Capacity of Wireless Channels

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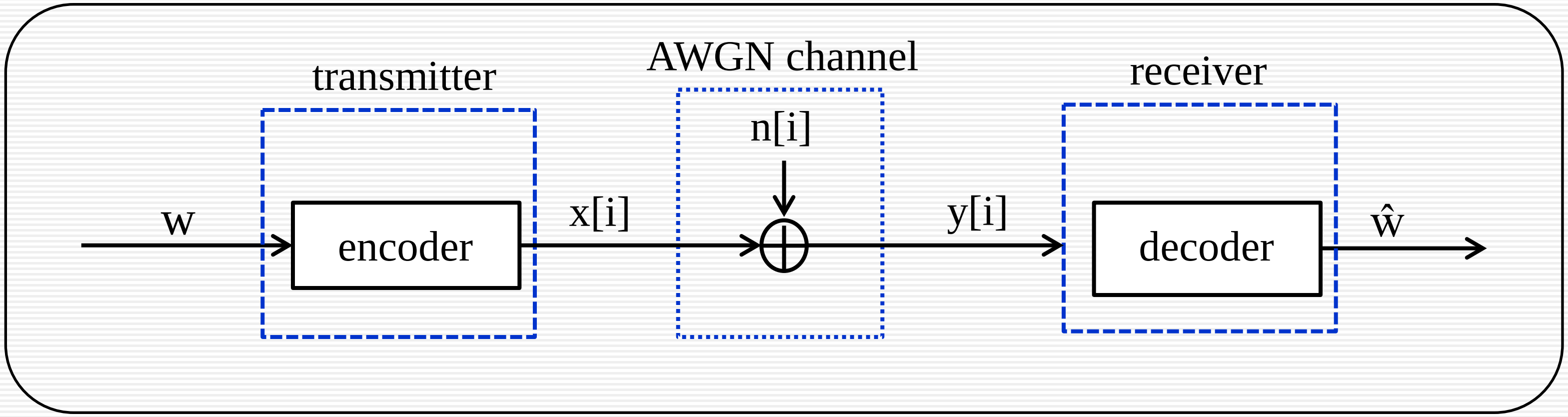
Introduction

- Channel capacity limit
  - The maximum channel rates that can be transmitted over the wireless channel with asymptotically small error probability, assuming no constraints on the delay or complexity of the encoder/decoder
- Scope
  - Capacity of a single-user wireless channel where the transmitter and/or receiver has a single antenna
    - a time-invariant additive white Gaussian Noise (AWGN) channel
    - a flat fading channel
    - a frequency selective fading channel

Capacity of AWGN Channel

Capacity in AWGN



- Shannon Capacity
  - $C = B\log_2(1+\gamma)$
  - $\gamma = P / N_0 B$
    - Received signal-to-noise ratio (SNR)
    - P : the transmitted signal power
    - N_0B: Noise power
  - Upper bound on the data rates that can be achieved under the real system constraints

Capacity of a discrete channel

- Mutual information
  - The average amount of information received over the channel per symbol
  - $\mathrm{I}(X;Y) = \mathrm{H}(X) - \mathrm{H}(X \mid Y)$
    - H(X): the average amount of information transmitted per symbol (entropy)
    - H(X|Y): the average uncertainty about a transmitted symbol when a symbol is received, and the average amount of information lost over noisy channel per symbol
    - $\mathrm{H}(X) = \sum_{x \in S_X} p(x) \log \frac{1}{p(x)}, \quad \mathrm{H}(X \mid Y) = \sum_{x \in S_X, y \in S_Y} p(x,y) \log \frac{1}{p(x \mid y)}$
  - $\mathrm{I}(X;Y) = \sum_{x \in S_X} p(x) \log \frac{1}{p(x)} - \sum_{x \in S_X, y \in S_Y} p(x,y) \log \frac{1}{p(x \mid y)}$

A	B	C	D
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

A: 0 (1 bit)
B: 10 (2 bits)
C: 110 (3 bits)
D: 110 (3 bits)
$\log_2 P(E)$

Capacity of a discrete channel

I(X; Y) = H(X) - H(X | Y)

$$= \sum_X p(x) \log \frac{1}{p(x)} - \sum_X \sum_Y p(x, y) \log \frac{1}{p(x|y)}$$

