Space Diversity

Wha Sook Jeon

Fall 2021

Mobile Computing and Communications Lab.

Introduction (1)

- Diversity techniques are based on the assumption that the probability that multiple statistically independent fading channels simultaneously experience deep fading is very low.
- The idea behind diversity is to send the same data over independent fading paths

- Macro-diversity
  - Diversity to mitigate the effects of shadowing
  - is generally implemented by combining signals received by several base stations or access points
  - requires coordination among the different base stations, which is implemented as a part of networking protocols in infrastructure-based wireless networks

Introduction (2)

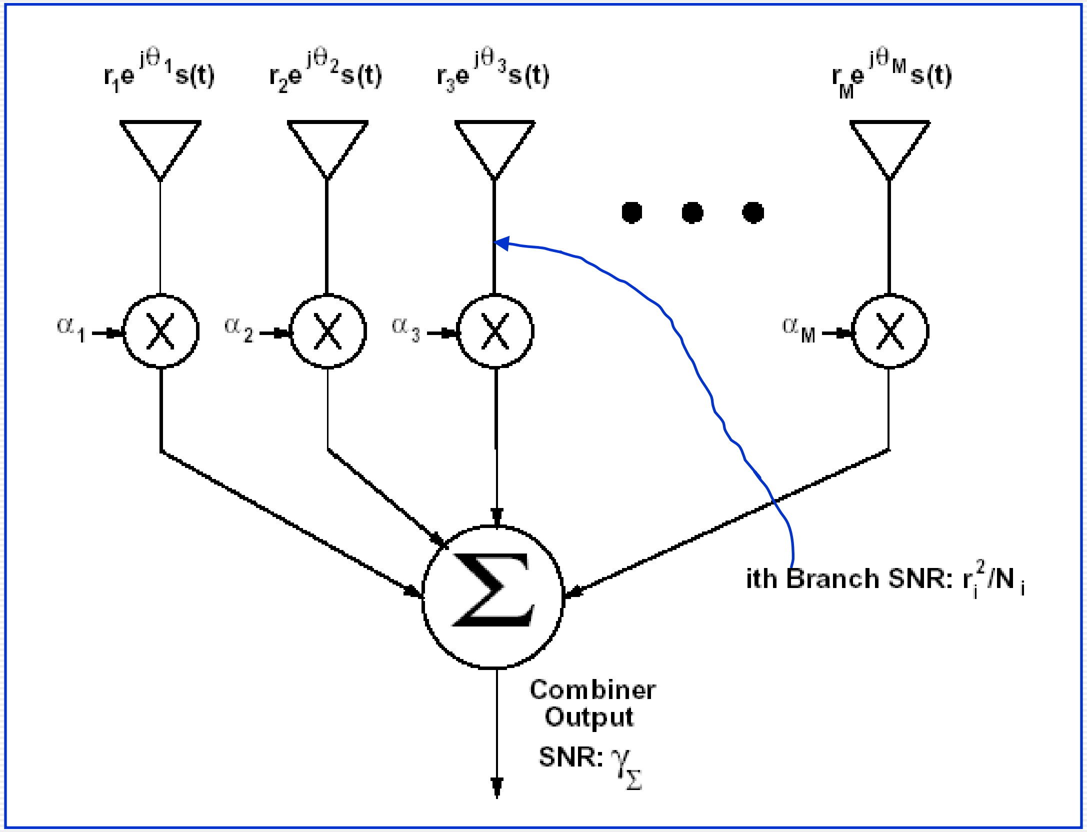
- Micro-diversity

  Diversity techniques that mitigate the effect of multipath fading

  – Space diversity: by using multiple transmit or receive antennas
  – Angle (or directional) diversity: with smart antennas which are antenna array with adjustable phase at each antenna element
  – Frequency diversity: by transmitting the same narrowband signal at different carrier frequencies
  – Path diversity: spread spectrum with RAKE receiver
  – Time diversity: by transmitting the same date at different time (coding or interleaving)

Receiver Diversity

System model for Receiver Diversity (1)



- Co-phasing:

  Removal of phase through multiplication by $\alpha_i = a_i e^{-j\theta_i}$

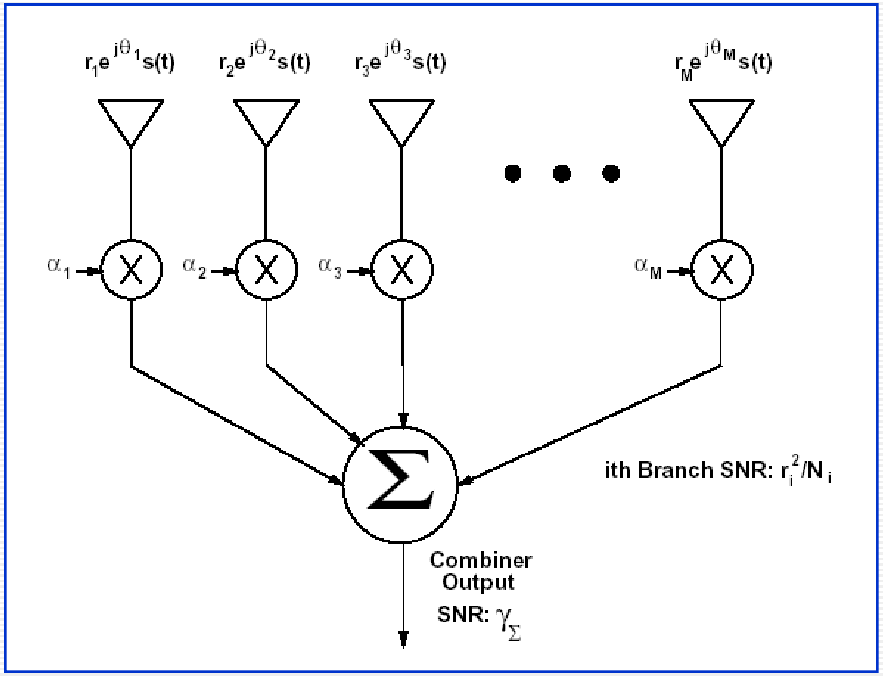
- $$\gamma_\Sigma = \frac{\left(\sum_{i=1}^{M} a_i r_i \sqrt{E_s}\right)^2 \Big/ T_s}{\sum_{i=1}^{M} a_i^2 N_0 B}$$

- Identical noise (one-sided) PSD N_0 on each branch and pulse shaping such that $BT_s=1$

System model for Receiver Diversity (2)

- Example (no fading, co-phasing: $r_i = 1, \quad \theta_j = 0$)
  - $$\gamma_\Sigma = \frac{\left(\sum_{i=1}^{M} a_i \sqrt{E_s}\right)^2}{\sum_{i=1}^{M} a_i^2 N_0}$$
  - when $a_i = 1 \big/ \sqrt{N_0}$

$$\gamma_\Sigma = \frac{\left(\sum_{i=1}^{M} \frac{\sqrt{E_s}}{\sqrt{N_0}}\right)^2}{N_0 \sum_{i=1}^{M} \frac{1}{N_0}} = \frac{ME_s}{N_0} = M\gamma$$



Diversity Gain

- With fading, the combining of multiple independent fading path leads to a more favorable distribution for γ_Σ
- Performance of a diversity system
  - Average symbol error probability
    - $\overline{P}_s = \int_0^\infty P_s(\gamma) f_{\gamma_\Sigma}(\gamma) d\gamma$

    where $P_s(\gamma)$ is a symbol error probability in AWGN channel with SNR γ
  - Outage probability
    - $P_{out} = p(\gamma_\Sigma \le \gamma_0) = \int_0^{\gamma_0} f_{\gamma_\Sigma}(\gamma) d\gamma$
- Diversity Gain
  - Performance advantage in $\overline{P}_s$ and P_{out} as a result of diversity combining

