Aeroelasticity

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Dynamic Aeroelasticity

Dynamic aeroelasticity

- Two principal phenomena

- Dynamic instability (flutter)
- Responses to dynamic load, or modified by aeroelastic effects

- Flutter ⋯ self-excited vibration of a structure arising from the interaction of aerodynamic elastic and internal loads

  "response" ⋯ forced vibration

  "Energy source" ⋯ flight vehicle speed

- Typical aircraft problems

- Flutter of wing
- Flutter of control surface
- Flutter of panel

Dynamic aeroelasticity

- Stability concept

If solution of dynamic system may be written or

$$y(x,t)=\sum_{k=1}^{N}\bar{y}_k(x)\,e^{(\sigma_k+i\omega_k)t}$$

a) $\sigma_k<0,\omega_k\neq 0\Rightarrow$ Convergent solution : “stable”

b) $\sigma_k=0,\omega_k\neq 0\Rightarrow$ Simple harmonic oscillation : ”stability boundary”

c) $\sigma_k>0,\omega_k\neq 0\Rightarrow$ Divergence oscillation : “unstable”

d) $\sigma_k<0,\omega_k=0\Rightarrow$ Continuous convergence : “stable”

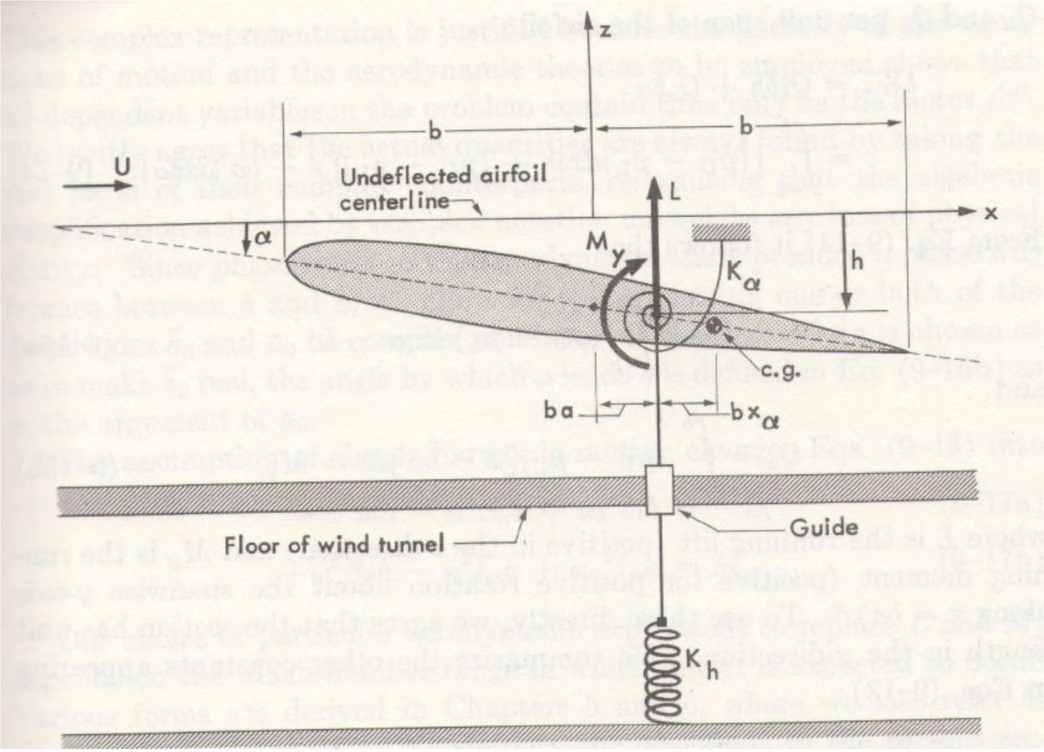
e) $\sigma_k=0,\omega_k=0\Rightarrow$ Time independent solution : ”stability boundary”

f) $\sigma_k>0,\omega_k=0\Rightarrow$ Continuous divergence : “unstable”

