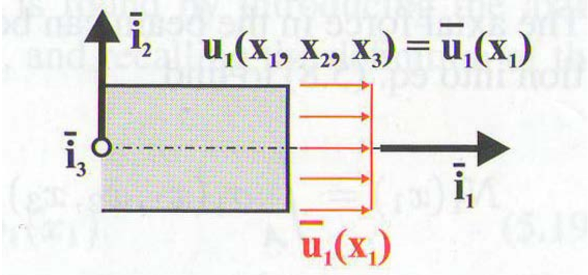


Fig. 5.7. Axial displacement distribution.

5.4.2 Sectional constitutive law

- **$\sigma_2 << \sigma_1$, $\sigma_3 << \sigma_2$ ⇨ transverse stress components ≈ 0, $\sigma_2 \approx 0$, $\sigma_3 \approx 0$**
  - Generalized Hooke's law → $\sigma_1(x_1, x_2, x_3) = E\varepsilon_1(x_1, x_2, x_3)$ (5.14)

    At the "infinitesimal" level

- **Inconsistency in E-B beam theory**

  Eq. (5.5a) → $\varepsilon_2 = 0, \varepsilon_3 = 0$

  Hooke's law → if $\sigma_2 = \sigma_3 = 0$, then $\varepsilon_2 = -\nu\sigma_1 / E, \quad \varepsilon_3 = -\nu\sigma_1 / E$

  (Poisson's effect) → very small effect, and assumed to vanish

  Eq.(5.13) → (5.14) : $\sigma_1(x_1, x_2, x_3) = E\varepsilon_1(x_1, x_2, x_3)$ (5.15)

  - Axial force

$$N_1(x_1) = \int_A \sigma_1(x_1, x_2, x_3)\, dA = \left[\int_A E dA\right] \bar{\varepsilon}_1(x_1) = S\bar{\varepsilon}(x_1) \quad (5.16)$$

Axial stiffness

$S = EA$ for homogeneous material

→ constitutive law for the axial behavior of the beam at the sectional level

5.4.3 Equilibrium eqns

❖ **Fig. 5.8** → infinitesimal slice of the beam of length dx_1

force equilibrium in axial dir. → $$\frac{dN_1}{dx_1} = -p_1 \qquad (5.18)$$

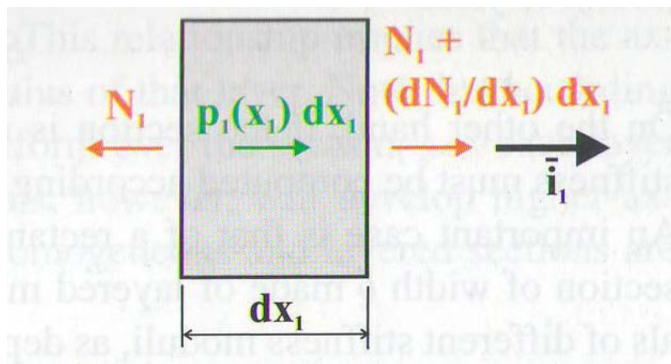


Fig. 5.8. Axial forces acting on an infinitesimal slice of the beam.

Eq. (1.4) → equilibrium condition for a differential element of a 3-D solid

Eq. (5.18) → equilibrium condition of a slice of the beam of differential length dx_1

5.4 Beams subjected to axial loads

5.4.4 Governing eqns

❖ **Eq (5.16) Eq. (5.18) and using Eq. (5.6)**

$$\frac{d}{dx_1}\left[S\frac{d\bar{u}_1}{dx_1}\right] = -p_1(x_1) \qquad (5.19)$$

❖ **3 B.C**

① Fixed(clamped) : $\bar{u}_1 = 0$

② Free (unloaded) : $N_1 = 0 \rightarrow \dfrac{d\bar{u}_1}{dx_1} = 0$

③ Subjected to a concentrated load P_1 : $N_1 = P_1 \rightarrow S\dfrac{d\bar{u}_1}{dx_1} = P_1$