Selection Combining (1)

- The combiner outputs the signal on the branch with the highest SNR
- Cumulative distribution function (cdf) of γ_Σ
  - $P_{\gamma_\Sigma}(\gamma) = p(\gamma_\Sigma < \gamma) = P(\max[\gamma_1, \gamma_2, ..., \gamma_M] < \gamma) = \prod_{i=1}^{M} p(\gamma_i < \gamma)$
- For M-branch diversity with uncorrelated Rayleigh fading amplitude,
  - On ith branch: $f(\gamma_i) = \frac{1}{\overline{\gamma}_i} e^{-\gamma_i/\overline{\gamma}_i}$, $\quad P_{out,i}(\gamma_0) = 1 - e^{-\gamma_0/\overline{\gamma}_i}$
  - Outage probability of the selection combiner for target γ_0
    - $P_{out}(\gamma_0) = p(\gamma_\Sigma < \gamma_0) = \prod_{i=1}^{M} (1 - e^{-\gamma_0/\overline{\gamma}_i}) = [1 - e^{-\gamma_0/\overline{\gamma}}]^M$

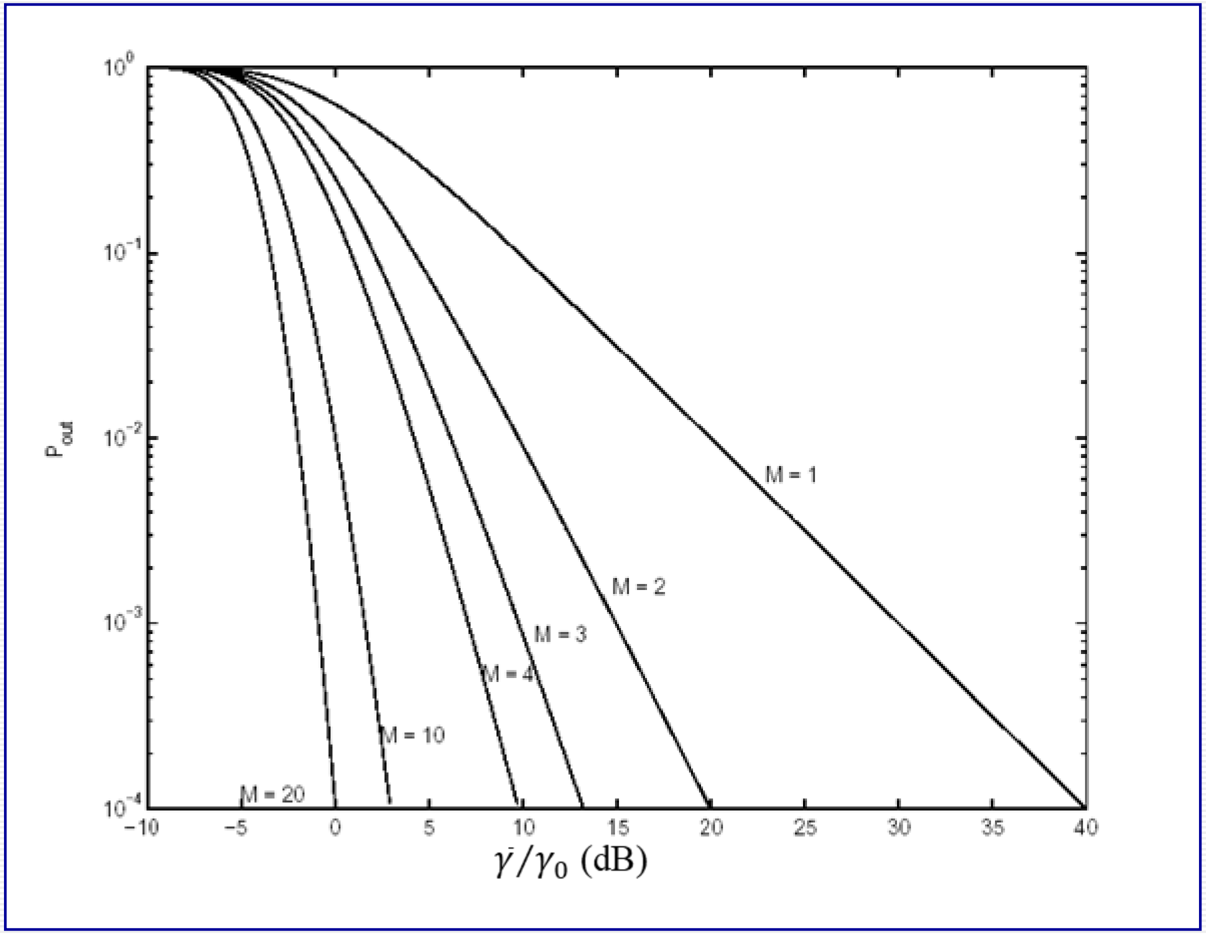
The average SNR for all branches are the same

  - pdf of γ_Σ : differentiating $P_{out}(\gamma_0)$ relative to γ_0
    - $f_{\gamma_\Sigma}(\gamma) = \frac{M}{\overline{\gamma}} [1 - e^{-\gamma/\overline{\gamma}}]^{M-1} e^{-\gamma/\overline{\gamma}}$

Selection Combining (2)