Dynamic aeroelasticity

- Flutter of a wing

Typical section with 2 D.O.F



K_α, K_h : torsional, bending stiffness

Dynamic aeroelasticity

- First step in flutter analysis

- Formulate eqns of motion
- The vertical displacement at any point along the mean aerodynamic chord from the equilibrium $z=0$ will be taken as $z_a(x,t)$

$$z_a(x,t) = -h - (x - x_{ea})\alpha$$

- The eqns of motion can be derived using Lagrange's eqn

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$$L = T - U$$

Dynamic aeroelasticity

- The total kinetic energy(T)

$$T = \frac{1}{2}\int_{-b}^{b} \rho \left(\frac{\partial z_a}{\partial t}\right)^2 dx$$

$$= \frac{1}{2}\int_{-b}^{b} \rho \left[\dot{h} + (x - x_{ea})\dot{\alpha}\right]^2 dx$$

$$= \frac{1}{2}\dot{h}^2 \underbrace{\int_{-b}^{b} \rho dx}_{\substack{m \\ \text{(airfoil mass)}}} + \dot{h}\dot{\alpha} \underbrace{\int_{-b}^{b} \rho (x - x_{ea}) dx}_{\substack{S_\alpha \\ \text{(static unbalance)}}} + \frac{1}{2}\dot{\alpha}^2 \underbrace{\int_{-b}^{b} (x - x_{ea})^2 dx}_{\substack{I_\alpha \\ \text{(mass moment of inertia} \\ \text{about c.g.)}}}$$

*Note) if $x_{ea} = x_{cg}$, then $S_\alpha = 0$ by the definition of c.g.

Therefore,

$$T = \frac{1}{2} m\dot{h}^2 + \frac{1}{2} I\dot{\alpha}^2 + S_\alpha \dot{h}\dot{\alpha}$$

Dynamic aeroelasticity

- The total potential energy (strain energy)

$$U = \frac{1}{2} k_h h^2 + \frac{1}{2} k_\alpha \alpha^2$$

- Using Lagrange's eqns with $L = T - U$

$$q_1 = h_1, q_2 = \alpha$$

$$\Rightarrow \begin{cases} m\ddot{h} + S_\alpha \ddot{\alpha} + k_h h = Q_h \\ S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + k_\alpha \alpha = Q_\alpha \end{cases}$$

Where Q_h, Q_α are generalized forces associated with two d.o.f's h, α respectively.

Dynamic aeroelasticity

$$Q_h = -L = -L(\alpha, h, \dot{\alpha}, \dot{h}, \ddot{\alpha}, \ddot{h}, \cdots)$$

$$Q_\alpha = M_{ea} = M_{ea}(\alpha, h, \dot{\alpha}, \dot{h}, \ddot{\alpha}, \ddot{h}, \cdots)$$

Governing eqn.

$$\Rightarrow \begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha \end{bmatrix} \begin{Bmatrix} h \\ \alpha \end{Bmatrix} = \begin{Bmatrix} -L \\ M_{ea} \end{Bmatrix}$$

- For approximation, let's use quasi-steady aerodynamics

$$L = qSC_{L_\alpha}(\alpha + \frac{\dot{h}}{U_\infty})$$

$$M_{ac} = qS_c C_{m_{\dot{\alpha}}} \dot{\alpha}$$

$$M_{ea} = (x_{ea} - x_{ac}) \bullet L + M_{ac} = eqSC_{L_\alpha}(\alpha + \frac{\dot{h}}{U_\infty}) + qS_c C_{m_{\dot{\alpha}}} \dot{\alpha}$$

Dynamic aeroelasticity

*Note) Three basic classifications of unsteadiness (linearized potential flow)

i) Quasi-steady aero: only circulatory terms due to the bound vorticity. Used for characteristic freq. below $2Hz$ (e.g., conventional dynamic stability analysis)

ii) Quasi-unsteady aero: includes circulatory terms from both bound and wake vorticities. Satisfactory results for $2Hz < \omega_\alpha, \omega_h < 10Hz$. Theodorsen is one that falls into here. (without apparent mass terms)

iii) Unsteady aero: "quasi-unsteady"+"apparent mass terms" (non-circulatory terms, inertial reactions: $\dot{\alpha}$, $\ddot{h}$) For $\omega > 10Hz$, for conventional aircraft at subsonic speed.

Dynamic aeroelasticity

Then, aeroelastic systems of equations becomes

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix}\begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix}+\begin{bmatrix} \dfrac{qSC_{L_\alpha}}{U_\infty} & 0 \\ -\dfrac{qSeC_{L_\alpha}}{U_\infty} & -qS_cC_{m_{\dot{\alpha}}} \end{bmatrix}\begin{Bmatrix} \dot{h} \\ \dot{\alpha} \end{Bmatrix}+\begin{bmatrix} K_h & qSC_{L_\alpha} \\ 0 & K_\alpha - qSeC_{L_\alpha} \end{bmatrix}\begin{Bmatrix} h \\ \alpha \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- For stability, we can obtain characteristic eqn. of the system and analyze the roots.