5.4.5 The sectional axial stiffness

❖ **Homogeneous material**

$$S = EA \qquad (5.20)$$

❖ **Rectangular section of width b made of layered material of different moduli(Fig. 5.9)**

$$S = \int_A E dA = \sum_{i=1}^{n} E^{[i]} \int_{A^{[i]}} dA^{[i]} = \sum_{i=1}^{n} E^{[i]} b\left(x_3^{[i+1]} - x_3^{[i]}\right)$$

weighting factor thickness

"weighted average" of the Young's modulus

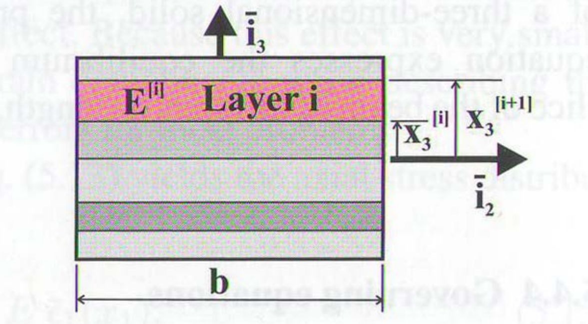


Fig. 5.9. Cross-section of a beam with various layered materials.

5.4.6 The axial stress distribution

❖ **Eliminating the axial strain form Eq.(5.15) and (5.16)**

$$\sigma_1(x_1, x_2, x_3) = \frac{E}{S} N_1(x_1) \tag{5.21}$$

➢ Homogeneous material

$$\sigma_1(x_1, x_2, x_3) = \frac{N_1(x_1)}{A} \tag{5.22}$$

→ Uniformly distributed over the section

➢ Sections made of layers presenting different moduli

$$\sigma_1^{[i]}(x_1, x_2, x_3) = E^{[i]} \frac{N_1(x_1)}{S} \tag{5.23}$$

→ Stress in layer I is proportional to the modulus of the layer

5.4 Beams subjected to axial loads

❖ **Eq (5.13) ➡ axial strain distribution is uniform over the section, i.e. each layer is equally strained (Fig. 5.10)**

➢ Strength criterion

$$\frac{E}{S}\left|N_{1\max}^{tens}\right| \le \sigma_{allow}^{tens}, \quad \frac{E}{S}\left|N_{1\max}^{comp}\right| \le \sigma_{allow}^{comp} \tag{5.24}$$

in case compressive, buckling failure mode may occur → Chap. 14

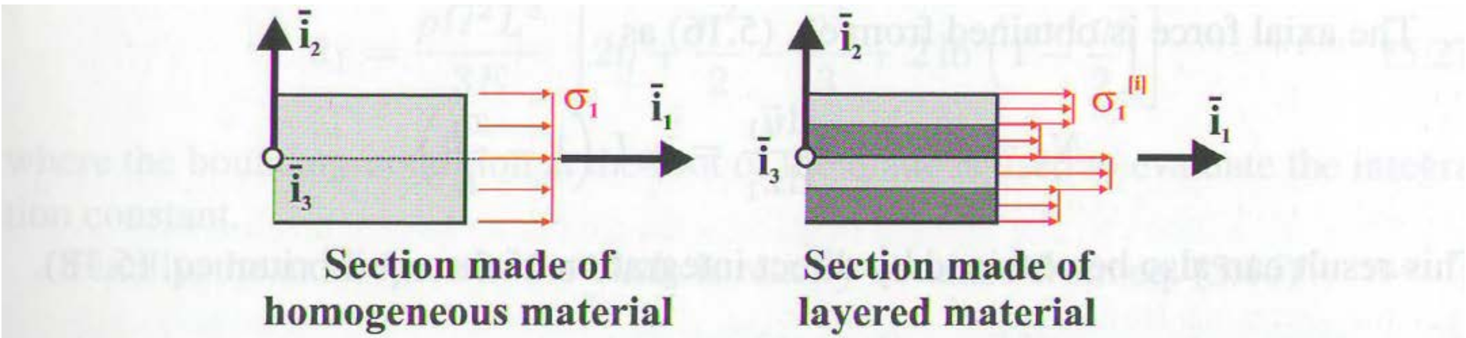


Fig. 5.10. Axial stress distribution for sections made of homogeneous and layered materials

5.5 Beams subjected to transverse loads

❖ **Fig. 5.14** → "transverse direction" distributed load, $p_2(x_1)$ [N/m]

concentrated load, P_2 [N]

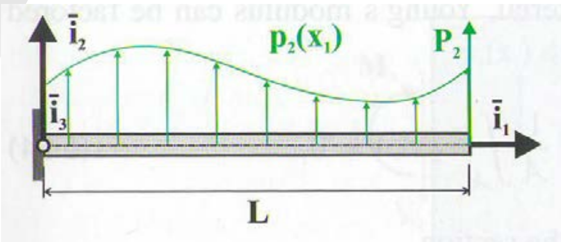


Fig. 5.14. Beam subjected to transverse loads.

