Multicarrier Modulation

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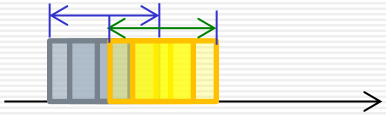
Data Transmission Using Multiple Carriers (1)

- The simplest form of multicarrier modulation
  - divides the data stream into multiple substreams to be transmitted over different orthogonal subchannels centered at different frequencies (FDM)
- Consider a system with data rate R and bandwidth B
  - If coherence bandwidth (B_c) is assumed to be $B_c < B$, the signal experiences frequency-selective fading
  - With a wide bandwidth, because a symbol time is usually shorter than delay spread, the effect of ISI cannot be negligible



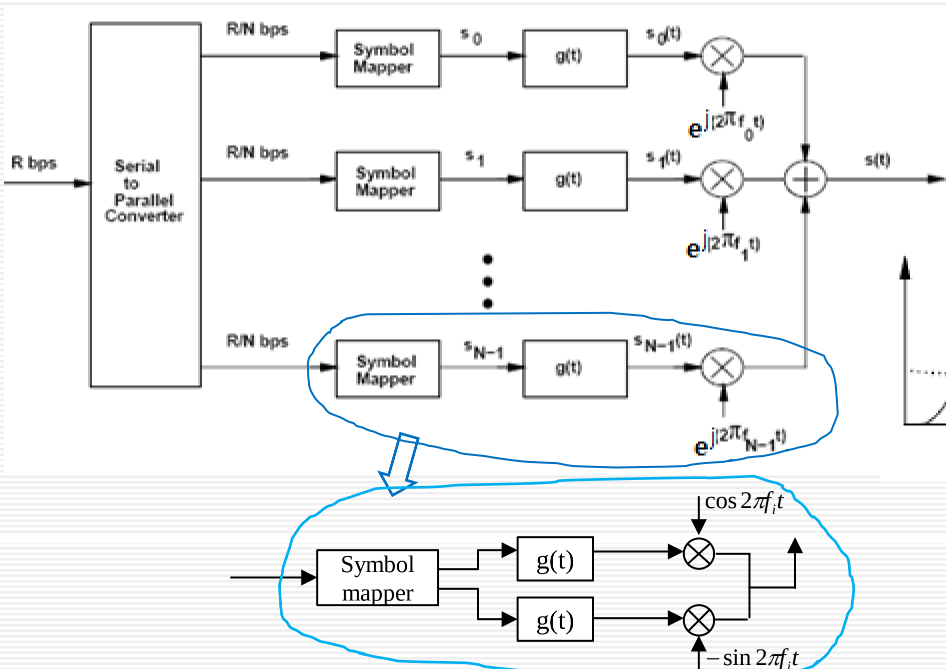
  - The wideband system is broken into N subsystems in parallel
    - Each with subchannel bandwidth $B_s = B / N$ and data rate $R_N = R / N$
    - For N sufficiently large, $B_s << B_c$,
      - relatively flat fading on each subchannel
      - The symbol time on each subchannel is much greater than the delay spread of the subchannel => experiences little ISI

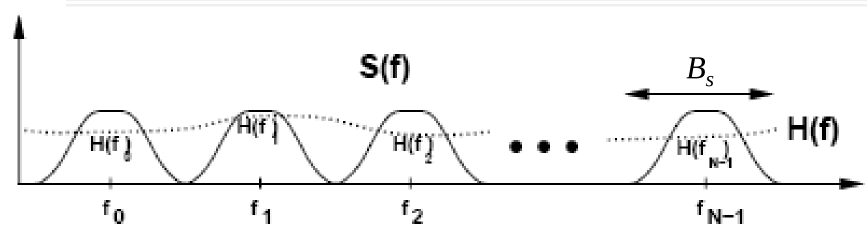
Data Transmission Using Multiple Carriers (2)

- Multicarrier transmitter
  - The ith substream is modulated via QAM or PSK with the subcarrier frequency f_i
  - The modulated signals associated with all the subcarriers are summed together

$$s(t)=\sum_{i=0}^{N-1}\mathrm{Re}\left\{s_i g(t)e^{j(2\pi f_i t+\phi_i)}\right\}$$

    where s_i is the complex symbol, ϕ_i is phase offset of the ith carrier, and $g(t)$ is pulse shaper
  - This system does not change the data rate or signal bandwidth relative to the original system but eliminates ISI for $B_s << B_c$





The ith subchannel is affected by the flat fading corresponding to a channel gain $H(f_i)$

Multicarrier Modulation with Overlapping Subchannels

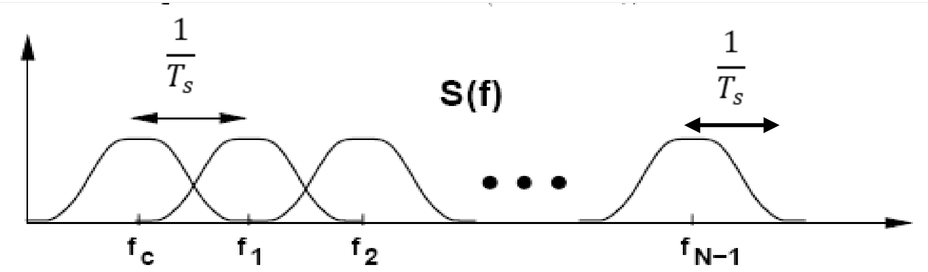
- The spectral efficiency of multicarrier modulation by overlapping the subchannel
  - The subcarrier must still be orthogonal so that they can be separated out by the demodulator in the receiver
- The subcarrier set: $\{\cos 2\pi(f_0 + i/T_s)t, i = 0, 1, 2, \cdots\}$
  - Orthonormal basis set:

$$\int_0^{T_s} \cos 2\pi(f_0 + i/T_s)t \times \cos 2\pi(f_0 + j/T_s)t \, dt = 0 \quad \text{if } i \neq j$$