neglect damping matrix for first,

$$\begin{bmatrix} m & S_\alpha \\ S_\alpha & I_\alpha \end{bmatrix}\begin{Bmatrix} \ddot{h} \\ \ddot{\alpha} \end{Bmatrix}+\begin{bmatrix} K_h & qSC_{L_\alpha} \\ 0 & K_\alpha - qSeC_{L_\alpha} \end{bmatrix}\begin{Bmatrix} h \\ \alpha \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Much insight can be obtained by looking at the undamped system (Dowell, pp. 83)

Dynamic aeroelasticity

Set $\alpha = \bar{\alpha} e^{pt}, h = \bar{h} e^{pt}$

$$\Rightarrow \begin{bmatrix} (mp^2 + K_h) & (S_\alpha p^2 + qSC_{L\alpha}) \\ S_\alpha p^2 & (I_\alpha p^2 + K_\alpha - qSeC_{L_\alpha}) \end{bmatrix} \begin{Bmatrix} \bar{h} \\ \bar{\alpha} \end{Bmatrix} e^{pt} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For non-trivial solution,

Characteristic eqn., $\det(\Delta) = 0$

$$\underbrace{(mI_\alpha - S_\alpha)}_{A} p^4 + \underbrace{[K_h I_\alpha + (K_\alpha - qSeC_{L_\alpha})m - qSC_{L\alpha} S_\alpha]}_{B} p^2 + \underbrace{K_h (K_\alpha - qSeC_{L_\alpha})}_{C} = 0$$

$$\therefore p^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Dynamic aeroelasticity

The signs of *A*, *B*, *C* determine the nature of the solution.

$A > 0, C > 0$ (if $q < q_D$)

B Either (+) or (-)

$$B = mK_\alpha + K_h I_\alpha - [me + S_\alpha] qSC_{L_\alpha}$$

- If $[me + S_\alpha] < 0, B > 0$ for all q
- Otherwise $B < 0$ when

$$\frac{K_\alpha}{e} + \frac{K_h I_\alpha}{me} - \left[1 + \frac{S_\alpha}{me}\right] qSeC_{L_\alpha} < 0$$

Dynamic aeroelasticity

- Two possibilities for *B* (*B*>0 and *B*<0)

i) *B*>0:

① $B^2 - 4AC > 0, p^2$ are real, negative, so *p* is pure imaginary → neutrally stable

② $B^2 - 4AC < 0, p^2$ is complex, at least one value should have a positive real part → unstable

③ $B^2 - 4AC = 0$ → stability boundary

- Calculation of q_F

$Dq_F^2 + Eq_F + F = 0$ ←(from $B^2 - 4AC = 0,$ stability boundary)

$$q_F = \frac{-E \pm \sqrt{E^2 - 4DF}}{2D}$$

Dynamic aeroelasticity

where,

$$D \equiv \left\{ \left[me + S_\alpha \right] S C_{L_\alpha} \right\}^2$$

$$E \equiv \left\{ -2\left[me + S_\alpha \right]\left[mK_\alpha + K_h I_\alpha \right] + 4\left[mI_\alpha - S_\alpha^2 \right] eK_h \right\} S C_{L_\alpha}$$

$$F \equiv \left[mK_\alpha + K_h I_\alpha \right]^2 - 4\left[mI_\alpha - S_\alpha^2 \right] K_h K_\alpha$$

① At least, one of the q_F must be real and positive in order for flutter to occur.

② If both are, the smaller is the more critical.

③ If neither are, flutter does not occur.

④ If $S_\alpha \le 0$ (c.g. is ahead of e.a), no flutter occurs(mass balanced)

Dynamic aeroelasticity

ii) $B<0$: B will become (-) only for large q

$B^2 - 4AC = 0$ will occur before B=0 since A>0,C>0

$\therefore$ To determine q_F, only B>0 need to be calculated.

