Tensor components

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ be an orthonormal basis to the Euclidean three-dimensional vector space $\mathcal{V}$. Let $\boldsymbol{T}$ be a second-order tensor from $\mathcal{V}$ to $\mathcal{V}$. Then, we have

$$\boldsymbol{T}\boldsymbol{u} = \boldsymbol{v}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are in $\mathcal{V}$. Note that $\boldsymbol{u}$ and $\boldsymbol{v}$ are linear combination of basis vectors as they are in $\mathcal{V}$. Choosing $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ for $\boldsymbol{u}$ in Eq. , we get

$$\begin{aligned}\boldsymbol{T}\boldsymbol{e}_1 &= \alpha_1\boldsymbol{e}_1 + \alpha_2\boldsymbol{e}_2 + \alpha_3\boldsymbol{e}_3\\ \boldsymbol{T}\boldsymbol{e}_2 &= \alpha_4\boldsymbol{e}_1 + \alpha_5\boldsymbol{e}_2 + \alpha_6\boldsymbol{e}_3\\ \boldsymbol{T}\boldsymbol{e}_3 &= \alpha_7\boldsymbol{e}_1 + \alpha_8\boldsymbol{e}_2 + \alpha_9\boldsymbol{e}_3,\end{aligned}$$

where α_i, $i = 1$ to 9, are scalars. Renaming α_i as $\boldsymbol{T}_{ij}$, $i = 1, 2, 3$; $j = 1, 2, 3$, we get

$$\boldsymbol{T}\boldsymbol{e}_j = T_{ij}\boldsymbol{e}_i.$$

Clearly, i is dummy index and j is the free index. The scalars T_{ij} are called components of the tensor $\boldsymbol{T}$ with respect to the base vectors $\boldsymbol{e}_i$, $i = 1, 2, 3$. Taking the dot product both sides of Eq. with $\boldsymbol{e}_k$, we get

$$\boldsymbol{e}_k \cdot \boldsymbol{T}\boldsymbol{e}_j = T_{ij}\boldsymbol{e}_k \cdot \boldsymbol{e}_i = T_{ij}\delta_{ik} = T_{kj}$$

Replacing k with i, we can write

$$T_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{T}\boldsymbol{e}_j.$$