  - Even if the subchannels overlap, the modulated signal transmitted in each subchannel can be separated out in the receiver
  - The total system bandwidth

$$B \approx N\frac{1}{T_s} = NB_s \qquad \text{for a large } N$$

  - Overlapping subcarriers



Multicarrier Modulation with Overlapping Subchannels

$2cosAcosB = \cos(A-B) + \cos(A+B)$

$$\frac{2}{T_s}\int_0^{T_s} \cos 2\pi(f_0 + i/T_s)t \cos 2\pi(f_0 + j/T_s)t\, dt$$

$$= \frac{1}{T_s}\int_0^{T_s} \cos 2\pi \frac{(i-j)t}{T_s}\, dt + \frac{1}{T_s}\int_0^{T_s} \cos 2\pi\left(2f_0 + \frac{i+j}{T_s}\right)t\, dt$$

$$\approx \frac{1}{T_s}\int_0^{T_s} \cos 2\pi \frac{(i-j)t}{T_s}\, dt$$

$$= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \Rightarrow \delta(j-i)$$

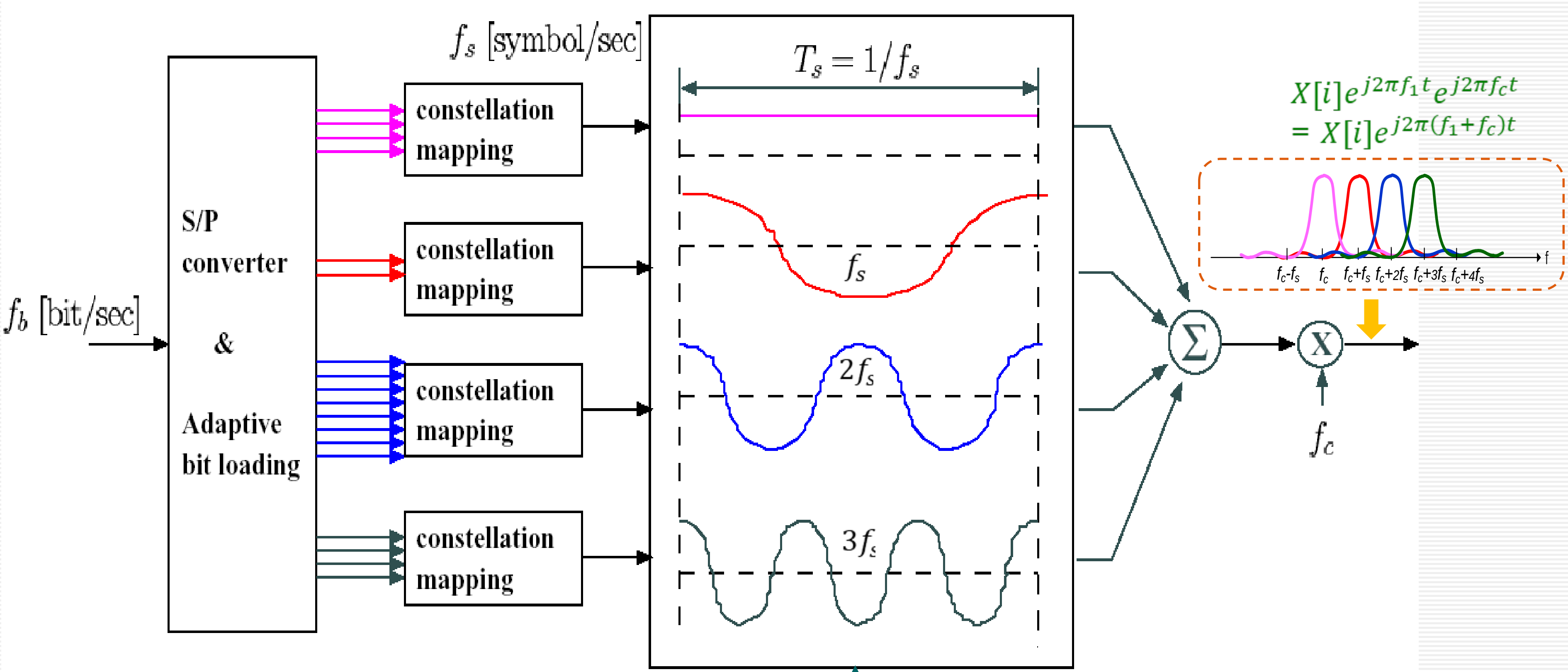
$$\hat{s}_i = \int_0^{T_s} \left(\sum_{j=0}^{N-1} s_j g(t) \cos 2\pi f_j t\right) g(t) \cos 2\pi f_i t\, dt$$

$$= \sum_{j=0}^{N-1} s_j \int_0^{T_s} g^2(t) \cos 2\pi(f_0 + j/T_s)t \times \cos 2\pi(f_0 + i/T_s)t\, dt$$