$\sum_Y p(x,y)$

$$= \sum_X \sum_Y p(x, y) \left(\log \frac{1}{p(x)} - \log \frac{1}{p(x|y)} \right)$$

$$= \sum_X \sum_Y p(x, y) \log \frac{p(x|y)}{p(x)}$$

$\log \frac{p(x,y)}{p(x)\,p(y)} = \log \frac{p(y|x)}{p(y)}$

$$= \sum_X \sum_Y p(x, y) \left(\log \frac{1}{p(y)} - \log \frac{1}{p(y|x)} \right)$$

$$= \sum_Y p(y) \log \frac{1}{p(y)} - \sum_X \sum_Y p(x, y) \log \frac{1}{p(y|x)}$$

= H(Y) – H(Y | X)

Capacity of a Continuous Channel

- Entropy of X

$$\mathrm{H(X)} = \int_{-\infty}^{\infty} p(x) \log \frac{1}{p(x)} \, dx$$

- Mutual Information I(X;Y)

$$\mathrm{I(X;\,Y)} = \mathrm{H(X)} - \mathrm{H(X \mid Y)}$$

$$= \int_{-\infty}^{\infty} p(x) \log \frac{1}{p(x)} \, dx - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \log \frac{1}{p(x|y)} \, dxdy$$

$$= \int_{-\infty}^{\infty} p(y) \log \frac{1}{p(y)} \, dy - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x, y) \log \frac{1}{p(y|x)} \, dxdy$$

$$= \mathrm{H(Y)} - \mathrm{H(Y \mid X)}$$

Capacity of a Band-limited AWGN Channel (1)

- Channel capacity
  - Maximum amount of mutual information I(X;Y) per second
  - Two steps
    - the maximum mutual information per sample
    - 2B samples per second (Nyquist's sampling theory)
- Maximum mutual information per sample
  - x, n, y: samples of the transmitted signal, noise, and received signal
  - H(y|x)
    - $$\mathrm{H(y \mid x)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y) \log \frac{1}{p(y|x)}\, dxdy$$
      $$= \int_{-\infty}^{\infty} p(x) \int_{-\infty}^{\infty} p(y \mid x) \log \frac{1}{p(y|x)}\, dydx$$
    - Because y=x+n, for a given x, y is equal to n plus a constant. The distribution of y is identical to that of n except for a translation by x
    - $p(y \mid x) = p_n(y - x)$, where $p_n(\cdot)$ is the PDF of noise sample

Capacity of a Band-limited AWGN Channel (1)

- $$\mathrm{H}(\mathrm{y} \mid \mathrm{x}) = \int_{-\infty}^{\infty} p(x) \int_{-\infty}^{\infty} p(y \mid x) \log \frac{1}{p(y|x)}\, dydx$$
- $$\int_{-\infty}^{\infty} p(y \mid x) \log \frac{1}{p(y|x)}\, dy$$

$$= \int_{-\infty}^{\infty} p_n(y - x) \log \frac{1}{p_n(y-x)}\, dy \qquad [\text{since} \quad p(y \mid x) = p_n(y-x)]$$

$$= \int_{-\infty}^{\infty} p_n(z) \log \frac{1}{p_n(z)}\, dz = H(\mathrm{n})$$

- $$\mathrm{H}(\mathrm{y} \mid \mathrm{x}) = \int_{-\infty}^{\infty} \mathrm{H}(\mathrm{n}) p(x)\, dx$$

$$= \mathrm{H}(\mathrm{n}) \int_{-\infty}^{\infty} p(x)\, dx = \mathrm{H}(\mathrm{n})$$

- I(x;y)=H(y) - H(n)

Capacity of a Band-limited AWGN Channel (2)

- Entropy of a band-limited white Gaussian noise with PSD N_0
  - Noise power : $N_0 B$
  - $$p(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2\sigma^2}, \quad \text{where } \sigma^2 = \int_{-\infty}^{\infty} z^2 p(z)dz$$
  - $$\mathrm{H(n)} = \int_{-\infty}^{\infty} P(n) \log \frac{1}{p(n)} dz = \frac{1}{2} \log(2\pi e \sigma^2)$$
  - N is the noise power ($N = \sigma^2$)

$$\boxed{\mathrm{H(n)} = \frac{1}{2} \log(2\pi e N)}$$

Capacity of a Band-limited AWGN Channel (3)

- When the signal power is S and the noise power is N, and the signal $s(t)$ and noise $n(t)$ are independent, the mean square value of y is $E[y^2] = S + N$
- We should know the maximum of H(Y)
  - The maximum entropy is obtained when the distribution of Y is Gaussian for a given E[Y²]
  - $p(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2/2\sigma^2}$, where $\sigma^2 = \int_{-\infty}^{\infty} y^2 p(y)dy = \mathrm{E}[Y^2]$
  - $\mathrm{H}_{\max}(Y) = \frac{1}{2}\log(2\pi e\sigma^2)$

$$\boxed{\mathrm{H}_{\max}(\mathrm{Y}) = \frac{1}{2}\log(2\pi e(S+N))}$$

Capacity of a Band-limited AWGN Channel (4)