Outage Probability in Rayleigh fading

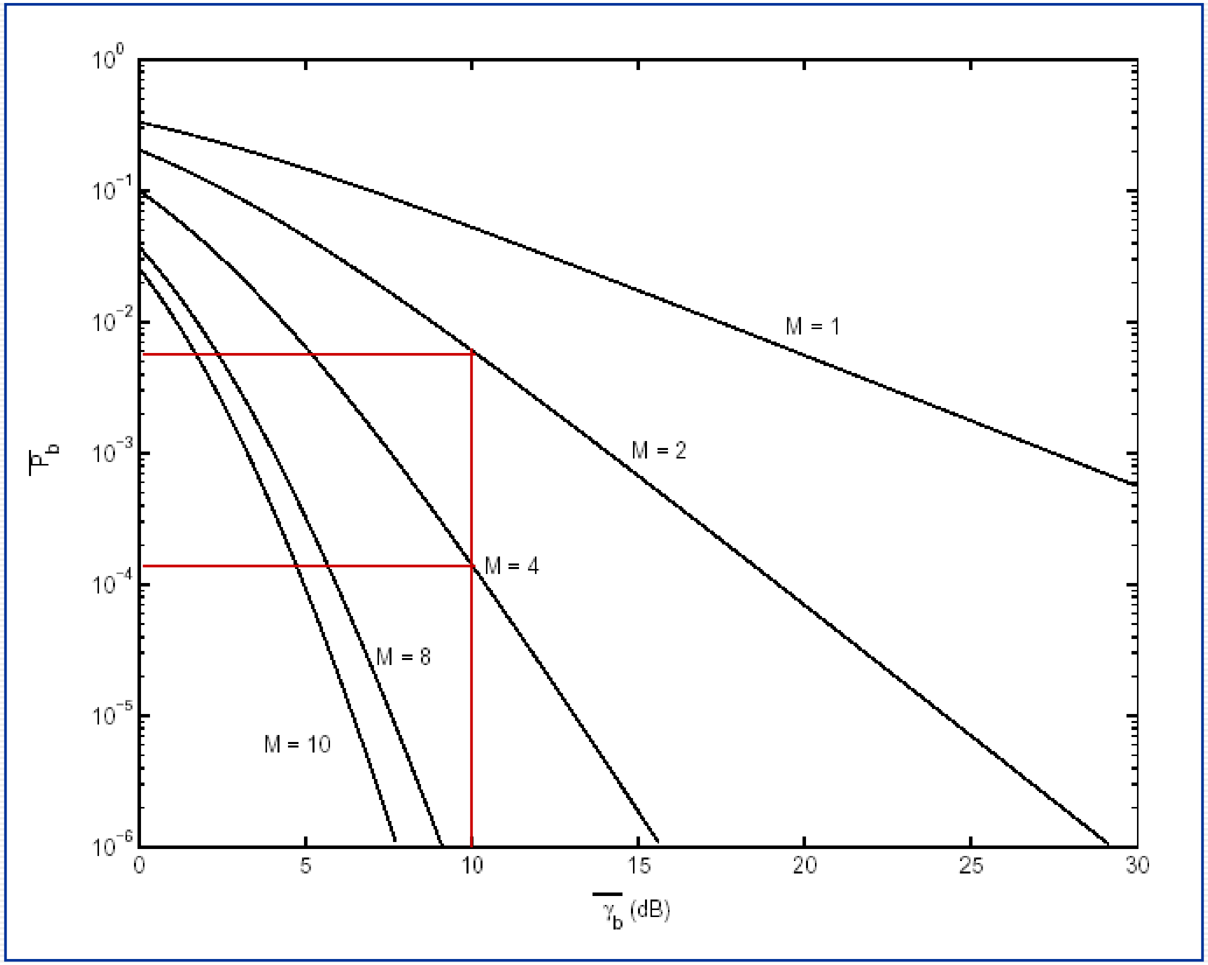
$$P_{out} = \left(1 - e^{-\gamma_0/\bar{\gamma}}\right)^M$$



Selection Combining (3)

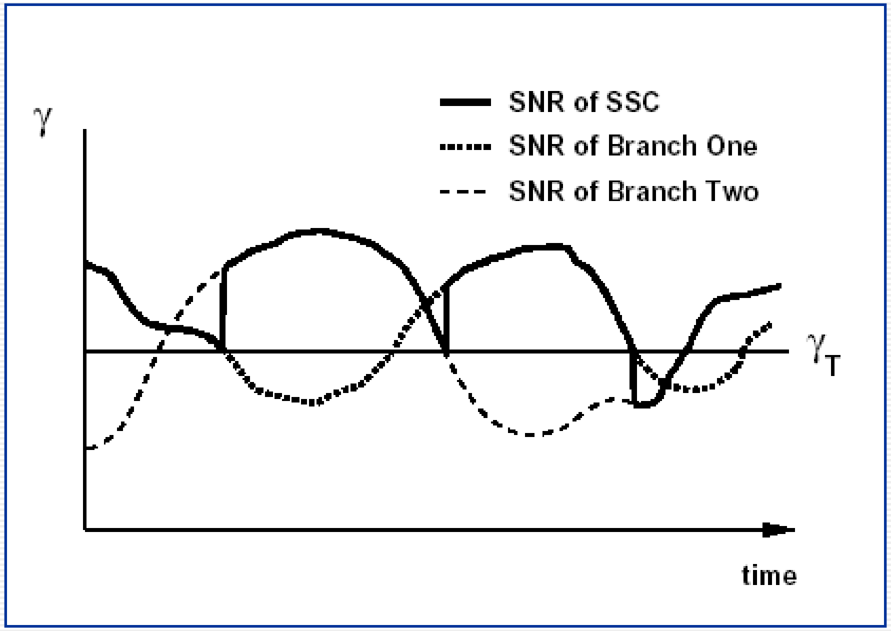
Average P_b of BPSK in Rayleigh fading

$$\int Q(\sqrt{2\gamma_b}) f_{\gamma_\Sigma}(\gamma_b) d\gamma$$



Threshold Combining (1)

- The combiner scans each branch in sequential order and outputs the first signal whose SNR is above a given threshold γ_T
- Co-phasing is not required because only one branch output is used at a time
- Switch-and-stay combining (SSC)
  - Once a branch is chosen, the combiner outputs that signal as long as the SNR on that branch remains the desired threshold.



two branches

Threshold Combining (2)

- Cdf of γ_Σ, the SNR of the combiner output with two branches:

$$P_{\gamma_\Sigma}(\gamma)=\begin{cases} P_{\gamma_1}(\gamma_T)P_{\gamma_2}(\gamma) & \gamma<\gamma_\mathrm{T}, \\ p(\gamma_T\le\gamma_1\le\gamma)+P_{\gamma_1}(\gamma_T)P_{\gamma_2}(\gamma) & \gamma\ge\gamma_\mathrm{T} \end{cases}$$

- For Rayleigh fading of each branch with
  –
$$P_{\gamma_\Sigma}(\gamma)=\begin{cases} (1-e^{-\gamma_T/\bar\gamma})(1-e^{-\gamma/\bar\gamma}) & \gamma<\gamma_\mathrm{T}, \\ \{1-e^{-\gamma/\bar\gamma}-(1-e^{-\gamma_T/\bar\gamma})\}+(1-e^{-\gamma_T/\bar\gamma})(1-e^{-\gamma/\bar\gamma}) & \gamma\ge\gamma_\mathrm{T}. \end{cases}$$

  – Outage probability for a given γ_0 : $P_{out}(\gamma_0)=P_{\gamma_\Sigma}(\gamma_0)$
  – Probability density function
    - $$f_{\gamma_\Sigma}(\gamma)=\begin{cases} (1-e^{-\gamma_T/\bar\gamma})(1/\bar\gamma)e^{-\gamma/\bar\gamma} & \gamma<\gamma_\mathrm{T} \\ (2-e^{-\gamma_T/\bar\gamma})(1/\bar\gamma)e^{-\gamma/\bar\gamma} & \gamma\ge\gamma_\mathrm{T} \end{cases}$$

Maximal Ratio Combining (1)

- Combiner output SNR: γ_Σ
  - Combiner output envelope: $r = \sum_{i=1}^{M} a_i r_i \sqrt{E_s}$
  - $$\gamma_\Sigma = \frac{\left(\sum_{i=1}^{M} a_i r_i \sqrt{E_s}\right)^2 / T_s}{\sum_{i=1}^{M} a_i^2 N_0 B} = \frac{\left(\sum_{i=1}^{M} a_i r_i \sqrt{E_s}\right)^2}{\sum_{i=1}^{M} a_i^2 N_0} \le \frac{\sum_{i=1}^{M} a_i^2 \sum_{i=1}^{M} r_i^2 E_s}{\sum_{i=1}^{M} a_i^2 N_0} = \sum_{i=1}^{M} \frac{r_i^2 E_s}{N_0} = \sum_{i=1}^{M} \gamma_i$$

    since $\left(\sum_{i=1}^{M} a_i r_i\right)^2 \le \sum_{i=1}^{M} a_i^2 \sum_{i=1}^{M} r_i^2$

- The goal is to choose the a_i to maximize γ_Σ
  - when $a_i^2 = r_i^2 / N_0$
  - $$\gamma_\Sigma = \frac{1}{N_0} \frac{\left(\sum_{i=1}^{M} a_i r_i \sqrt{E_s}\right)^2}{\sum_{i=1}^{M} a_i^2} = \sum_{i=1}^{M} \frac{r_i^2 E_s}{N_0} = \sum_{i=1}^{M} \gamma_i$$