➡ bending moments transverse shear forces,

and $\left\{ \begin{array}{l} \text{axial} \\ \text{transverse shearing} \end{array} \right\}$ stresses will be generated

5.5.1 Kinematic description

❖ **Assumption** → transverse loads only cause $\left\{ \begin{array}{l} \text{transverse displacement} \\ \text{curvature of the section} \end{array} \right.$

➢ General displacement field (Eq.(5.6))

$$u_1(x_1, x_2, x_3) = -x_2 \frac{d\bar{u}_1(x_1)}{dx} \quad (5.29a)$$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1) \quad (5.29b)$$

$$u_3(x_1, x_2, x_3) = 0 \quad (5.29c)$$

➡

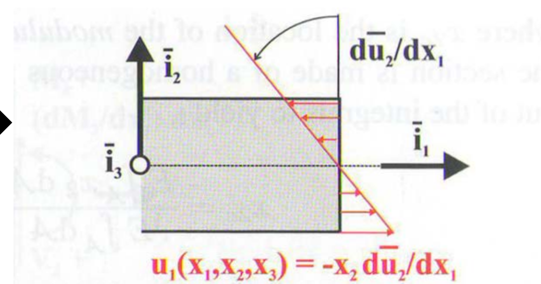


Fig. 5.15. Axial displacement distribution on cross-section.

→ linear distribution of the axial displacement component over the x-s

5.5 Beams subjected to transverse loads

- Only non-vanishing strain component

$$\varepsilon_1\left(x_1,x_2,x_3\right)=-x_2\kappa_3(x_1) \quad (5.36)$$ → linear distribution of the axial strain

5.5.2 Sectional constitutive law

- Linearly elastic material, axial stress distribution

$$\sigma_1\left(x_1,x_2,x_3\right)=-Ex_2\kappa_3(x_1) \qquad (5.31)$$

- Sectional axial force by Eq. (5.8)

$$N_1(x_1)=\int_A \sigma_1\left(x_1,x_2,x_3\right)dA=-\left[\int_A Ex_2 dA\right]\kappa_3(x_1) \qquad (5.32)$$

- Axial force = 0 since subjected to transverse loads only

$$\kappa_3 \neq 0, \quad then \quad \left[\int_A Ex_2 dA\right]=0$$

$$\Rightarrow \quad x_{2c}=\frac{1}{S}\int_A Ex_2 dA=\frac{S_2}{S}=0 \qquad (5.33)$$

Location of the “modulus-weighted centroid” of the x-s

5.5 Beams subjected to transverse loads

- If homogeneous material

$$x_{2c} = \frac{E\int_A x_2 dA}{E\int_A dA} = \frac{1}{A}\int_A x_2 dA = 0 \qquad (5.34)$$

$\rightarrow x_2$ is simply the area center of the section

➡ The axis system is located at the $\begin{cases} \text{modulus-weighted centroid} \\ \text{area center if homogeneous material} \end{cases}$

↑ center of mass – 3 coincide

✓ Center of mass $\quad x_{2nn} = \dfrac{\rho\int_A x_2 dA}{\rho\int_A dA} = \dfrac{\int_A x_2 dA}{\int_A dA} = x_{2c}$

- Bending moment by Eq.(5.31)

$$M_3(x_1) = \left[\int_A E x_2{}^2 dA\right]\kappa_3(x_1) = H_{33}^c \kappa_3(x_1) \qquad (5.35)$$

↑ "centroid bending stiffness" about axis $\bar{i}_3$

Constitutive law for the bending behavior of the beam bending moment $\propto$ the curvature

↑ Bending stiffness ("flexural rigidity")