- $I_{max}(x;y) = H_{max}(y) - H(n)$

$$= \frac{1}{2}\log[2\pi e(S+N)] - \frac{1}{2}\log(2\pi e N)$$

$$= \frac{1}{2}\log(1 + {}^{S}\!/_{N})$$

- Channel capacity: $2 \times B \times I_{max}(x;y)$

$$\boxed{C = B\log(1+\frac{S}{N})}$$

- Reference :B. P. Lathi, *Modern Digital and Analog Communication System*, 3rd Ed., Oxford. (Chapter 15)

Capacity of Flat-Fading Channels

Capacity of Flat-Fading Channels

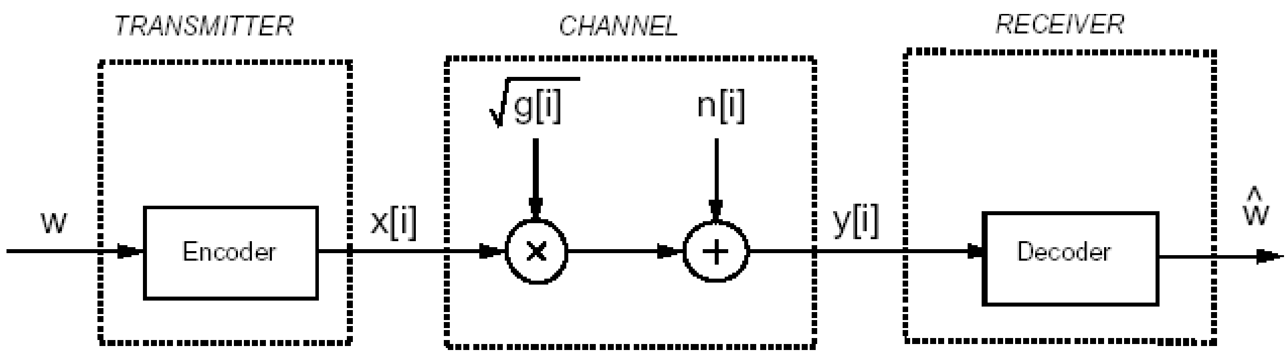


Figure 4.1: Flat-Fading Channel and System Model.

- The channel capacity depends on the information about $g[i]$
  - Channel side information (the value of $g[i]$) known to the receiver
  - If the receiver reports CSI to the transmitter, the transmitter can know CSI

CSI at Transmitter and Receiver (1)

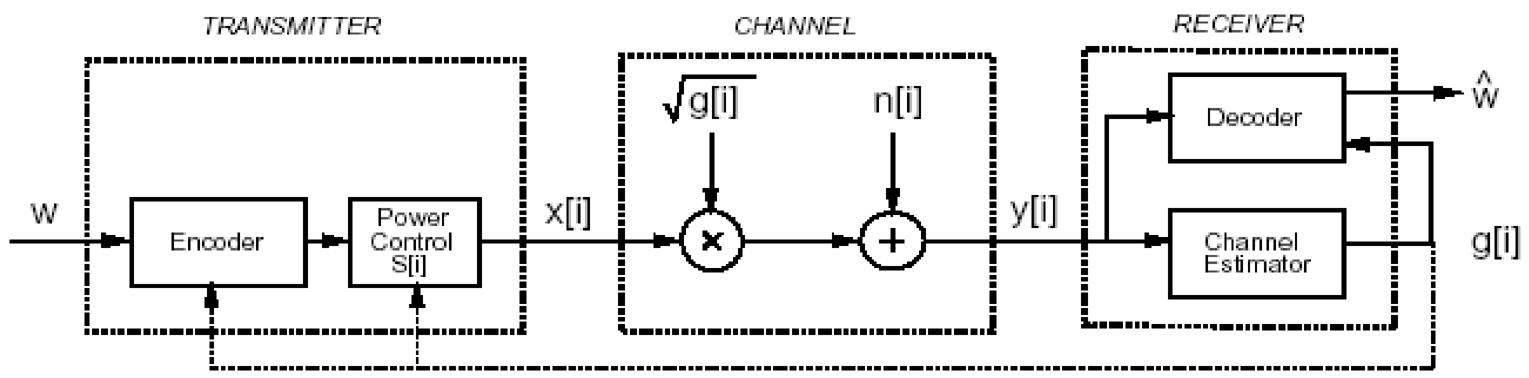


Figure 4.3: System Model with Transmitter and Receiver CSI.