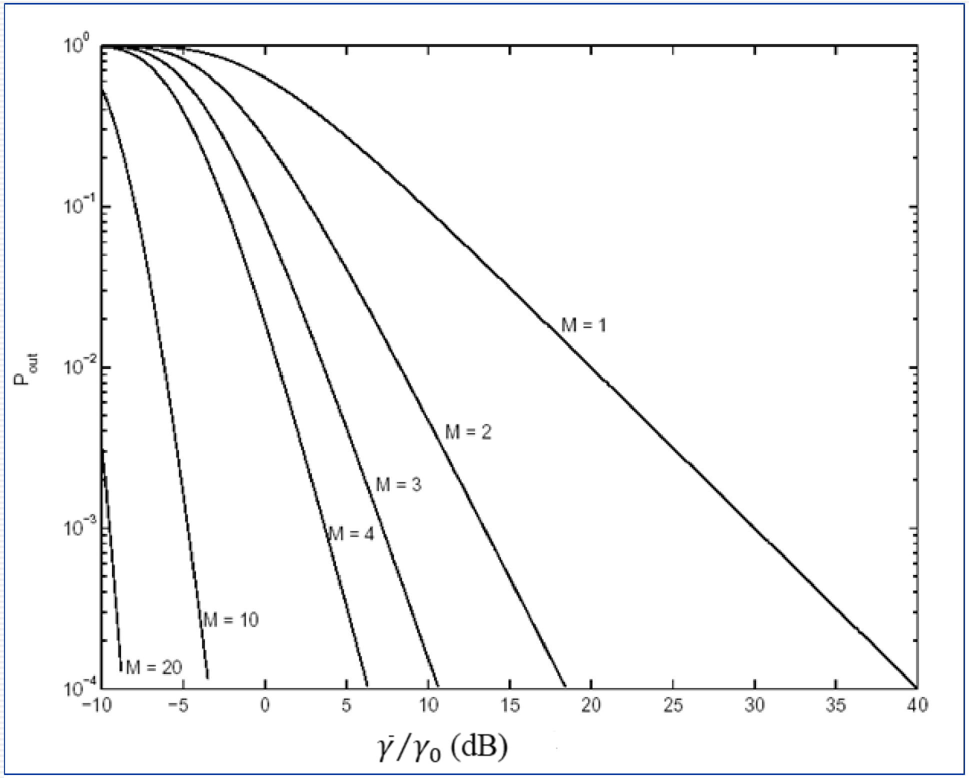
Maximal Ratio Combining (2)

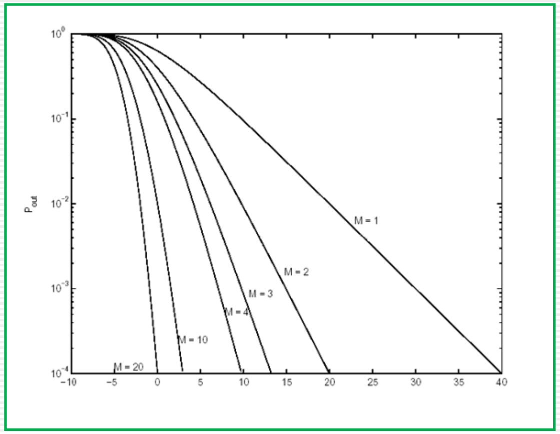
Assume i.i.d Rayleigh fading on each branch with the same average SNR $\bar{\gamma}$

- Probability density function (pdf) of $\gamma_\Sigma \quad (= \gamma_1 + \gamma_2 + \cdots + \gamma_M)$
  - M-stage Erlang distribution with mean $M\bar{\gamma}$
    - $$f_{\gamma_\Sigma}(\gamma) = \frac{\gamma^{M-1} e^{-\gamma/\bar{\gamma}}}{\bar{\gamma}^M (M-1)!} \qquad \gamma \geq 0$$
- Outage probability for a given γ_0
  - $$P_{out} = p(\gamma_\Sigma < \gamma_0)$$
    $$= \int_0^{\gamma_0} f_{\gamma_\Sigma}(\gamma) d\gamma = 1 - e^{-\gamma/\bar{\gamma}} \sum_{k=1}^{M} \frac{(\gamma_0/\bar{\gamma})^{k-1}}{(k-1)!}$$
- Average symbol (bit) error probability for BPSK modulation
  - $$\overline{P}_b = \int_0^\infty Q(\sqrt{2\gamma})\, f_{\gamma_\Sigma}(\gamma) d\gamma$$

Maximal Ratio Combining (3)