$$= \sum_{j=0}^{N-1} s_j\, \delta(j-i) = s_i$$

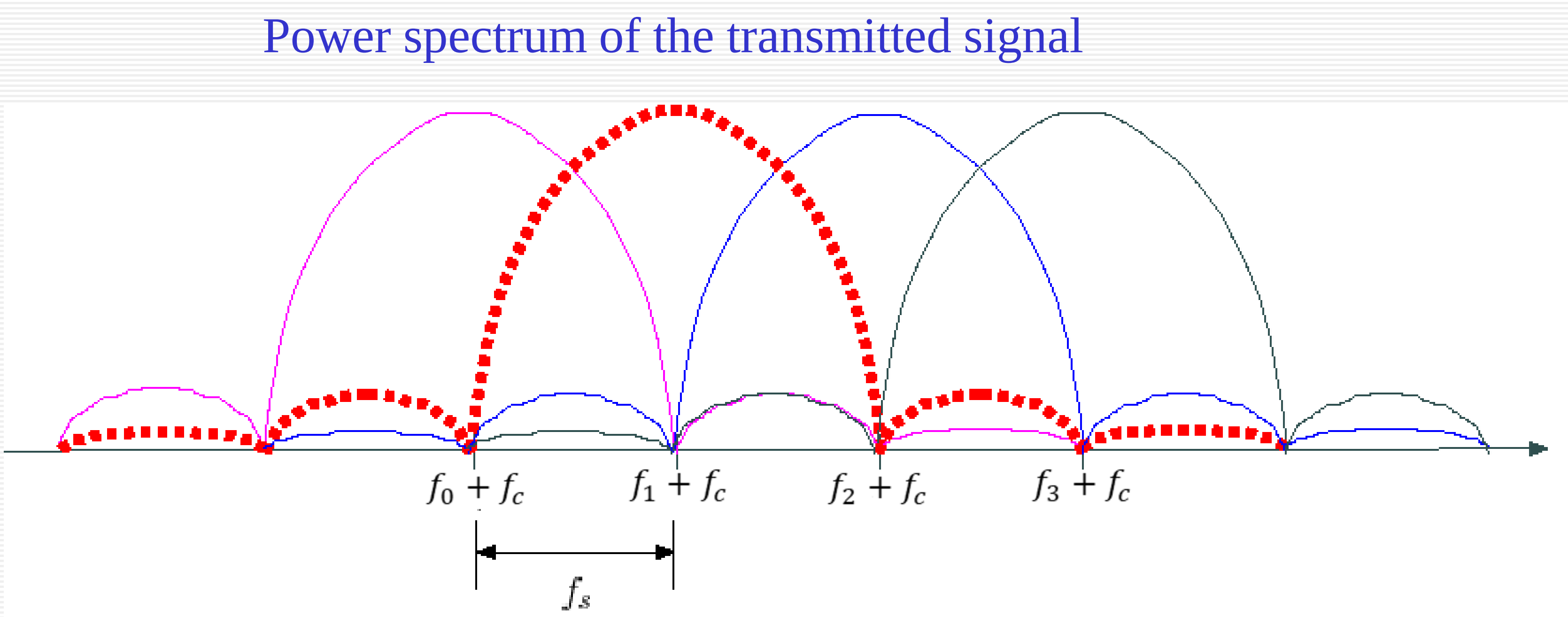
Multicarrier Modulation (1)

- Multicarrier modulation, which is invented in the 1950s, can be used widely by the development of the discrete Fourier transform and inverse DFT
- Multicarrier Modulation



each waveform is orthogonal with each other for a symbol interval (T_s)

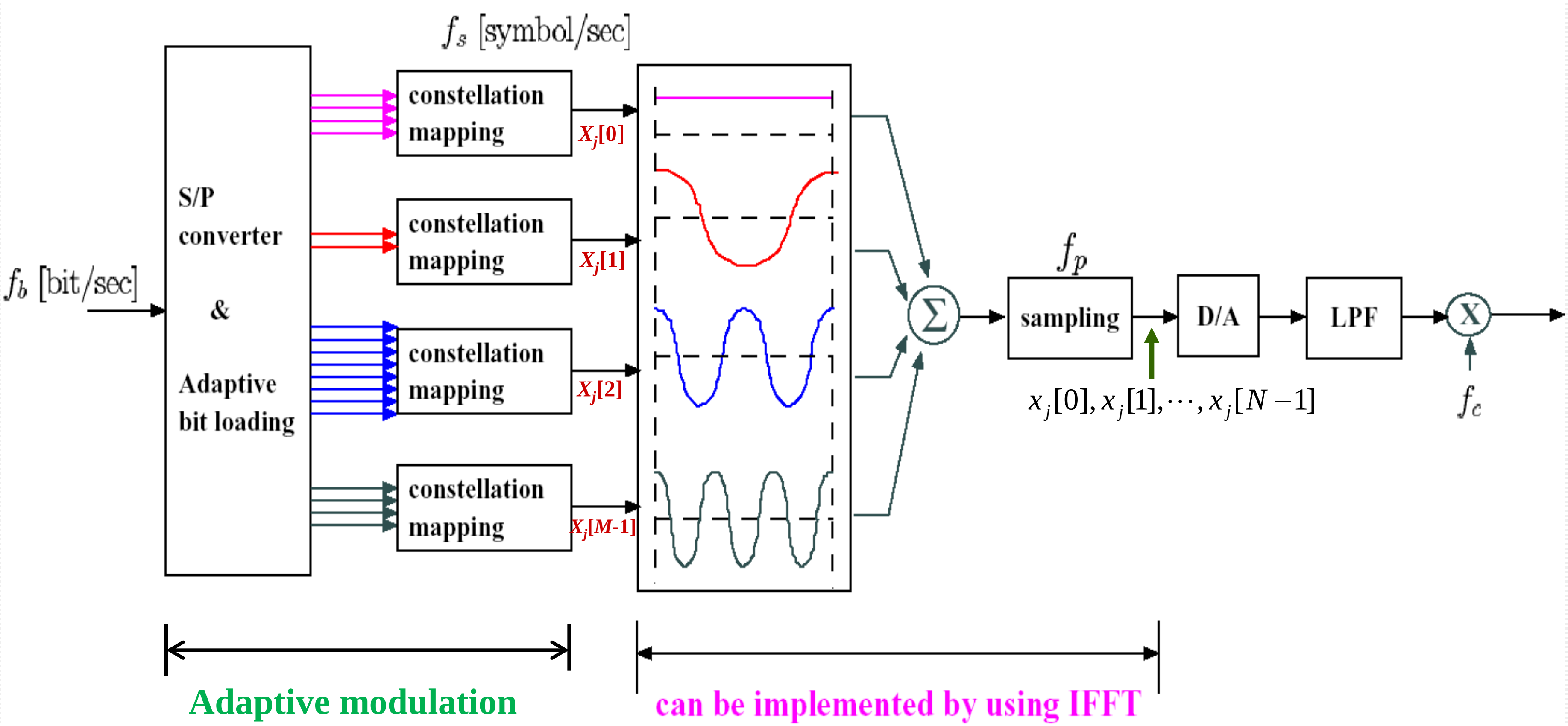
Multicarrier Modulation (2)



- Signal spectrum on each subcarrier has sinc function for a rectangular pulse shape
- Implementation via FFT/IFFT