- The transmitter may adjust the rate and/or power based on the reported CSI
- When the transmitter controls merely the rate with a fixed transmission power,

$$C = \int_0^\infty B \log_2(1+\gamma) f(\gamma) d\gamma$$

CSI at Transmitter and Receiver (2)

- Transmission power as well as rate can be adapted.
- Adaptation of transmission power $P_t(\gamma)$ to the received SNR γ subject to an average power constraint Φ
- average power constraint: $\int_0^\infty P_t(\gamma)\, f(\gamma)\, d\gamma \le \Phi$
- The (time varying) fading channel capacity with average power constraint

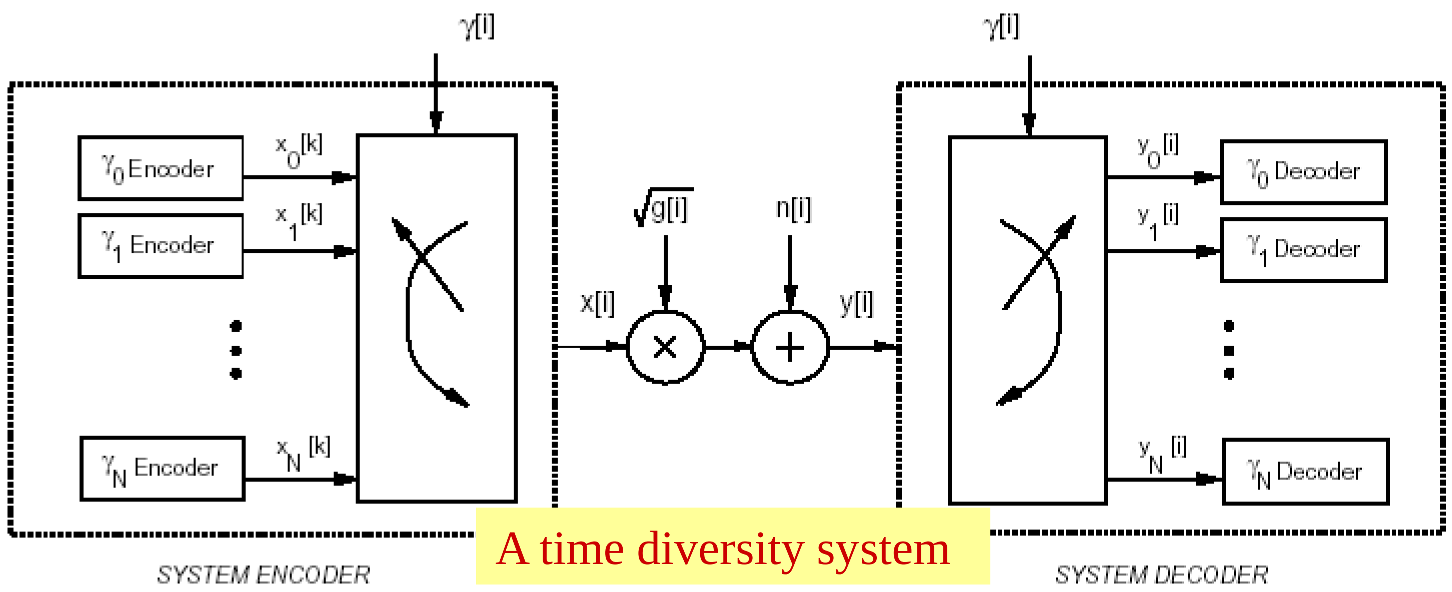
$$C = \max_{P_t(\gamma):\int P_t(\gamma)\, f(\gamma) d\gamma = \Phi} \int_0^\infty B \log_2\left(1 + \frac{P_t(\gamma)\gamma}{\Phi}\right) f(\gamma)\, d\gamma$$

* γ: the received SNR at transmission power Φ

$$\gamma = \frac{\Phi g}{N} \Rightarrow g = \frac{N\gamma}{\Phi}\;; \quad \frac{P_t(\gamma) g}{N} = \frac{P_t(\gamma)\gamma}{\Phi}$$

SNR for tx power $P_t(\gamma)$ by reported γ

CSI at Transmitter and Receiver (3)