Outage Probability
in Rayleigh fading

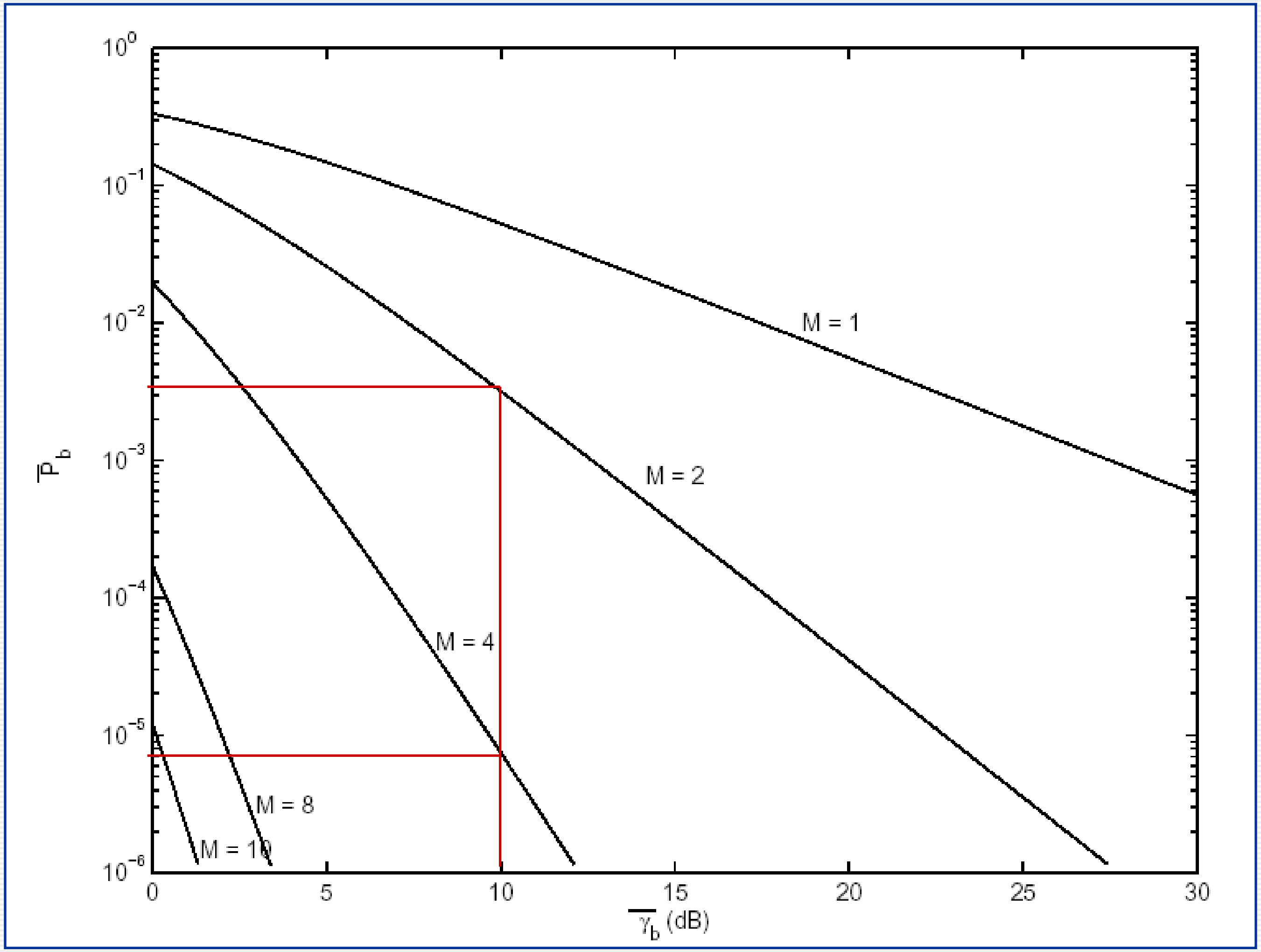




Selection combining

Maximal Ratio Combining (4)

Average P_b
of BPSK in
Rayleigh fading



Equal Gain Combining

- Simple technique which co-phases the signal on each branch and then combines them with equal weighting, $\alpha_i = e^{-j\theta_i}$
- Combiner output SNR γ_Σ, assuming the same noise PSD in each branch
  - $\gamma_\Sigma = \dfrac{1}{N_0 M}\left(\sum_{i=1}^{M} r_i \sqrt{E_s}\right)^2$
- For i.i.d. Rayleigh fading with two branches having average branch SNR $\bar{\gamma}$
  - Cdf of γ_Σ: $P_{\gamma_\Sigma}(\gamma) = 1 - e^{-2\gamma/\bar{\gamma}} - \sqrt{\pi\gamma/\bar{\gamma}}\, e^{-\gamma/\bar{\gamma}}\left\{1 - 2Q\left(\sqrt{2\gamma/\bar{\gamma}}\right)\right\}$
  - Outage Probability: $P_{out} = P_{\gamma_\Sigma}(\gamma_0)$
  - Pdf of γ_Σ: $f_{\gamma_\Sigma}(\gamma) = \dfrac{1}{\bar{\gamma}} e^{-2\gamma/\bar{\gamma}} - \sqrt{\pi}\, e^{-\gamma/\bar{\gamma}}\left(\dfrac{1}{\sqrt{4\gamma\bar{\gamma}}} - \dfrac{1}{\bar{\gamma}}\sqrt{\dfrac{\gamma}{\bar{\gamma}}}\right)\left(1 - 2Q\left(\sqrt{\dfrac{2\gamma}{\bar{\gamma}}}\right)\right)$
  - Average bit error rate for BPSK

$$\bar{P}_b = \int_0^\infty Q(\sqrt{2\gamma})\, f_{\gamma_\Sigma}(\gamma)\, d\gamma = 0.5\left(1 - \sqrt{1 - \left(1 + \bar{\gamma}\right)^{-2}}\right)$$

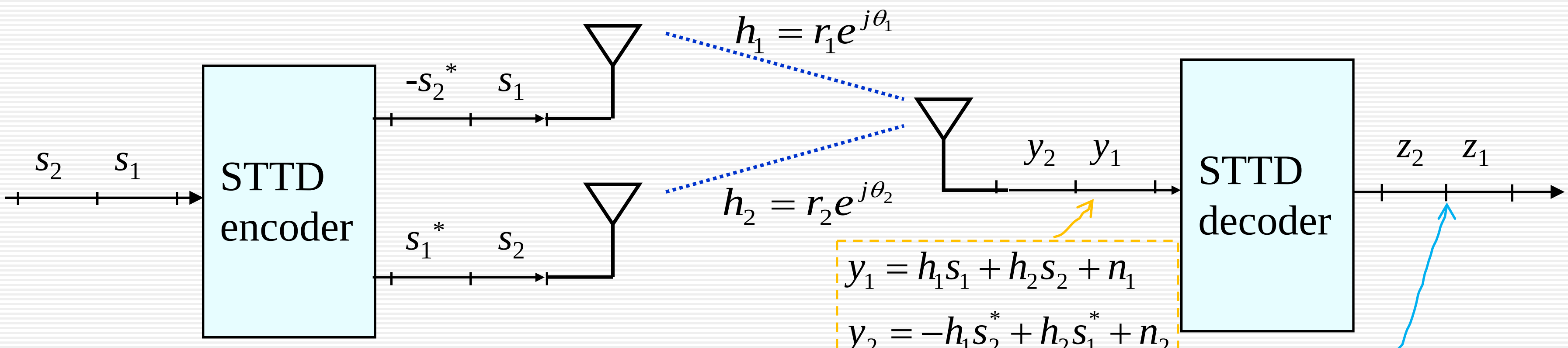
Transmit Diversity

Channel Known at Transmitter

- A transmit diversity system with M transmit antennas and one receive antenna is considered
- We assume that the path gain $r_i e^{j\theta_i}$ of the ith antenna is known at transmitter.
- The signal is multiplied by $\alpha_i = a_i e^{-j\theta_i}$ and then sent through the ith antenna.
- Because the symbol energy E_s in the transmitted signal $s(t)$ is a constant, $\sum_{i=1}^{M} a_i^2 = 1$
- Received signal: $r(t) = \sum_{i=1}^{M} a_i r_i s(t)$
- The weights a_i to achieve the maximum SNR: $a_i = \dfrac{r_i}{\sqrt{\sum_{i=1}^{M} r_i^2}}$
- The resulting SNR: $\gamma_\Sigma = \dfrac{\left(\sum_{i=1}^{M} r_i a_i \sqrt{E_s}\right)^2 / T_s}{N_0 B} = \sum_{i=1}^{M} \dfrac{r_i^2 E_s}{N_0} = \sum_{i=1}^{M} \gamma_i$
  - When the channel gains are known at transmitter, the transmit diversity is similar to the receiver diversity with MRC
  - If all antennas has the same gain $r_i = r$, $\gamma_\Sigma = M r^2 E_s / N_0$
  - There is an array gain of M corresponding to an M-fold increase in SNR over a single antenna transmitting with full power

Channel Unknown at Transmitter-Alamouti Scheme

- The transmitter no longer knows the channel gain
  - If the transmit energy is divided equally among antenna, no performance advantage is obtained
- Alamouti Scheme
  - This scheme is designed for a digital communication system with two antennas
  - The scheme to combine both space and time diversity (STTD)



$$y_1 = h_1 s_1 + h_2 s_2 + n_1$$

$$y_2 = -h_1 s_2^* + h_2 s_1^* + n_2$$

$$z_1 = \boxed{h_1^* y_1 + h_2 y_2^*} = (|h_1|^2 + |h_2|^2) s_1 + h_1^* n_1 + h_2 n_2^*$$

$$z_2 = \boxed{h_2^* y_1 - h_1 y_2^*} = (|h_1|^2 + |h_2|^2) s_2 + h_2^* n_1 - h_1 n_2^*$$

$$h_i^* = r_i e^{-j\theta_i}$$

STTD-Alamouti Scheme

- Channel estimation with known data (x_1, x_2)

$$\hat{h}_1 = y_1 x_1^* - y_2 x_2 = (|x_1|^2 + |x_2|^2) h_1 + n_1 x_1^* - n_2 x_2$$
$$\hat{h}_2 = y_1 x_2^* - y_2 x_1 = (|x_1|^2 + |x_2|^2) h_2 + n_1 x_2^* - n_2 x_1$$