➡ $M_1(x_1) = H_{33}^c \kappa_3(x_1) \qquad (5.37)$

"moment-curvature" relationship

5.5 Beams subjected to transverse loads

5.5.3 Equilibrium eqns

❖ **Fig. 5.16** → infinitesimal slice of the beam of length dx_1

$M_3(x_1), V_2(x_1)$ acting at a face at location x_1

@ $x_1 + dx_1$, evaluated using a Taylor series expansion, and H.O terms ignore

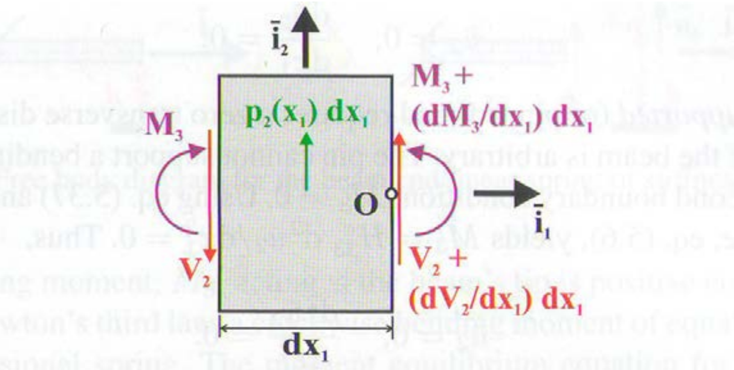


Fig. 5.16. Equilibrium of an infinitesimal slice of the beam.

➡ 2 equilibrium eqns

- vertical force → $\dfrac{dV_2}{dx_1} = -p_2(x_1)$ (5.38a)
- moment about O → $\dfrac{dM_3}{dx_1} + V_2 = 0$ (5.38b)

→ $\dfrac{d^2M_3}{dx_1^{\,2}} = p_2(x_1)$ (5.39)

5.5.4 Governing eqns

- **Eq. (5.37) Eq. (5.39), and recalling Eq. (5.6)**

$$\frac{d^2}{dx_1^{\,2}}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^{\,2}}\right] = p_2(x_1) \qquad (5.40)$$

4th order DE

- 4 B.C
  - ① Clamped end $\bar{u}_2 = 0, \quad \frac{d\bar{u}_2}{dx_1} = 0$
  - ② Simply supported(pinned) $\bar{u}_2 = 0, \quad \frac{d^2\bar{u}_1}{dx_1^{\,2}} = 0$
  - ③ Free(or unloaded) end $\frac{d^2\bar{u}_2}{dx_1^{\,2}} = 0, \quad -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^{\,2}}\right] = 0$
  - ④ End subjected to a concentrated transverse load P_2 $\quad P_2 = V_2 = -\frac{dM_3}{dx_1}$

$$\frac{d^2\bar{u}_2}{dx_1^{\,2}} = 0, \quad -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^{\,2}}\right] = P_2$$

5.5 Beams subjected to transverse loads

⑤ Rectilinear spring(Fig. 5.17) $\underset{\uparrow}{-}V_2(L) = k\bar{u}_2(L)$

sign convention

$$\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^{\ 2}}\right]_{x_1=L} \underset{\uparrow}{-} k\bar{u}_2(L) = 0, \qquad \frac{d^2\bar{u}_2}{dx_1^{\ 2}} = 0$$

(+) when the spring is located at the left end

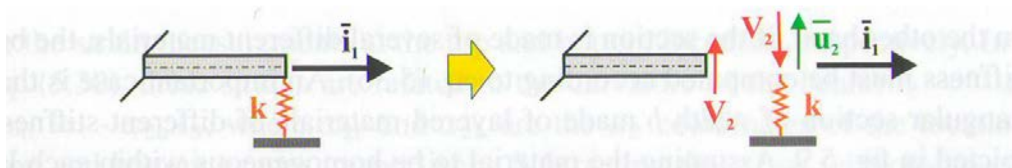


Fig. 5.17. Free body diagram for the beam end linear spring of stiffness constant k.