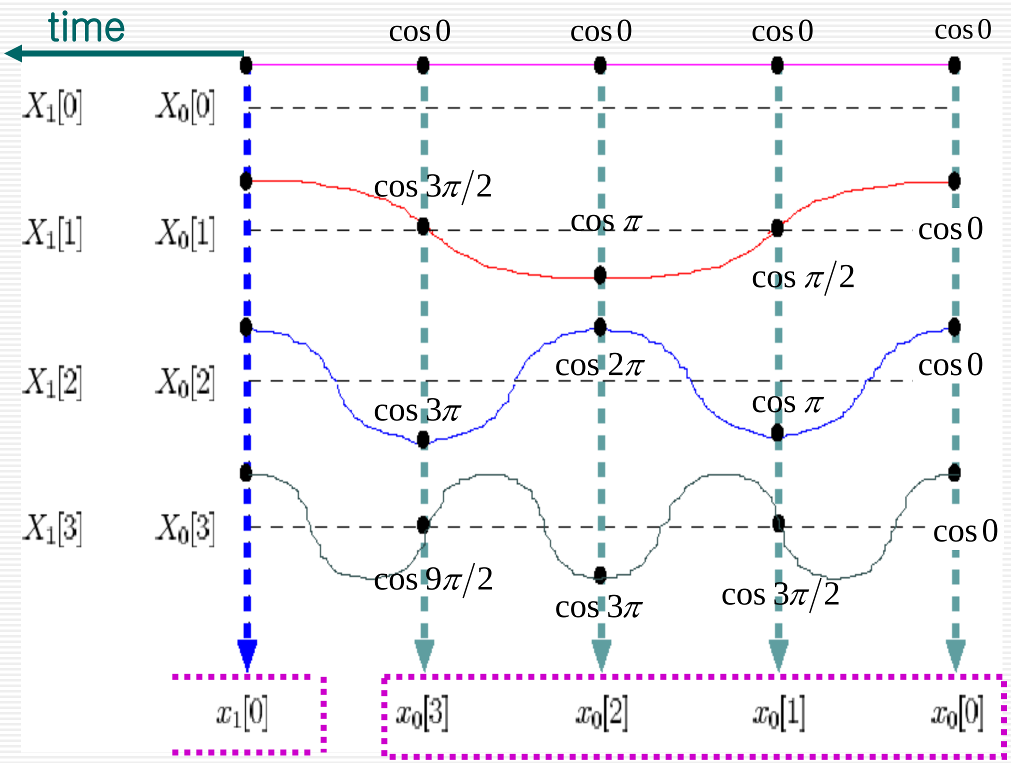
Discrete Implementation of Multicarrier Modulation (1)

- Implementation



Discrete Implementation of Multicarrier Modulation (3)

- M=4, N =4,



1-th OFDM symbol

0-th OFDM symbol

$$x[0] = \frac{1}{\sqrt{4}}\left\{X[0] + X[1] + X[2] + X[3]\right\}$$

$$x[1] = \frac{1}{\sqrt{4}}\left\{X[0] + X[1]e^{j2\pi/4} + X[2]e^{j4\pi/4} + X[3]e^{j6\pi/4}\right\}$$

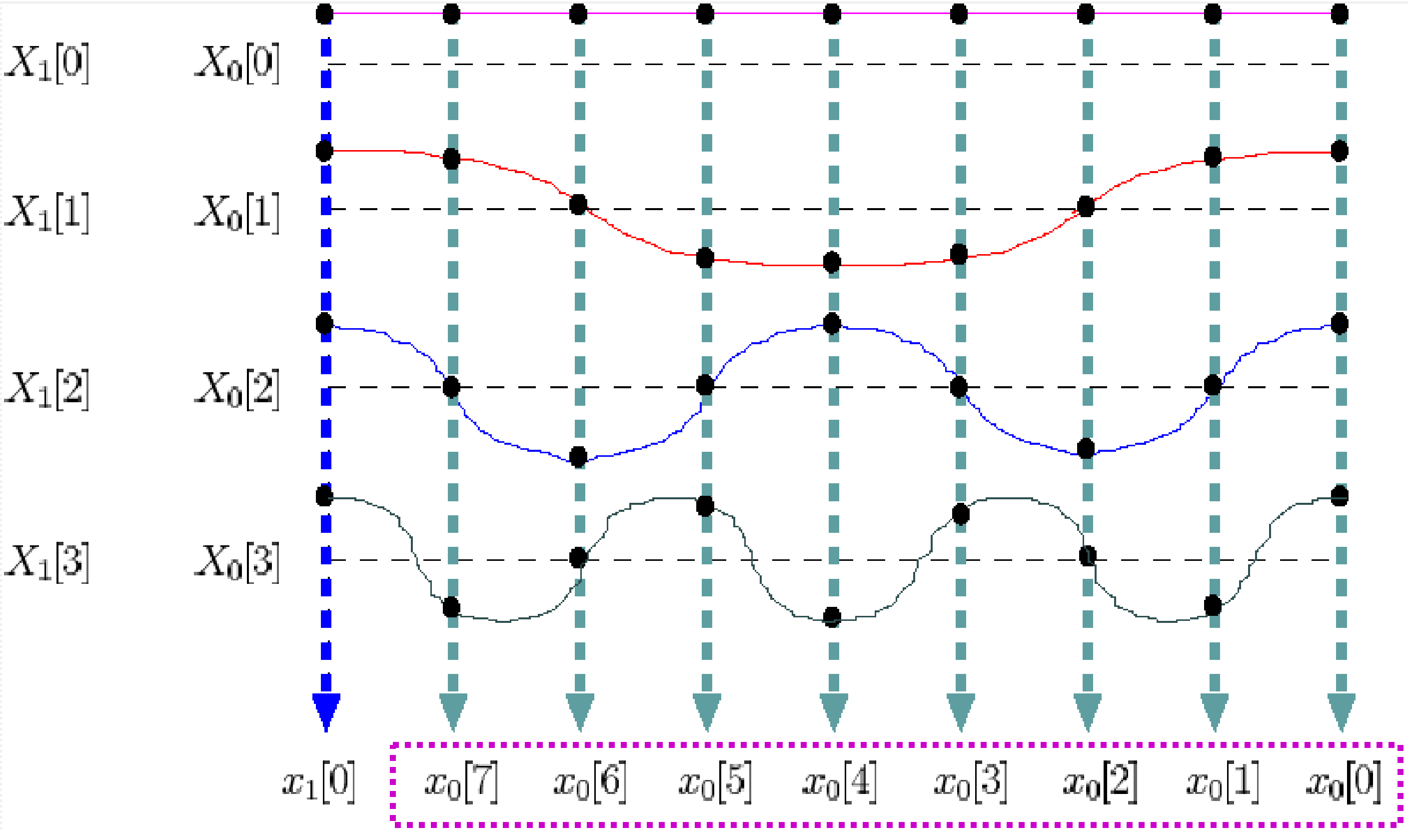
$$x[2] = \frac{1}{\sqrt{4}}\left\{X[0] + X[1]e^{j4\pi/4} + X[2]e^{j8\pi/4} + X[3]e^{j12\pi/4}\right\}$$