Examine p as q increases

Low $q \rightarrow p = \pm i\omega_1, \pm i\omega_2 (B^2 - 4AC > 0)$

Higher $q \rightarrow p = \pm i\omega_1, \pm i\omega_2 (B^2 - 4AC = 0) \rightarrow$ stability boundary

More higher $q \rightarrow p = -\sigma_1 \pm i\omega_1, \sigma_2 \pm i\omega_2 (B^2 - 4AC < 0) \rightarrow$

dynamic instability

Even more higher $q \rightarrow p = 0, \pm i\omega_1 (C = 0) \rightarrow$ stability boundary

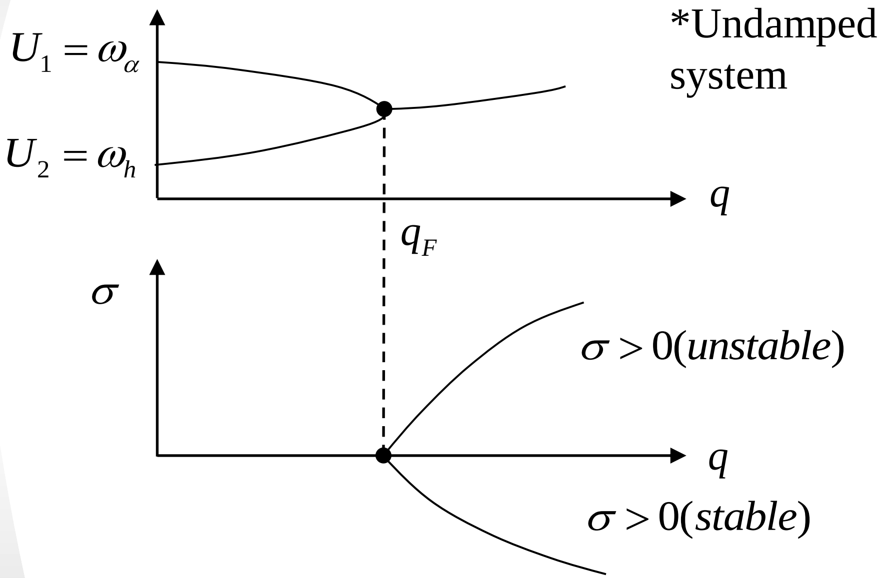
$\therefore$ Flutter condition: $B^2 - 4AC = 0$

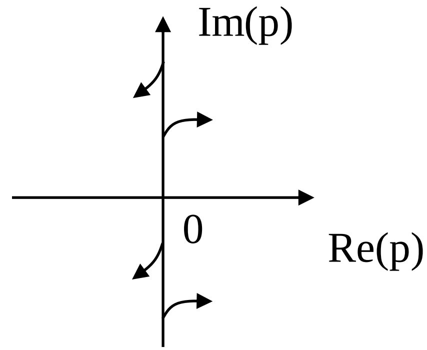
Torsional divergence: $C = 0$

Dynamic aeroelasticity

Graphically,

$$\omega_\alpha^2 = \frac{K_\alpha}{I_\alpha}, \omega_h^2 = \frac{K_h}{m}$$





- Effect of static unbalance

In Dowell's book, after Pines[1958]

$S_\alpha \leq 0 \rightarrow$ avoid flutter, if $S_\alpha = 0, \dfrac{q_F}{q_D} = 1 - \dfrac{\omega_h^2}{\omega_\alpha^2}$

Dynamic aeroelasticity

If $q_D < 0 (e < 0)$ $\dfrac{\omega_h}{\omega_\alpha} < 1.0 \Rightarrow q_F < 0$ no flutter

If $q_D > 0$ and $\dfrac{\omega_h}{\omega_\alpha} > 1.0 \Rightarrow$ no flutter

- Inclusion of damping→ "can be a negative damping"

for better accuracy,

$m\ddot{q} + c\dot{q} + Kq = 0,$ where

$$c = \begin{bmatrix} \dfrac{qSC_{L_\alpha}}{U_\infty} & 0 \\ -\dfrac{qSC_{L_\alpha}}{U_\infty} & -qScC_{m_{\dot{\alpha}}} \end{bmatrix}$$

The characteristic equation is now in the form of

$$A_4 p^4 + A_3 p^3 + A_2 p^2 + A_1 p + A_0 = 0$$

Dynamic aeroelasticity

$$A_4 p^4 + A_3 p^3 + A_2 p^2 + A_1 p + A_0 = 0 \cdots *$$

- Routh criteria for stability

; At critical position, the system real part becomes zero, damping becomes zero.