A time diversity system

- The range of fading values is quantized to a finite set $\{\gamma_j : 1 \le j \le N\}$
- For each γ_j , an encoder-decoder pair for the AWGN channel with SNR $P_t(\gamma_j)\gamma_j/\Phi$
- The codeword from the corresponding encoder, x_j , is transmitted with power $P_t(\gamma_j)$ at data rate $C_j = B\log_2\left(1 + P_t(\gamma_j)\gamma_j/\Phi\right)$

CSI at Transmitter and Receiver (4)

$$\max \int_0^\infty B\log_2\left(1+\frac{P_t(\gamma)\gamma}{\Phi}\right) f(\gamma)d\gamma$$

$$\text{subject to} \int_0^\infty P_t(\gamma)f(\gamma)d\gamma \le \Phi$$

Original problem

- Optimal power allocation
  - Lagrangian

$$J(P_t(\gamma),\lambda)=\int_0^\infty B\log_2\left(1+\frac{P_t(\gamma)\gamma}{\Phi}\right) f(\gamma)\,d\gamma+\lambda\left[\Phi-\int_0^\infty P_t(\gamma)\,f(\gamma)\,d\gamma\right]$$

  - Differentiate the Lagrangian and set the derivate to zero

$$\frac{\partial J(P_t(\gamma),\lambda)}{\partial P_t(\gamma)}=\left[\left(\frac{B/\ln 2}{1+\gamma P_t(\gamma)/\Phi}\right)\frac{\gamma}{\Phi}-\lambda\right]f(\gamma)=0$$

  - Solve for $P_t(\gamma)$ with the constraint that $P_t(\gamma)>0$

$$\boxed{\frac{P_t(\gamma)}{\Phi}=\begin{cases}1/\gamma_0-1/\gamma & \gamma\ge\gamma_0\\ 0 & \gamma<\gamma_0\end{cases}}$$

$= 0$

$$\frac{P_t(\gamma)}{\Phi}=\frac{B}{\lambda\ln 2\,\Phi}-\frac{1}{\gamma}$$

$\frac{1}{\gamma_0}$

CSI at Transmitter and Receiver (5)

- Capacity

$$1 + \frac{P_t(\gamma)\gamma}{\Phi} = 1 + \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right)\gamma = \frac{\gamma}{\gamma_0}$$

$$\boxed{C = \int_{\gamma_0}^{\infty} B \log_2\left(\frac{\gamma}{\gamma_0}\right) f(\gamma)\, d\gamma} \quad \ldots (4.13)$$

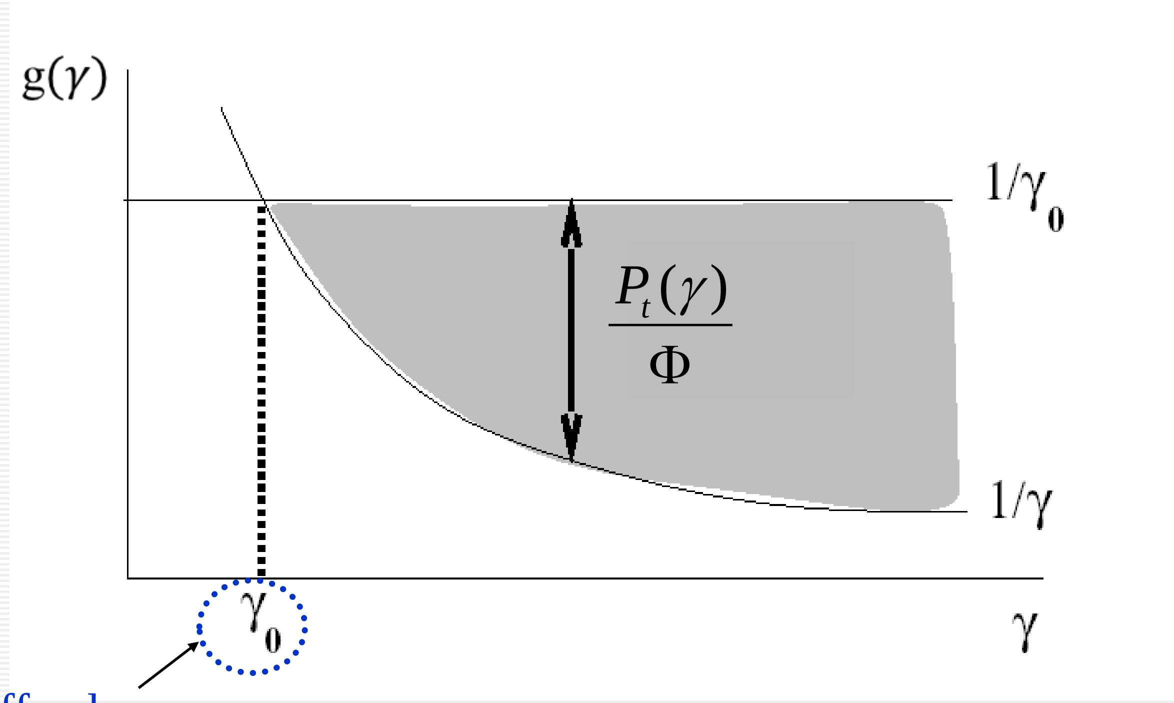
  - Time-varying data rate : the rate corresponding to the instantaneous SNR γ is $B\log_2(\gamma/\gamma_0)$
  - Transmission power adaption
    - Optimal power allocation (Water filling)