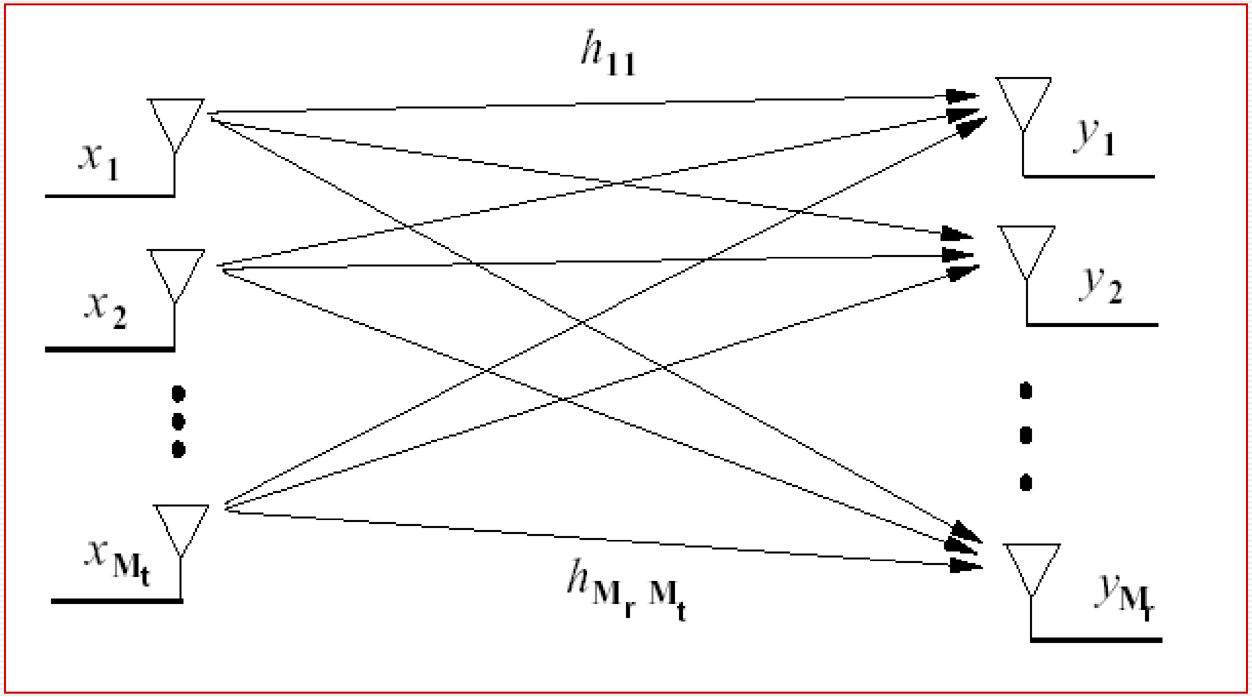
- Diversity gain

$$z_1 = (|h_1|^2 + |h_2|^2) s_1 + \tilde{n}_1$$
$$z_2 = (|h_1|^2 + |h_2|^2) s_2 + \tilde{n}_2$$

Multiple Input and Multiple Output

- Narrowband MIMO Model
  - Parallel Decomposition
  - MIMO Multiplexing Gain
- MIMO Channel Capacity
- MIMO Diversity Gain

Narrowband MIMO Model



$$\begin{bmatrix} y_1 \\ \vdots \\ y_{M_r} \end{bmatrix} = \begin{bmatrix} h_{11} & \cdots & h_{1M_t} \\ \vdots & \vdots & \vdots \\ h_{M_r1} & \cdots & h_{M_rM_t} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{M_t} \end{bmatrix} + \begin{bmatrix} n_1 \\ \vdots \\ n_{M_r} \end{bmatrix}$$

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}$$

Parallel Decomposition of MIMO Channel (1)

- An MIMO channel with $M_r \times M_t$ channel gain matrix **H** that is known to both the transmitter and receiver
- SVD (singular value decomposition) of the matrix **H**

$$\mathbf{H} = \mathbf{U}\Sigma\mathbf{V}^H$$

  - Σ: $M_r \times M_t$ diagonal matrix of singular values $\sigma_1, \ldots, \sigma_{R_H}$ of **H**, where

    $\sigma_i = \sqrt{\lambda_i}$ for λ_i which is the ith largest eigenvalue of $\mathbf{HH}^H$

    Positive real number

    > $\mathbf{A}^H = (A^*)^T$: Hermitian (i.e., conjugate transpose) of matrix **A**
    > An eigenvalue of **A**: λ such that $\det(\mathbf{A}-\lambda\mathbf{I})=0$
    > An eigenvector **z** of **A**: $\mathbf{Az} = \lambda\mathbf{z}$

  - U: $M_r \times M_r$ unitary matrix, i.e, $\mathbf{U}^H\mathbf{U}=I$. => each column of U is orthonormal vector

    V: $M_t \times M_t$ unitary matrix, $\mathbf{V}^H\mathbf{V}=\mathbf{I}$. => each column of V is an orthonormal vector
  - each column of **U**: left singular vector of **H** (eigenvector of $\mathbf{HH}^H$)

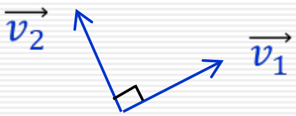
    each column of **V**: right singular vector of **H** (eigenvector of $\mathbf{H}^H\mathbf{H}$)

    $$\mathrm{H}v = \sigma_i u \quad \text{and} \quad \mathrm{H}^H u = \sigma_i v$$

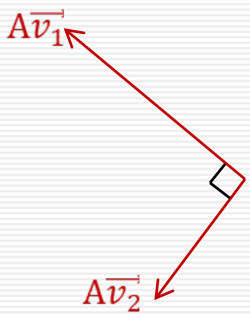
    $v_1.v_2=0$ (orthogonal) & $v_1.v_1=1$ (i.e., $\|v_1\|=1$) (unit size)

Preliminary: Singular Value Decomposition

unitary matrix $V = [\overrightarrow{v_1} \quad \overrightarrow{v_2}]$



$\xrightarrow{A}$



$v_1.v_2=0$ (orthogonal)
& $v_1.v_1=1$ (unit size)

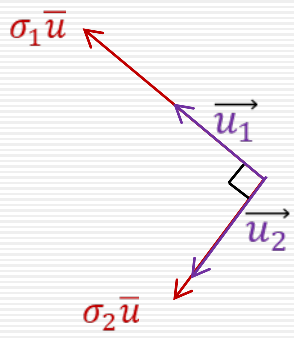
$$AV = [A\overrightarrow{v_1} \quad A\overrightarrow{v_2}]$$

$$AV = U\Sigma$$

$$AVV^T = U\Sigma V^T$$

Since V is an orthogonal matrix, $VV^T = I$

$$\boxed{A = U\Sigma V^T}$$



unitary matrix $U = [\overrightarrow{u_1} \quad \overrightarrow{u_2}]$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$U\Sigma = [\sigma_1 \overrightarrow{u_1} \quad \sigma_2 \overrightarrow{u_2}]$$

$A\overrightarrow{v_i} = \sigma_i \overrightarrow{u_i}$: ($\overrightarrow{u_i}$: left singular vector of A, $\overrightarrow{v_i}$: right singular vector of A)