$$x[3] = \frac{1}{\sqrt{4}}\left\{X[0] + X[1]e^{j6\pi/4} + X[2]e^{j12\pi/4} + X[3]e^{j18\pi/4}\right\}$$

$$x[n] = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1} X[i]e^{j\frac{2\pi i}{N}n}$$

Discrete Implementation of Multicarrier Modulation (4)

- M=4, N=8



⇨ " $M \le N$ " **:** if M<N, X[M]=0, … X[N-1]=0

Discrete Implementation of Multicarrier Modulation (5)

N-point DFT

$$x[n],\quad n=0,\cdots,N-1 \;\underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}}\; X[k],\quad k=0,\cdots,N-1$$

IDFT

$$x[n]=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1} X[i]\times e^{j(2\pi i/N)n}$$

DFT

$$X[i]=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} x[n]\times e^{-j(2\pi i/N)n}$$

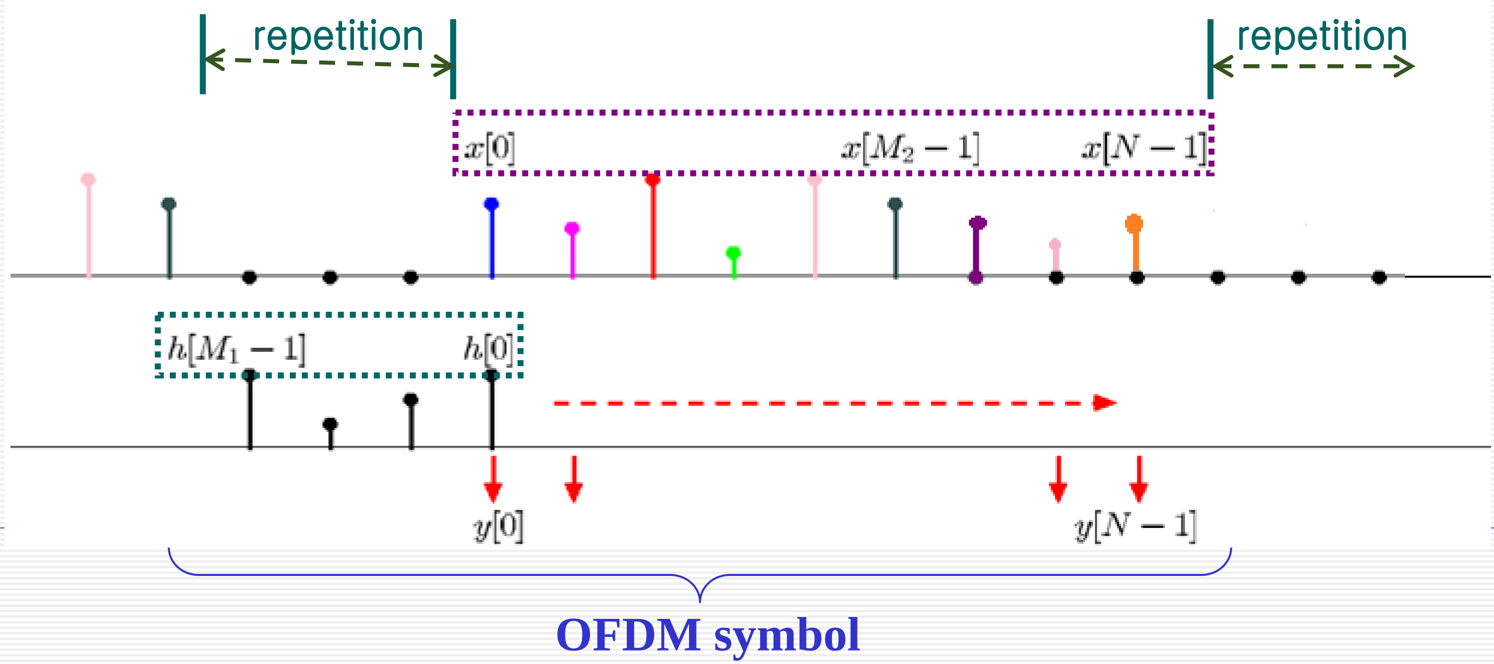
⇨ "N = power of two" : FFT/IFFT (fast processing)

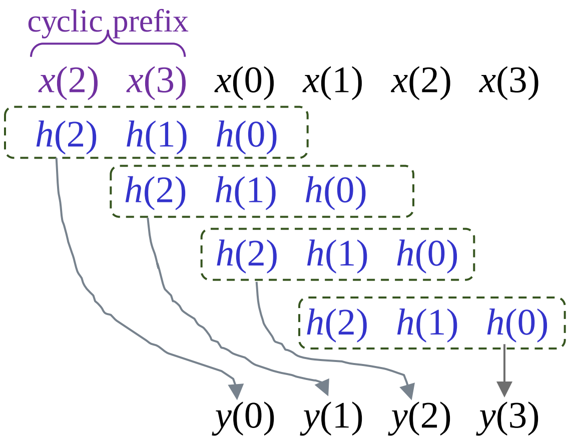
Discrete Implementation of Multicarrier Modulation (6)