⑥ Rotational spring(Fig. 5.18) $-M_3(L) = k\Phi_3(L)$

$$\left. H_{33}^c \frac{d^2\bar{u}_2}{dx_1^{\ 2}}\right|_{x_1=L} \underset{\uparrow}{+} k\frac{d^2\bar{u}_2}{dx_1^{\ 2}} = 0, \qquad -\frac{d}{dx_1}\left[H_{33}^c \frac{d^2\bar{u}_2}{dx_1^{\ 2}}\right]_{x_1=L} = 0$$

(-) when at the left end

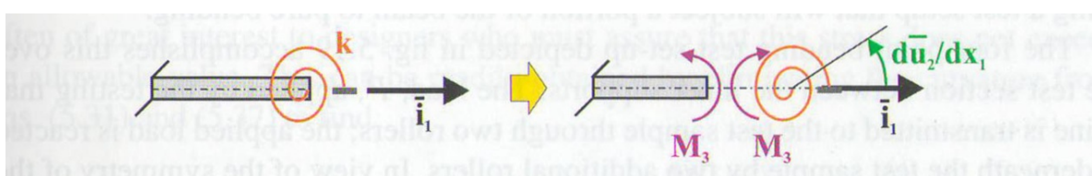


Fig. 5.18. Free body diagram for a beam with end rotational spring of stiffness constant k.

5.5.5 The sectional bending stiffness

- **Homogeneous material**

$$H_{33}^{c} = EI^{c}{}_{33} \tag{5.41}$$

$$I^{c}{}_{33} = \int_A x_2{}^2 dA \tag{5.42}$$

: purely geometric quantity, the area second moment of the section computed about the area center

- Rectangular section of width b made of layered materials (Fig. 5.9)

$$H_{33}^{c} = \int_A E x_2{}^2 dA = \sum_{i=1}^{n} E^{[i]} \int_{A^{[i]}} x_2{}^2 dA^{[i]} = \frac{b}{3} \sum_{i=1}^{n} E^{[i]} \left[(x_2{}^{[i+1]})^3 - (x_2{}^{[i]})^3 \right] \tag{5.43}$$

"weighted average" of the Young's moduli

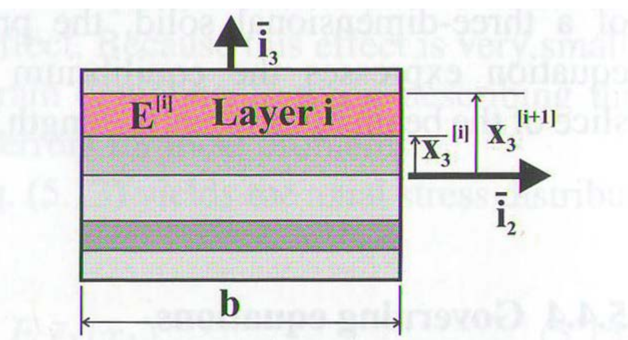


Fig. 5.9. Cross-section of a beam with various layered materials.

5.5 Beams subjected to transverse loads

5.5.6 The axial stress distribution

- **Local axial stress** → eliminating the curvature from Eq.(5.3), (5.37)

$$\sigma_1(x_1,x_2,x_3) = -Ex_2\frac{M_3(x_1)}{H_{33}^c} \qquad (5.44)$$

- homogeneous material

$$\sigma_1(x_1,x_2,x_3) = -x_2\frac{M_3(x_1)}{I_{33}} \qquad (5.45)$$

→ linearly distributed over the section, independent of Young's modulus

- various layer of materials

$$\sigma_1^{[i]}(x_1,x_2,x_3) = -E^{[i]}x_2\frac{M_3(x_1)}{H_{33}^c} \qquad (5.46)$$

→ axial STRAIN distribution is linear over the section ← Eq.(5.30)
axial stress distribution → piecewise linear (Fig. 5.20)

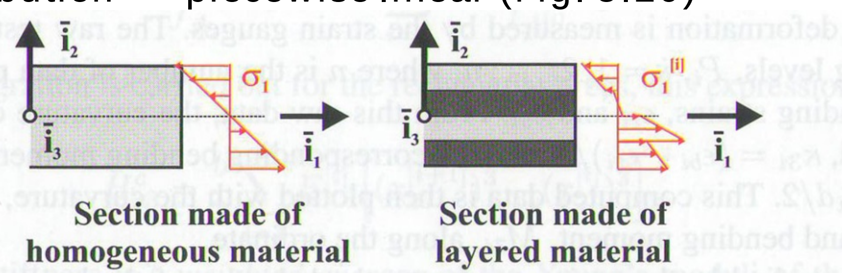


Fig. 5.20. Axial stress distributions in homogeneous and layered sections.