$$\frac{P_t(\gamma)}{\Phi} = \begin{cases} 1/\gamma_0 - 1/\gamma & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

CSI at Transmitter and Receiver (6)

- Water filling

$g(\gamma) = {}^{1}\!/_{\gamma}$



cutoff value

$$\boxed{\gamma_0 \text{ such that } \int_{\gamma_0}^{\infty} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma} \right) f(\gamma)\, d\gamma = 1}$$

the better channel, the more power and the higher data rate

CSI at Transmitter and Receiver (7)

- Channel inversion and zero outage
  - The transmitter controls the transmission power using CSI so as to maintain a constant received power (inverts the channel fading)
  - The channel appears to the encoder and decoder as a time-invariant AWGN channel
  - transmission power: $P_t(\gamma)/\Phi = \sigma/\gamma$
    - $\sigma = 1/\mathrm{E}[1/\gamma]$ from $\int_0^\infty (\sigma/\gamma)\, f(\gamma)\, d\gamma = 1$
  - Fading channel capacity with channel inversion is equal to the AWGN channel capacity with SNR σ

$$P_R = gP_t(\gamma) = \frac{N\gamma}{\Phi} P_t(\gamma)$$

$$\frac{P_t(\gamma)}{\Phi} = \frac{P_R}{N}\frac{1}{\gamma}$$

σ

$$C = B\log_2\left(1 + \frac{P_t(\gamma)\gamma}{\Phi}\right) = B\log_2\left(1 + \frac{\sigma\gamma}{\gamma}\right)$$

$$\boxed{C = B\log_2(1+\sigma) = B\log_2\left(1 + \frac{1}{\mathrm{E}[1/\gamma]}\right)} \quad \ldots.\ (4.18)$$

CSI at Transmitter and Receiver (8)

- Channel inversion and zero outage
  - A fixed data rate regardless of channel condition
  - One pair of encoder and decoder is designed for an AWGN channel with SNR σ and Tx power is adjusted as $P_t(\gamma) = \sigma\Phi/\gamma$

    => the simplest scheme to implement
  - zero outage:
    - Should maintain a constant data rate in all fading states
    - Zero outage capacity is significantly smaller than Shannon capacity on fading channel
      - In Rayleigh fading, the zero outage capacity is zero
  - Channel inversion is common in spread-spectrum system with near-far interference imbalances

CSI at Transmitter and Receiver (9)

- Truncated channel inversion
  - Suspending transmission in bad fading states
  - Truncated channel inversion
    - Power adaptation policy that compensates only for fading above a cutoff γ_0
    - $\frac{P_t(\gamma)}{\Phi} = \begin{cases} \sigma/\gamma & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$ where $\sigma = \left(\mathrm{E}_{\gamma_0}[1/\gamma]\right)^{-1} = \left(\int_{\gamma_0}^{\infty} \frac{1}{\gamma} f(\gamma)d\gamma\right)^{-1}$
    - Outage probability $P_{out} = p(\gamma < \gamma_0)$
    - Outage capacity for a given P_{out} and corresponding cutoff γ_0

  - $C(P_{out}) = B\log_2\left(1+\frac{1}{\mathrm{E}_{\gamma_0}[1/\gamma]}\right)p(\gamma \geq \gamma_0)$

    - Maximum outage capacity

$$\boxed{C = \max_{\gamma_0} B\log_2\left(1+\frac{1}{\mathrm{E}_{\gamma_0}[1/\gamma]}\right)p(\gamma \geq \gamma_0)} \quad \ldots. \ (4.22)$$