◆ Circular convolution

⇨
- h[n] with duration M1,   x[n] with duration N
- DFT of N-point circular convolution of h[n] and x[n] is equal to the multiplication of DFT(h[n]) and DFT(x[n])

$$y[n] = h[n] \circledN x[n] \quad (n = 0, \cdots, N-1) \quad \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}} \quad Y[k] = H[k]X[k] \quad (k = 0, \cdots, N-1)$$





$$y(0) = x(0)h(0) + x(3)h(1)+x(2)h(2)$$
$$y(1) = x(1)h(0) + x(0)h(1)+x(3)h(2)$$
$$y(2) = x(2)h(0) + x(1)h(1)+x(0)h(2)$$
$$y(3) = x(3)h(0) + x(2)h(1)+x(1)h(2)$$

$$\xrightarrow{\text{FT}}$$

$$H[0] = [h(0) + h(1)+ h(2)]$$
$$H[1] = [h(0)+h(1)e^{-j2\pi/4}+h(2)e^{-j4\pi/4}]$$
$$H[2] = [h(0)+h(1)e^{-j4\pi/4}+h(2)e^{-j8\pi/4}]$$
$$H[3] = [h(0)+h(1)e^{-j6\pi/4}+h(2)e^{-j12\pi/4}]$$

$$X[0] = 1/2\,[x(0) + x(1)+ x(2) + x(3)]$$
$$X[1] = 1/2\,[x(0)+x(1)e^{-j2\pi/4}+x(2)e^{-j4\pi/4}+x(3)e^{-j6\pi/4}]$$
$$X[2] = 1/2\,[x(0)+x(1)e^{-j4\pi/4}+x(2)e^{-j8\pi/4}+x(3)e^{-j12\pi/4}]$$
$$X[3] = 1/2\,[x(0)+x(1)e^{-j6\pi/4}+x(2)e^{-j12\pi/4}+x(3)e^{-j18\pi/4}]$$

$$
\begin{aligned}
X[0]H[0] &= \tfrac{1}{2}\,[x(0)+x(1)+x(2)+x(3)] \times [h(0)+h(1)+h(2)] \\
&= \tfrac{1}{2}\,[\; x(0)h(0) + x(1)h(0) + x(2)h(0) + x(3)h(0) \\
&\quad + x(0)h(1) + x(1)h(1) + x(2)h(1) \\
&\quad + x(3)h(1) + x(0)h(2) + x(1)h(2) \\
&\quad + x(2)h(2) + x(3)h(2) \;] \\
&= 1/2\,[\; y(0) + y(1) + y(2) + y(3) \;] \\
&= Y[0]
\end{aligned}
$$

$$Y[0] = 1/2\,[y(0) + y(1)+ y(2) + y(3)]$$
$$Y[1] = 1/2\,[y(0)+y(1)e^{-j2\pi/4}+y(2)e^{-j4\pi/4}+y(3)e^{-j6\pi/4}]$$
$$Y[2] = 1/2\,[y(0)+y(1)e^{-j4\pi/4}+y(2)e^{-j8\pi/4}+y(3)e^{-j12\pi/4}]$$
$$Y[3] = 1/2\,[y(0)+y(1)e^{-j6\pi/4}+y(2)e^{-j12\pi/4}+y(3)e^{-j18\pi/4}]$$

$$\boxed{Y[i] = H[i]\, X[i]}$$

Discrete Implementation of Multicarrier Modulation (7)

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-2] \\ y[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & \bullet & \bullet & \bullet & 0 & h[2] & h[1] \\ h[1] & h[0] & 0 & \bullet & \bullet & \bullet & 0 & h[2] \\ h[2] & h[1] & h[0] & \bullet & \bullet & \bullet & 0 & 0 \\ & & & & & & & \\ 0 & \bullet & \bullet & 0 & h[2] & h[1] & h[0] & 0 \\ 0 & \bullet & \bullet & \bullet & 0 & h[2] & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-2] \\ x[N-1] \end{bmatrix}$$

$y(0) = x(0)h(0) + x(3)h(1) + x(2)h(2)$
$y(1) = x(1)h(0) + x(0)h(1) + x(3)h(2)$
$y(2) = x(2)h(0) + x(1)h(1) + x(0)h(2)$
$y(3) = x(3)h(0) + x(2)h(1) + x(1)h(2)$

Diagonal matrix

$$\begin{bmatrix} Y[0] \\ Y[1] \\ \vdots \\ Y[N-2] \\ Y[N-1] \end{bmatrix} = \begin{bmatrix} H[0] & & & & \\ & H[1] & & & \\ & & \ddots & & \\ & & & H[N-2] & \\ & & & & H[N-1] \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-2] \\ X[N-1] \end{bmatrix}$$

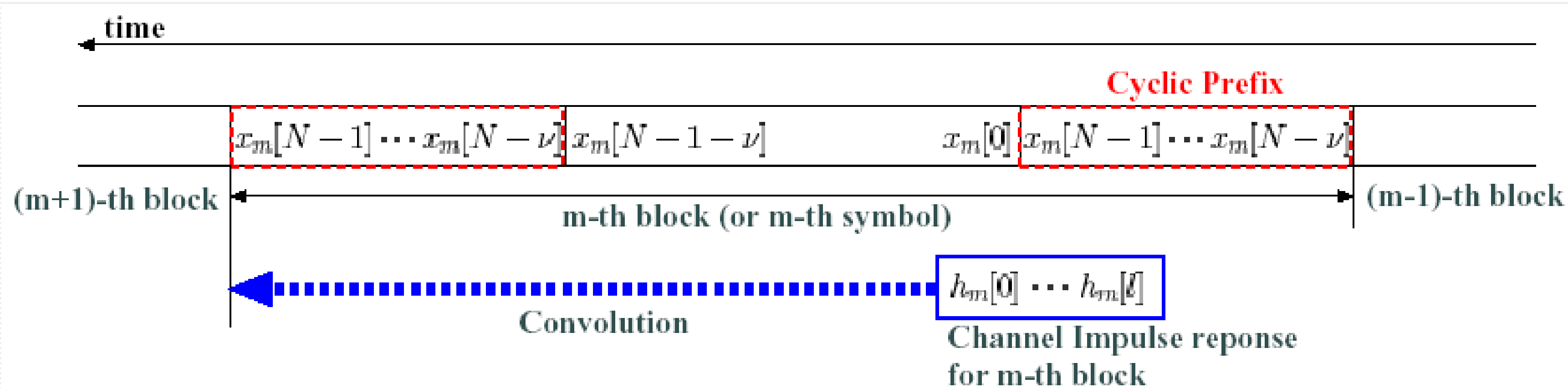
: Y[i] depends on only X[i]