Substitute $p = i\omega$ into (*), we get,

$$\begin{cases} A_4\omega^4 - A_2\omega^2 + A_0 = 0 \\ i(-A_3\omega^3 + A_1\omega) = 0 \end{cases}$$

From the second eqn, $\omega^2 = \frac{A_1}{A_3}$, substitute into first equation, then,

$$A_4\left(\frac{A_1}{A_3}\right)^2 - A_2\left(\frac{A_1}{A_3}\right) + A_0 = 0 \quad \text{or} \quad A_4A_1^2 - A_1A_2A_3 + A_0A_3^2 = 0$$

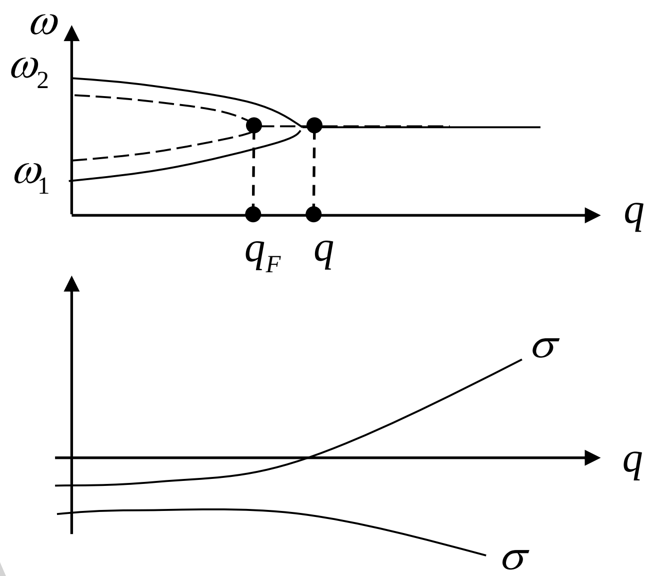
Dynamic aeroelasticity

And, we can examine p as q increases,

Low $q \rightarrow p = -\sigma_1 \pm i\omega_1, -\sigma_2 \pm i\omega_2 \rightarrow$ damped natural freq.

Higher $q \rightarrow p = -\sigma_1 \pm i\omega_1, \pm i\omega_2$

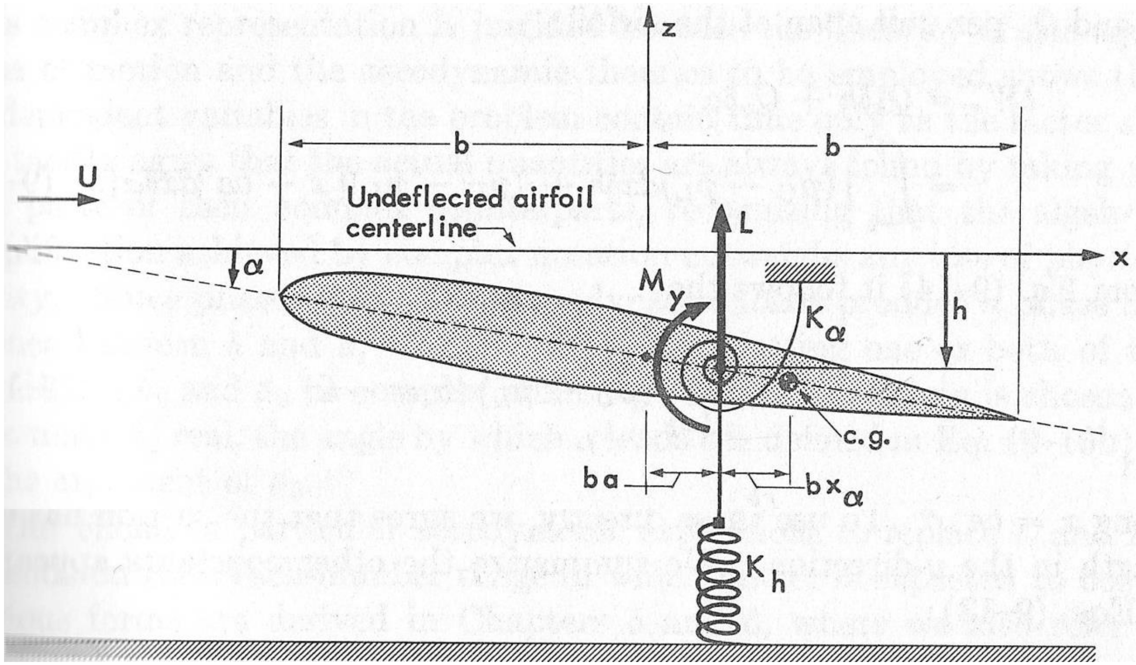
More higher $q \rightarrow p = -\sigma_1 \pm i\omega_1, \pm\sigma_2 \pm i\omega_2 \rightarrow$ dynamic instability.