Capacity Comparison

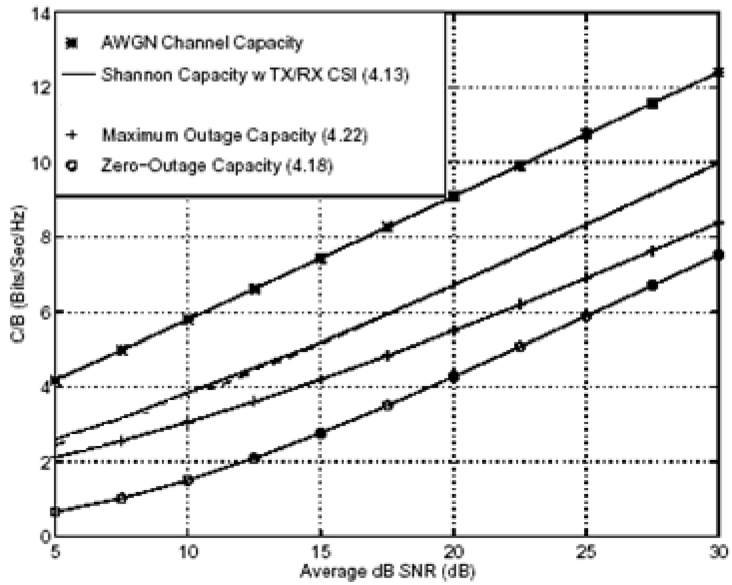


Figure 4.6: Capacity in Log-Normal Shadowing.

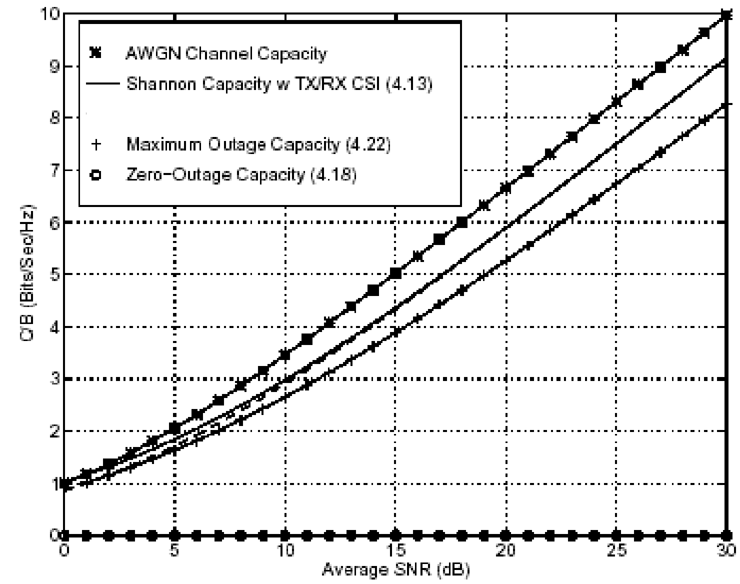


Figure 4.7: Capacity in Rayleigh Fading.

AWGN Channel Capacity: $C = \int_0^\infty B\log_2(1+\gamma)f(\gamma)d\gamma$

Shannon Capacity w Tx/Rx CSI: $C = \int_0^\infty B\log_2(\gamma/\gamma_0)f(\gamma)d\gamma$

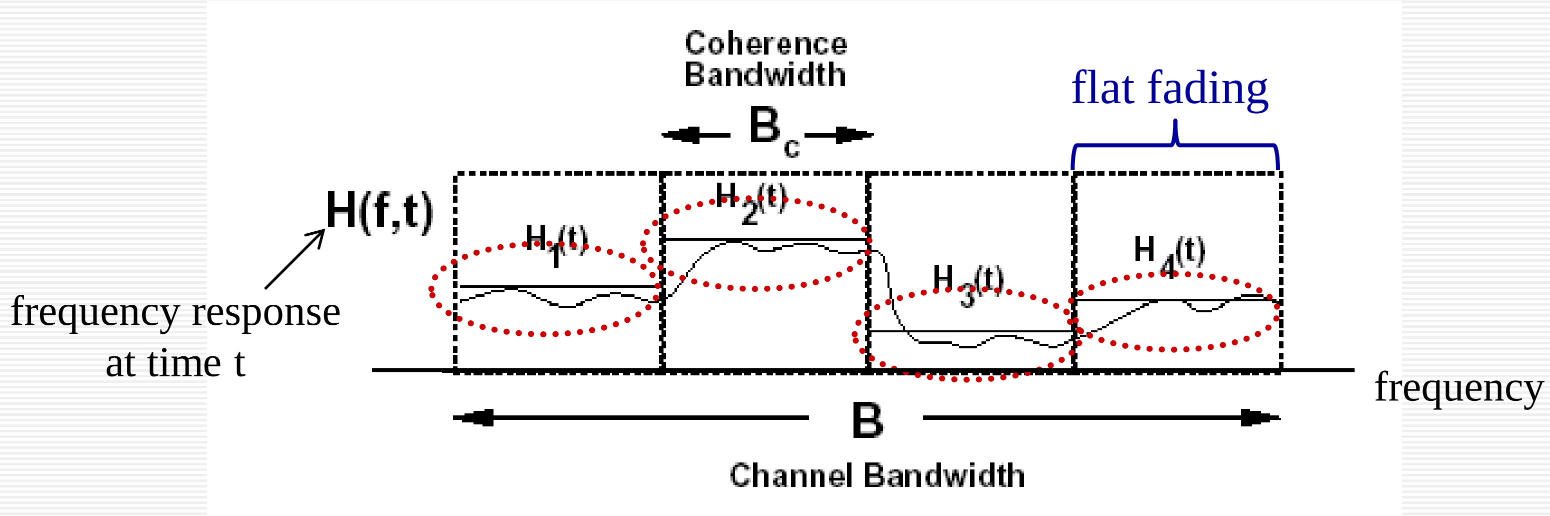
Zero-Outage Capacity: $C = B\log_2(1+1/E[1/\gamma])$