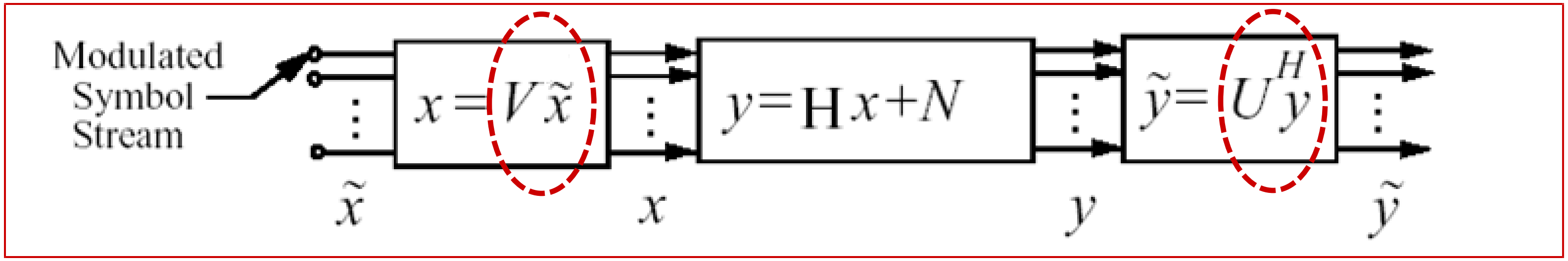
Parallel Decomposition of MIMO Channel (2)

- The transmit precoding ($x = V\tilde{x}$) and receiver shaping ($\tilde{y} = U^H y$) transform the MIMO channel into $\boldsymbol{R}_{\mathbf{H}}$ (the number of nonzero singular values σ_i of H) parallel single-input and single-output (SISO)

$$
\begin{aligned}
\tilde{y} = \mathrm{U}^{\mathrm{H}}(\mathrm{Hx} + \mathrm{n}) \quad &= \mathrm{U}^{\mathrm{H}}(\mathrm{U}\Sigma\mathrm{V}^{\mathrm{H}}\mathrm{x} + \mathrm{n}) \\
&= \mathrm{U}^{\mathrm{H}}(\mathrm{U}\Sigma\mathrm{V}^{\mathrm{H}}\mathrm{V}\tilde{\mathrm{x}} + \mathrm{n}) \\
&= \mathrm{U}^{\mathrm{H}}\mathrm{U}\Sigma\mathrm{V}^{\mathrm{H}}\mathrm{V}\tilde{\mathrm{x}} + \mathrm{U}^{\mathrm{H}}\mathrm{n} \\
&= \Sigma\tilde{\mathrm{x}} + \tilde{\mathrm{n}}
\end{aligned}
$$

$$
\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \\ \tilde{n}_3 \end{bmatrix} \qquad \tilde{y}_i = \sigma_i \tilde{x}_i + \tilde{n}_i
$$

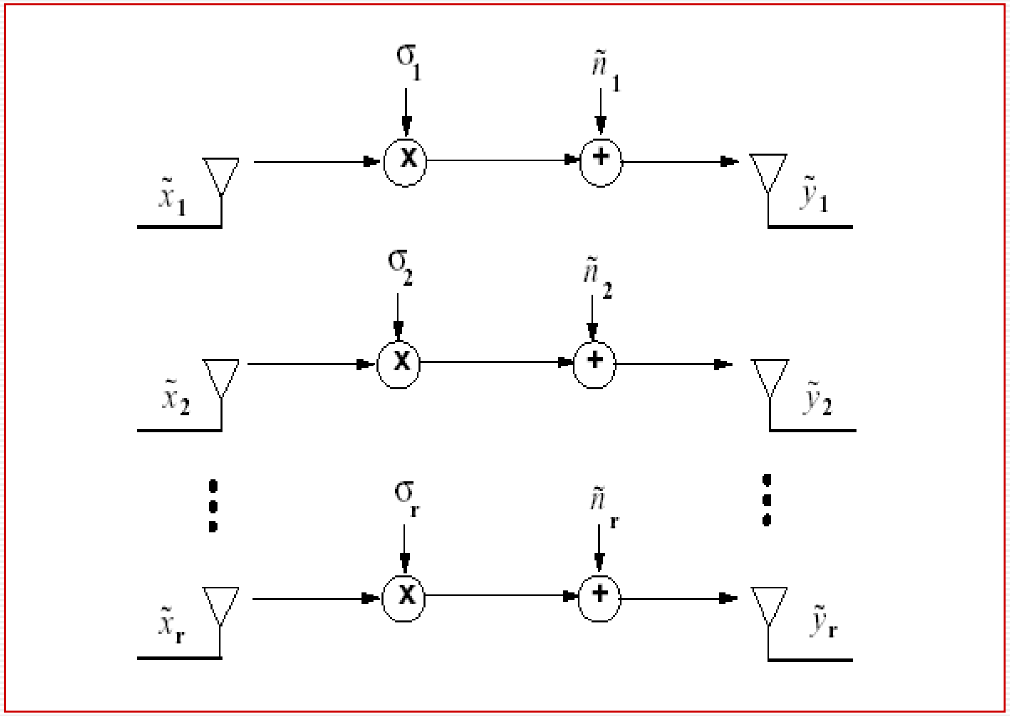
Parallel Decomposition of MIMO Channel (2)



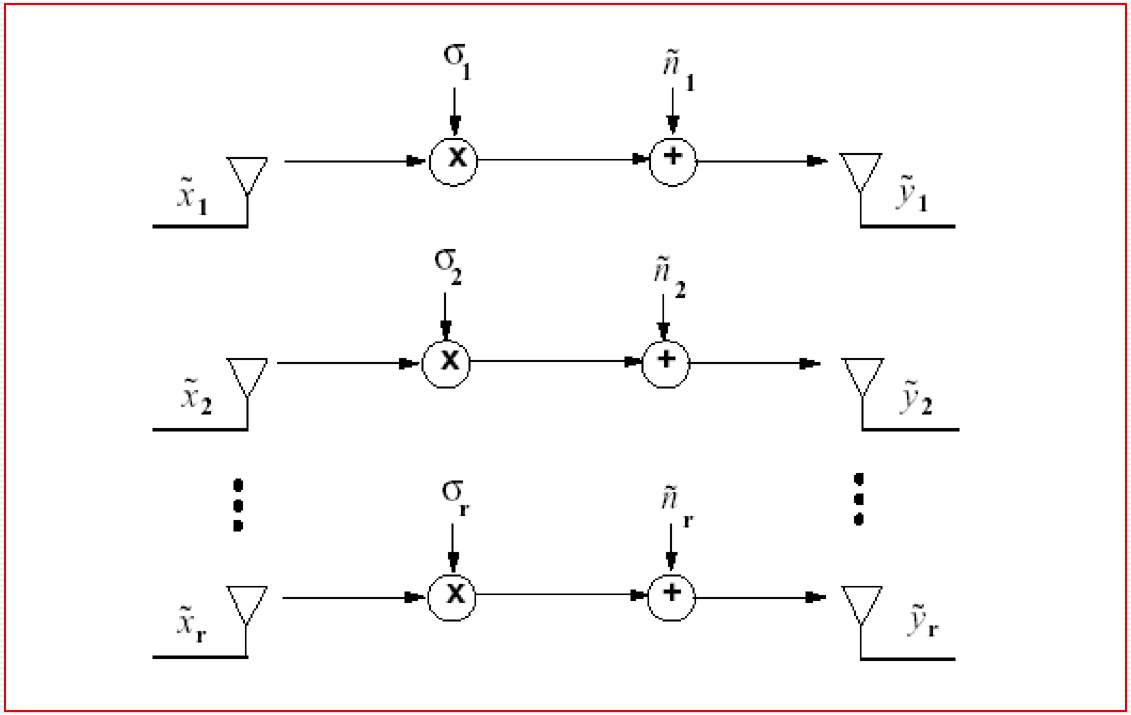
- R_{H} parallel SISO channel

  By sending independent data across each of the parallel channel, the MIMO channel can support R_{H} (=r) times the data rate