❖ **Strength criterion**

$$\frac{\left|x_2^{\max}\right|}{H_{33}^c} E\left|M_3^{\max}\right| \le \sigma_{allow}^{comp}, \quad \frac{\left|x_2^{\max}\right|}{H_{33}^c} E\left|M_3^{\max}\right| \le \sigma_{allow}^{ten},$$

Maximum (+) bending moment in the beam

➢ Layers of various material

→ must be computed at the $\left\{ \begin{matrix} \text{top} \\ \text{bottom} \end{matrix} \right\}$ locations of each ply

5.5.7 Rational design of beams under bending

❖ **"Neutral axis"** → along axis $\bar{i}_3$ which passes through the section's centroid

➢ Material located near the N.A carries almost no stress

➢ Material located near the N.A contributes little to the bending stiffness

➡ Rational design → removal of the material located at and near the N.A and relocation away from that axis

5.5 Beams subjected to transverse loads

❖ **Fig. 5.21** → { rectangular, ideal } section, same mass $m = bh\rho$

↑ a thin web would be used to keep the 2 flanges

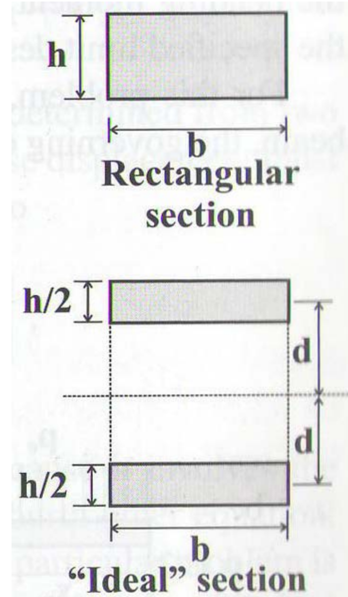


Fig. 5.21. A rectangular section, and the ideal section.

➢ Ratio of bending stiffness

$$\frac{H_{ideal}}{H_{rect}} = \frac{E \bullet 2\left[\frac{b(h/2)^2}{12} + \frac{bh}{2}d^2\right]}{E\frac{bh^3}{12}} = \frac{1}{4} + 12\left(\frac{d}{h}\right)^2$$

For $d/h = 10,\quad \frac{H_{ideal}}{H_{rect}} \cong 1200$

➢ Ratio of max. axial stress

$$\frac{\sigma_{ideal}}{\sigma_{rect}} = \frac{E\frac{h}{2}M_3 I_{ideal}}{I_{ideal}E\left(d + \frac{h}{4}\right)M_3} = \frac{\frac{1}{4} + 12\left(\frac{d}{h}\right)^2}{\frac{1}{2} + 2\left(\frac{d}{h}\right)}$$

For $d/h = 0,\quad \frac{\sigma_{rect}^{max}}{\sigma_{ideal}^{max}} \cong 6(d/h) = 60$

→ ideal section can carry a 60 times larger bending moment

➢ Ideal section ➡ "I beam," but prone to instabilities of web and flange buckling

5.6 Beams subjected to combined and transverse loads

- Sec. 5.4, 5.5 → convenient to locate the origin of the axes system at the centroid of the beam's x-s

5.6.1 Kinematic description

location of centroid

$$u_1\left(x_1,x_2,x_3\right)=\bar{u}_1(x_1)-(x_2-x_{2C})\frac{d\bar{u}_2(x_1)}{dx_1} \qquad (5.73a)$$

$$u_2\left(x_1,x_2,x_3\right)=\bar{u}_2(x_1) \qquad (5.73b)$$

$$u_3\left(x_1,x_2,x_3\right)=0 \qquad (5.73c)$$

- Strain field

$$\varepsilon_1\left(x_1,x_2,x_3\right)=\bar{\varepsilon}_1(x_1)-(x_2-x_{2C})\kappa_3(x_1) \qquad (5.74)$$

5.6 Beams subjected to combined and transverse loads

5.6.2 Sectional constitutive law

❖ Axial stress distribution

$$\sigma_1(x_1, x_2, x_3) = E\bar{\varepsilon}_1(x_1) - E(x_2 - x_{2C})\kappa_3(x_1) \qquad (5.75)$$

➢ Axial force

$$N_1 = \int_A \left[E\bar{\varepsilon}_1(x_1) - E(x_2 - x_{2C})\kappa_3(x_1)\right] dA$$

$$= \underbrace{\left[\int_A E dA\right]}_{S \text{ (axial stiffness)}} \bar{\varepsilon}_1(x_1) + \underbrace{\left[\int_A E(x_2 - x_{2C}) dA\right]}_{= \int_A E x_2 dA - x_{2C}\int_A E dA = S_2 - S x_{2C} = 0} \kappa_3(x_1)$$