- Static instability $\cdots |K| = 0$
- Dynamic instability
  a) frequency coalescence (unsymmetric K)
  b) Negative damping $(C_{ij} < 0)$
  c) Unsymmetric damping (gyroscopic)

Straight Aircraft Wing

Consider disturbance from equilibrium



Using modal method, the displacement $\left(w_{ea}\right)$ and rotation $\left(\theta_{ea}\right)$ at elastic axis can be expressed as

$$\begin{cases} w_{ea} = \sum_{r=1}^{N} h_r(y) q_r(t) \\ \theta_{ea} = \sum_{r=1}^{N} \alpha_r(y) q_r(t) \end{cases}$$

where

$q_r(t)$: generalized (modal) coordinate

$h_r(y), \alpha_r(y)$: mode shape

N: total number of modes

Straight Aircraft Wing

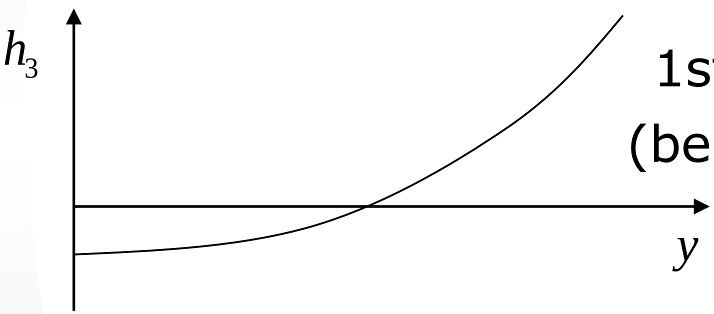
For $N = 4$,

$a)\ h_1 = 1, \alpha_1 = 0:$ rigid translation mode $(\omega_1 = 0)$

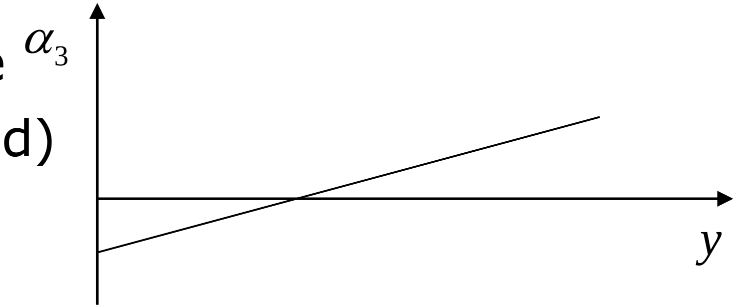
$b)\ h_2 = x_0, \alpha_2 = 0:$ rigid pitch mode about c.g. $(\omega_2 = 0)$

$c)\ h_3(y), \alpha_3(y):$ 1st bending of wing $(\omega_3 \neq 0)$

$d)\ h_4(y), \alpha_4(y):$ 1st torsion of wing $(\omega_4 \neq 0)$



1st bending mode
(bending dominated)



Modes can be assumed, or calculated from mass-spring representation. The displacements and rotations at any point

$$w(x,y,t) = w_{ea} + (x - x_0)\theta_{ea} = \sum_{r=1}^{N}\left[h_r + (x - x_0)\alpha_r\right]q_r(t)$$

$$\theta(x,y,t) = \theta_{ea} = \sum_{r=1}^{N}\alpha_r q_r(t)$$

Straight Aircraft Wing

The kinetic energy (T) is

$$T = \frac{1}{2}\iint_{\frac{1}{2}aircraft} m\left(\dot{w}\right)^2 dxdy$$

$$= \frac{1}{2}\iint m\sum_{r=1}^{N}\left[h_r + \left(x - x_0\right)\alpha_r\right]\dot{q}_r \sum_{s=1}^{N}\left[h_s + \left(x - x_0\right)\alpha_s\right]\dot{q}_s dxdy$$

$$= \frac{1}{2}\sum_{r=1}^{N}\sum_{s=1}^{N} m_{rs}\dot{q}_r\dot{q}_s$$

where, $m_{rs} = \int_0^l \left[Mh_r h_s + I_\alpha \alpha_r \alpha_s + S_\alpha\left(h_r\alpha_s + h_s\alpha_r\right)\right]dy$

$M = \int_{LE}^{TE} mdx$: mass/unit span

$S_\alpha = \int_{LE}^{TE}\left(x - x_0\right)mdx$: static unblance/unit span

$I_\alpha = \int_{LE}^{TE}\left(x - x_0\right)^2 mdx$: moment of inertia about E.A./unit span