Zero-Outage Capacity: $C = \max_{\gamma_0} B\log_2\left(1+1/E_{\gamma_0}[1/\gamma]\right)p(\gamma \geq \gamma_0)$

Capacity of frequency-selective fading channel

Frequency Selective Fading Channel

Channel division into flat fading subchannel

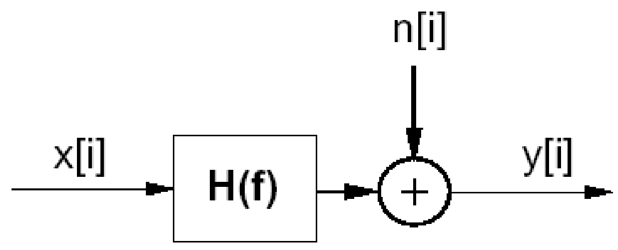


- Analysis is like that of flat fading subchannel with frequency axis

Time-invariant Channel (1)

- A time-invariant channel with frequency response $H(f)$ that is known to both transmitter and receiver
- Block fading
  - Frequency is divided into subchannels of bandwidth B_c with constant frequency response H_j over each subchannel
  - P_j: Tx power on the jth subchannel
  - A set of AWGN channels in parallel with SNR ($|H_j|^2 P_j / N_0 B_c$) on the jth channel
  - Total power constraint: $\sum_j P_j \le P$

total transmission power



Time-Invariant Frequency-Selective Fading Channel.

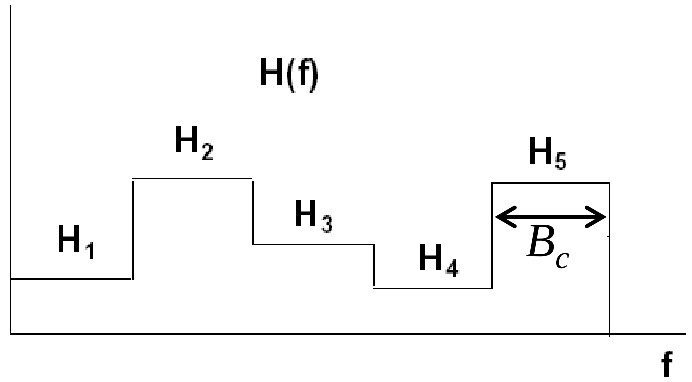


Figure 4.10: Block Frequency-Selective Fading

Time-invariant Channel (2)

- Capacity under block fading
  - $$C = \sum_{\max P_j : \sum_j P_j \le P} B_c \log_2\left(1 + \frac{|H_j|^2 P_j}{N_0 B_c}\right)$$
  - $$\frac{P_j}{P} = \begin{cases} 1/\gamma_0 - 1/\gamma_j & \gamma_j \ge \gamma_0 \\ 0 & \gamma_j < \gamma_0 \end{cases}$$
  - $$\gamma_j := \frac{|H_j|^2 P}{N_0 B_c}$$
  - $$\gamma_0 \text{ such that } \sum_j \left(1/\gamma_0 - 1/\gamma_j \right) = 1$$
  - $$\boxed{C = \sum_{j:\gamma_j \ge \gamma_0} B_c \log_2\left(\frac{\gamma_j}{\gamma_0}\right)}$$