Y[*i*] = H[*i*] X[*i*]

Cyclic Prefix (CP) (1)

- Cyclic Prefix is added so that the received signal can be N-point circular convolution of the transmitted signal and the channel impulse response

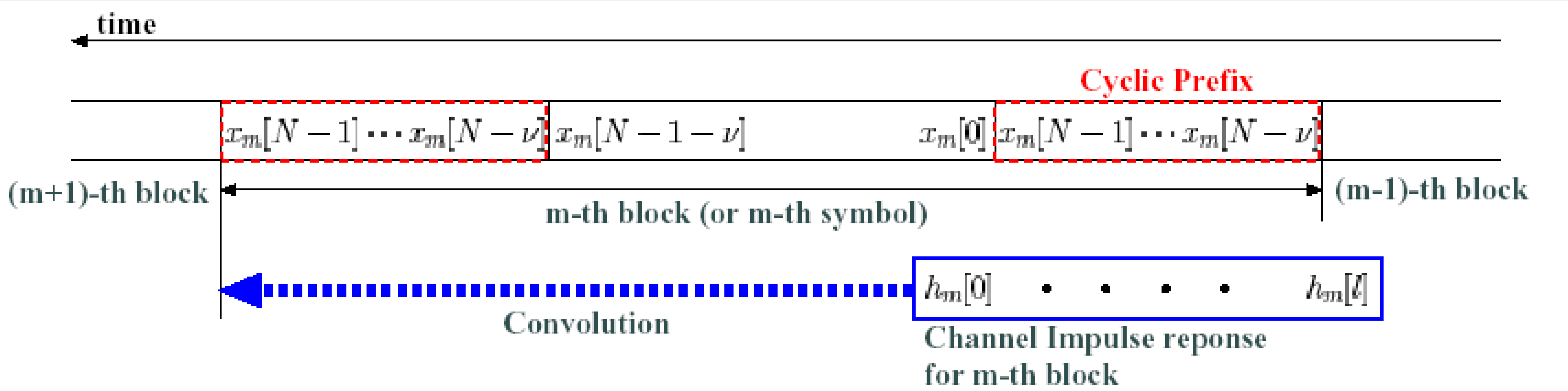
⇨ When the CP is longer than the length of channel impulse response



$$y_m[n] = h_m[n] \circledN x_m[n] + \eta_m[n] \xrightarrow{\text{DFT of the received signal}} Y_m = \underbrace{H_m}_{\text{Diagonal matrix}} X_m + \Upsilon_m$$

Cyclic Prefix (CP) (2)