Straight Aircraft Wing

The potential energy (U) is

$$U = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w_{ea}}{\partial y^2}\right)^2 dy + \frac{1}{2}\int_0^l GJ\left(\frac{\partial \theta_{ea}}{\partial y}\right)^2 dy$$

$$= \frac{1}{2}\int_0^l EI\sum_{r=1}^{N} h_r'' q_r \sum_{s=1}^{N} h_s'' q_s dy + \frac{1}{2}\int_0^l GJ\sum_{r=1}^{N} \alpha_r' q_r \sum_{s=1}^{N} \alpha_s' q_s dy$$

$$= \frac{1}{2}\sum_{r=1}^{N}\sum_{s=1}^{N} K_{rs} q_r q_s$$

where, $K_{rs} = \int_0^l EI h_r'' h_s'' dy + \int_0^l GJ \alpha_r' \alpha' dy$

[Note] $K_{rs} = 0$ for rigid modes 1,2, since $h_1'' = h_2'' = 0$ and $\alpha_1' = \alpha_2' = 0$

Straight Aircraft Wing

Finally, the work done by airloads,

$$\delta W = -\int_0^l L_{ea}\delta w_{ea} dy + \int_0^l M_{ea}\delta\theta_{ea} dy - L_{HT}\delta w_{HT} + M_{HT}\delta\theta_{HT} = \sum_{r=1}^{N} Q_r \delta q_r$$

subscript HT : horizontal tail contribution (rigid fuselage assumption)

where, $Q_r = \int_0^l \left(-h_r L_{ea} + \alpha_r M_{ea}\right) dy - h_{r(HT)} L_{HT} + \alpha_{r(HT)} M_{HT}$

[Note] $r = 1 \rightarrow Q_1 = -\int_0^l L_{ea} dy - L_{HT} = -\frac{1}{2} L_{Total}$

$r = 2 \rightarrow Q_2 = \frac{1}{2} M_{Total} (C.G)$

place $T, U,$ and Q_r into the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} + \frac{\partial U}{\partial q_r} = Q_r$$

yield the equation of motion

Straight Aircraft Wing

Equation of motion in matrix form

$$[m_{rs}]\{\ddot{q}_r\}+\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&K_{33}&K_{34}\\0&0&K_{43}&K_{44}\end{bmatrix}\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix}=\{Q_r\}$$

zeros are associated with rigid body modes

[Note] If we used normal modes, $w(x,y,t)=\sum_{r=1}^{N}\phi_r(x,y)q_r(t)$

free-free normal mode

The equation of motion would be uncoupled

$$[m_{rs}]\rightarrow\begin{bmatrix}\ddots&&\\&m_{rr}&\\&&\ddots\end{bmatrix},\quad [K_{rs}]\rightarrow\begin{bmatrix}\ddots&&\\&m_{rr}\omega_r^2&\\&&\ddots\end{bmatrix}$$

Straight Aircraft Wing

[Note] Free-free normal mode vs Uncoupled modes

from entire structures

$$M_r \ddot{q}_r + M_r \omega_r^2 q_r = Q_r$$

(more accurate)

for individual components

then, combine together by

Rayleigh-Ritz method,

$$\sum m_{rs} \ddot{q}_s + \sum k_{rs} q_s = 0$$

(more versatile)

Straight Aircraft Wing

Now, let's introduce the aerodynamic load by considering 2-D, incompressible, strip theory

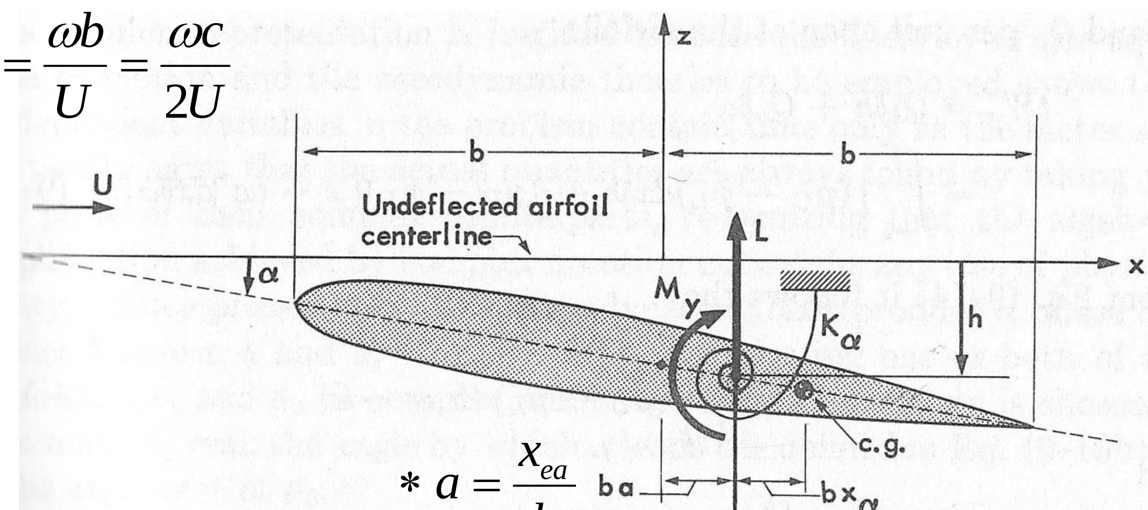
$$L_{ea} = \pi\rho b^2\left[\ddot{w}_{ea} + U\dot{\theta}_{ea} - ba\ddot{\theta}_{ea}\right] + 2\pi\rho UbC(k)\left[\dot{w}_{ea} + U\theta_{ea} - b\left(\tfrac{1}{2} - a\right)\theta_{ea}\right]$$