  (Multiplexing gain)



MIMO Channel Capacity in Static Channels



Channel Known at Transmitter (1)

- *Notation*
  - B: channel bandwidth
  - P: transmit power constraint

- MIMO capacity with CSIT and CSIR
  - $C = \max_{P_i : \sum_i P_i = P} \sum_{i=1}^{R_{\mathrm{H}}} B \log_2 \left(1 + \frac{\sigma_i^2 P_i}{N_0 B} \right)$

    where P_i: the tx power allocated to the *i*th parallel channel

Channel Known at Transmitter (2)

- Water-filling
  - When $\gamma_i = \sigma_i^2 P / N_0 B$ is the SNR of ith channel at full power

$\sigma^2 = \frac{\gamma_i N_0 B}{P}$

$$C = \max_{P_i : \sum_i P_i = P} \sum_{i=1}^{R_H} B \log_2 \left(1 + \sigma_i^2 \frac{P_i}{N_0 B} \right)$$

$$= \max_{P_i : \sum_i P_i = P} \sum_{i=1}^{R_H} B \log_2 \left(1 + \frac{P_i \gamma_i}{P} \right)$$

  - Water-filling power allocation for the MIMO channel: optimal
    - $\frac{P_i}{P} = \begin{cases} 1/\gamma_0 - 1/\gamma_i & \gamma_i \geq \gamma_0 \\ 0 & \gamma_i < \gamma_0 \end{cases}$

$$\boxed{C = \sum_{i:\gamma_i \geq \gamma_0}^{R_H} B \log_2 \left(\frac{\gamma_i}{\gamma_0} \right)}$$

Channel Unknown at Transmitter

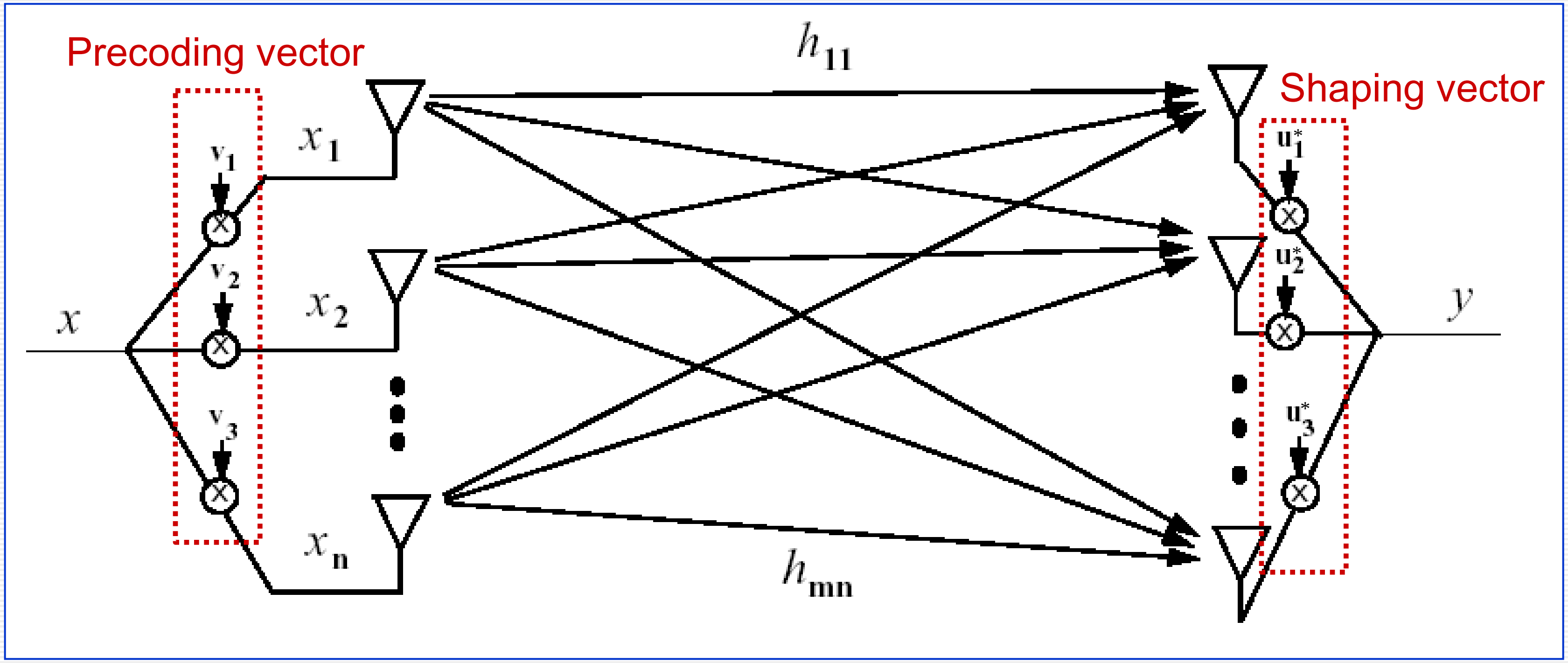
- Uniform power allocation
  - Mutual information for MIMO channel

$$C_{EqPw} = \sum_{i=1}^{R_{\mathrm{H}}} B\log_2\left(1+\sigma_i^2\frac{P_i}{N_0B}\right) = \sum_{i=1}^{R_{\mathrm{H}}} B\log_2\left(1+\frac{\sigma_i^2 P/M_t}{N_0B}\right)$$

$$= \sum_{i=1}^{R_{\mathrm{H}}} B\log_2\left(1+\frac{\gamma_i}{M_t}\right)$$

where $\gamma_i = \sigma_i^2 P / N_0 B$ and M_t is the number of tx antennas

MIMO Diversity Gain: Beamforming



MIMO Diversity Gain

- The same symbol x is sent over the ith antenna with weight v_i and the signal received on the jth antenna of the receiver is weighted by u_j^*
- The resulting received signal: $\mathrm{y} = \mathrm{u}^H(\mathrm{Hvx} + \mathrm{n})$
  - For the maximum singular value $\sigma_{\max}$ of H, u and v are the first columns of U and V, (principal left and right singular vector)

    $\mathrm{Hv} = \sigma_{\max}\mathrm{u}$ and $\mathrm{H}^{\mathrm{H}}\mathrm{u} = \sigma_{\max}\mathrm{v}$
  - $$\begin{aligned} y &= \mathrm{u}^H\mathrm{Hv}x + \mathrm{u}^H\mathrm{n} = \mathrm{u}^H\sigma_{\max}\mathrm{u}x + \mathrm{u}^H\mathrm{n} \\ &= \sigma_{\max}\mathrm{u}^H\mathrm{u}x + \mathrm{u}^H\mathrm{n} = \sigma_{\max}x + \mathrm{u}^H\mathrm{n} \\ &= \underline{\sigma_{\max}x + n} \end{aligned}$$
- When channel knowledge at both receiver and transmitter,

$$C = B\log_2\left(1 + \frac{\sigma_{\max}^2 P}{N_0 B}\right)$$