⇨ **When CP is shorter than the channel impulse response length.**

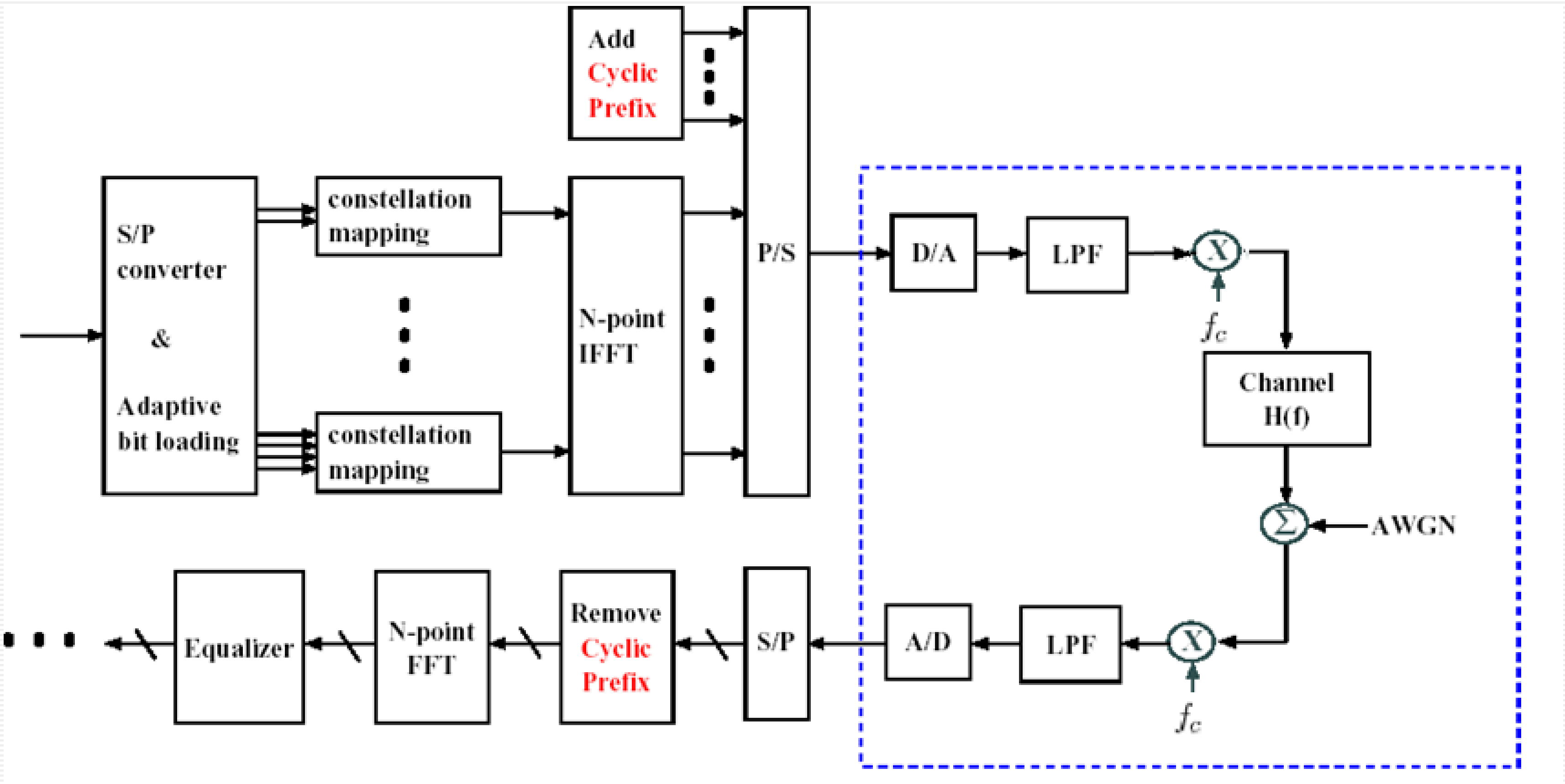


$$Y_m = H_m X_m + \Upsilon_m + \underbrace{H_{\mathrm{ISI}} X_{m-1}}_{\text{Inter Symbol Interference}}$$

⇨ **To avoid ISI,**

the CP length should be set to be longer than the channel impulse response length.

Transmitter and Receiver (OFDM)



Challenges in Multicarrier Systems (1)

- **Peak-to-Average Power Ratio**
  - PAR grows with the number of subcarriers
    - Average power

$$\mathrm{E}\left[\left(\frac{1}{\sqrt{N}}\left|x_0+x_1+\cdots+x_{N-1}\right|\right)^2\right]=\frac{1}{N}\mathrm{E}\left[\left|x_0+x_1+\cdots+x_{N-1}\right|^2\right]=\frac{\mathrm{E}[|x_0|^2]+\mathrm{E}[|x_1|^2]+\cdots+\mathrm{E}[|x_{N-1}|^2]}{N}=1$$

    - Maximum power: when all the x_i add coherently

$$max\left[\left(\frac{1}{\sqrt{N}}|x_0+x_1+\cdots+x_N|\right)^2\right]=\left|\frac{N}{\sqrt{N}}\right|^2=N$$

    - The maximum PAR is N for N subcarriers
    - The observed PAR is typically less than N because full coherent addition of all N symbols is highly improbable.

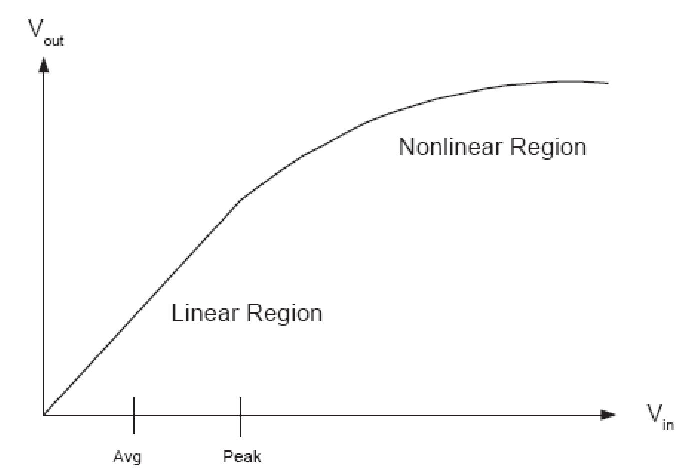


Figure 12.11: A typical power amplifier response.

Challenges in Multicarrier Systems (2)

- **Frequency Offset**
  - Orthogonality is assured by the subcarrier separation $\Delta f = 1/T_N$
  - In practice, the frequency separation of the subcarriers is imperfect
    - Due to mismatched oscillators, Doppler shifts, or timing sync error
  - Intercarrier interference (ICI)
    - The received samples of the FFT will contain interference from adjacent subcarriers due to the degradation in the orthogonality of the subcarriers
- **Timing Offset**
  - The effect of timing error is less than that from the frequency offset due to guard time (cyclic prefix), that is, as long as a full N-sample OFDM symbol is used at the receiver without interference from previous or subsequent symbols

Multiuser OFDM

- **OFDM-TDMA**
  - Channel time is divided into time slots with a fixed length
  - Each user is assigned the time slot(s).
  - The user uses all sub-carriers during its time slots
- **OFDMA**
  - The sub-carriers is grouped into subchannels
  - Each user is assigned the subchannels
  - The user uses its assigned subchannels until they are released
  - ✓ OFDM-FDMA
    - A subchannel is composed of a band of contiguous sub-carriers
    - In frequency selective fading environment, each subchannel has a different channel quality
    - Performance improvement with high reporting overhead (resource management)
  - ✓ OFDM-Interleaved-FDMA
    - A subchannel is composed of non-contiguous sub-carriers
    - Most subchannels have almost the same quality
    - Low CSI reporting overhead
- **OFMA-CDMA**

Case Study I – IEEE 802.11a

- IEEE 802.11a, which occupies a bandwidth of 20 MHz in the 5GHz band, is based on OFDM
  – $N = 64$ subcarriers are generated, although only 48 are actually used for data transmission
  – The cyclic prefix length is 1/4 of OFDM symbol time
  – Possible coding rates are 1/2, 2/3, 3/4
  – The modulation types can be used are BPSK, QPSK, 16QAM, 64QAM
- The subcarrier bandwidth $B_s = 20\text{ MHz}/64 = 312.5\text{ kHz}$
- Symbol time per subcarrier is $T_s = 1/B_s \times 5/4 = 4\ \mu\text{s}$
- Maximum data rate is $R_{Max} = 48 \times \frac{3}{4} \times 6 \times \frac{1}{4 \times 10^{-6}} = 54\,\text{Mbps}$