$$M_{ea} = \pi\rho b^3\left[a\ddot{w}_{ea} + U\left(\tfrac{1}{2} - a\right)\dot{\theta}_{ea} - b\left(\tfrac{1}{8} + a^2\right)\ddot{\theta}_{ea}\right]$$

$$+2\pi\rho Ub^2\left(\tfrac{1}{2} + a\right)C(k)\underbrace{\left[\dot{w}_{ea} + U\theta_{ea} - b\left(\tfrac{1}{2} - a\right)\theta_{ea}\right]}$$

lift deficiency fn. $\quad \frac{3}{4}c$ airspeed (downwash)

$$* \; k = \frac{\omega b}{U} = \frac{\omega c}{2U}$$



$$* \; a = \frac{x_{ea}}{b}$$

Unsteady Aeroelasticity

- **Unsteady Aeroelasticity in Incompressible Flow (B.A.H p.272 and B.A. p.119)**

  - For incompressible flow $\left(M << 1\right)$

    a separation can be made between circulatory and non-circulatory airloads

  - When the airfoil performs chordwise rigid motion.

    the circulatory lift depends only on the downwash at the $\frac{3}{4}c$ station

$$w_{\frac{3}{4}c} = \left[\dot{w}_{ea} + U\theta_{ea} - b\left(\tfrac{1}{2} - a\right)\theta_{ea}\right]: \text{ downwash at } \tfrac{3}{4}c$$

$$L_{ea} = \pi\rho b^2\left[\ddot{w}_{ea} + U\dot{\theta}_{ea} - ba\ddot{\theta}_{ea}\right] + 2\pi\rho UbC(k)\left[\dot{w}_{ea} + U\theta_{ea} - b\left(\tfrac{1}{2} - a\right)\theta_{ea}\right]$$

"always acts at $\frac{1}{4}c$ "

lift deficiency fn.

Unsteady Aeroelasticity

However, $\begin{cases} w_{ea} = \sum_s h_s q_s \\ \theta_{ea} = \sum_s \alpha_s q_s \end{cases}$

and placing these into L_{ea}, M_{ea} yields

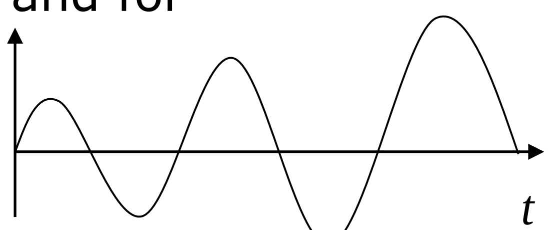
$$Q_r = \int_0^l \left(-h_r L_{ea} + \alpha_r M_{ea}\right) dy + H.O.T = Q_r\left(q_s, \dot{q}_s, \ddot{q}_s\right)$$

coupled set of homogeneous differential equations.

For stability analysis, assume $q_r(t) = \bar{q}_r e^{pt}$

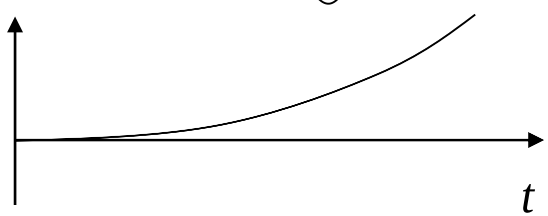
where $p = \sigma + i\omega$, and for

$a) + \sigma, \omega \neq 0$



"flutter"

$b) + \sigma, \omega = 0$



"divergence"