➡ $N_1 = S\bar{\varepsilon}_1$

➢ Bending moment

$$M_3{}^C = -\int_A (x_2 - x_{2C})\left[E\bar{\varepsilon}_1(x_1) - E(x_2 - x_{2C})\kappa_3(x_1)\right] dA$$

$$= -\underbrace{\left[\int_A E(x_2 - x_{2C}) dA\right]}_{=0} \bar{\varepsilon}_1(x_1) + \underbrace{\left[\int_A E(x_2 - x_{2C})^2 dA\right]}_{H^c{}_{33} \text{ (bending stiffness)}} \kappa_3(x_1)$$

➡ $M_3{}^C = H_{33}^c \kappa_3$

➡ "decoupled sectional constitutive law"

2 crucial steps {
① Displacement field must be in the form of Eq.(5.73)
② Bending moment must be evaluated w.r.t. the centroid

→ Thus, centroid plays a crucial rule

5.6.3 Equilibrium eqns

❖ **Fig. 5.47** → infinitesimal slice of the beam of length dx_1

- Force equilibrium in horizontal dir. $$\frac{dN_1}{dx_1} = -p_1$$ ➡ (5.18)
- Vertical equilibrium $$\frac{dV_2}{dx_1} = -p_2$$ ➡ (5.38a)
- Equilibrium of moments about the centroid

$$\frac{dM_3}{dx_1} + V_2 = \underline{(x_{2a} - x_{2C})}\, p_1 \qquad (5.77)$$

Moment arm of the axial load w.r.t the centroid

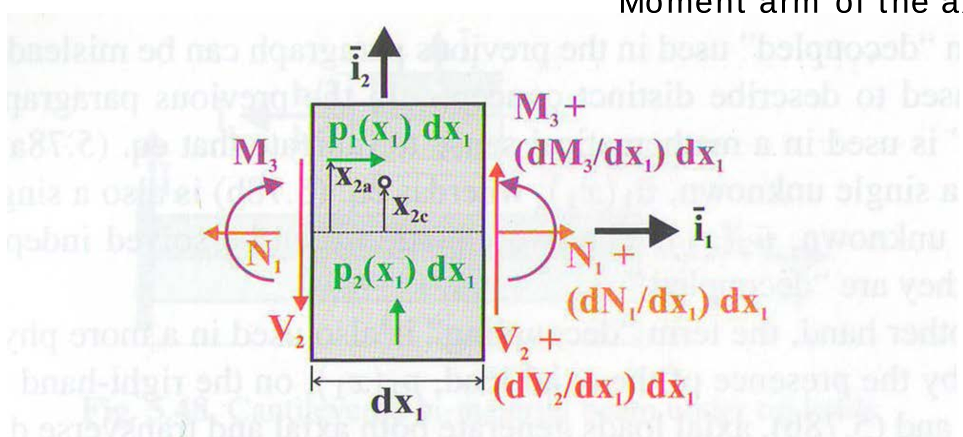


Fig. 5.47. Axial forces acting on an infinitesimal slice of the beam.

5.6 Beams subjected to combined and transverse loads

5.6.4 Governing eqns

$$\begin{cases} \dfrac{d}{dx_1}\left[S \dfrac{d\bar{u}_1}{dx_1} \right] = -p_1(x_1) \\ \dfrac{d^2}{dx_1{}^2}\left[H_{33}^c \dfrac{d^2\bar{u}_2}{dx_1{}^2} \right] = p_2(x_1) + \underline{\dfrac{d}{dx_1}\left[(x_{2a} - x_{2C}) p_1(x_1) \right]} \end{cases}$$

(5.78a) ➡ (5.19)

(5.78b) ➡ almost similar to (5.19) concept

→ "decoupled" eqns $\left\{ \begin{array}{l} (5.78a) \rightarrow \bar{u}_1(x_1) \\ (5.78b) \rightarrow \bar{u}_2(x_1) \end{array} \right\}$ can be independently solved

{ If axial loads are applied @ centroid, extension and bending are "decoupled"
If axial loads are **not** applied @ centroid, extension and bending are "coupled"