

Chapter 12. LINEAR PROGRAMMING

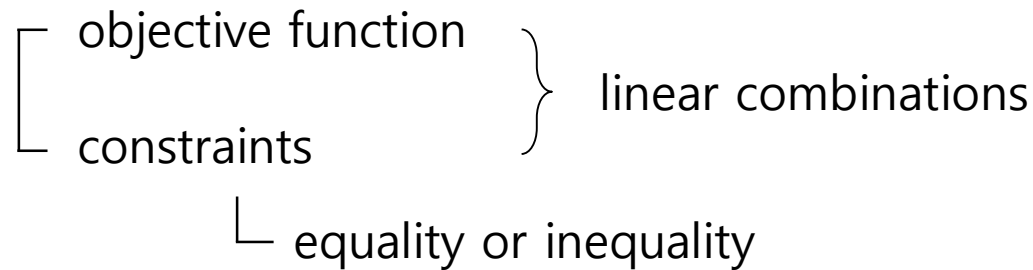
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Chapter 12. LINEAR PROGRAMMING

12.1 The origins of Linear Programming



1930s, economic models

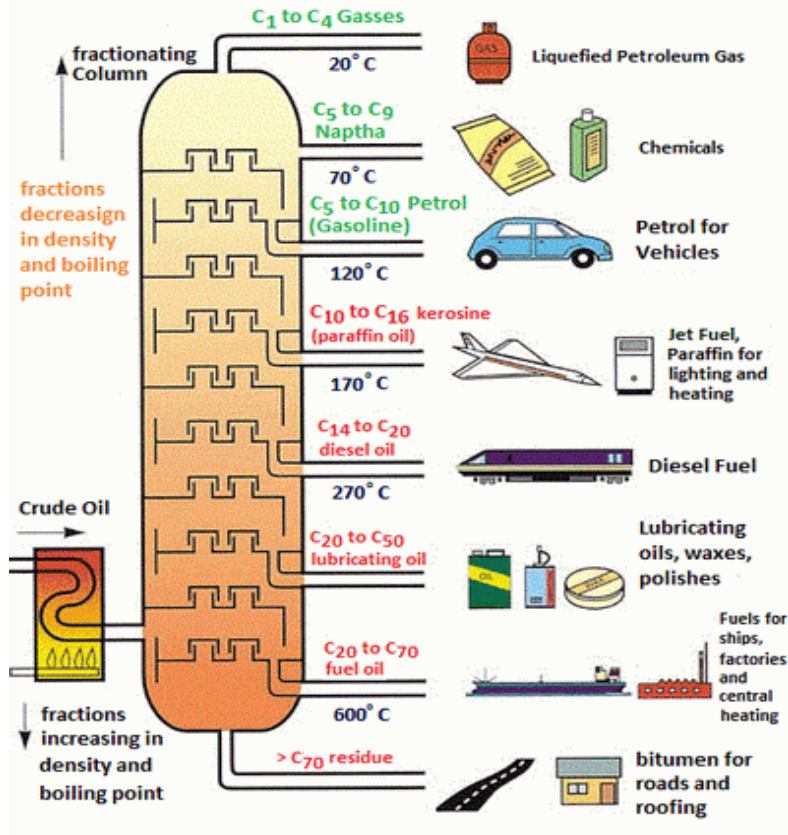
1947 USAF simplex method

└ (United States Air Force)

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12.2 Some examples

(1) blending application – oil company (2) machine allocation – manufacturing plant

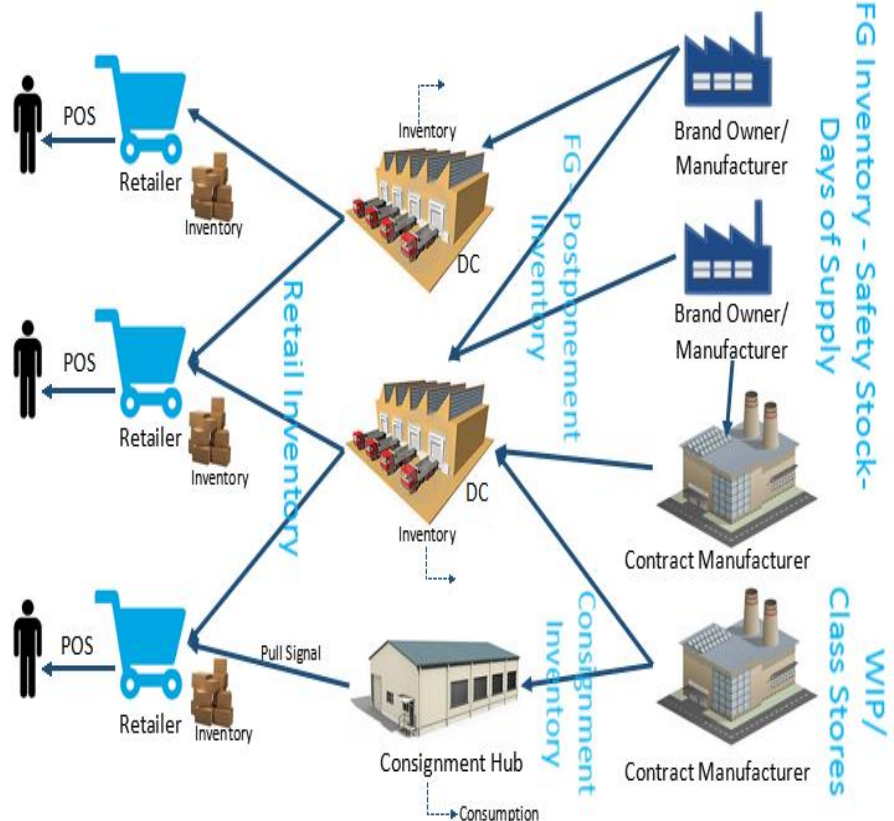


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12.2 Some examples

(3) inventory and production planning

(4) transportation



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12.3 Mathematical statement

objective function

$$y = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Constraints

$$\phi_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq r_1$$

\vdots

$$\phi_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq r_m$$

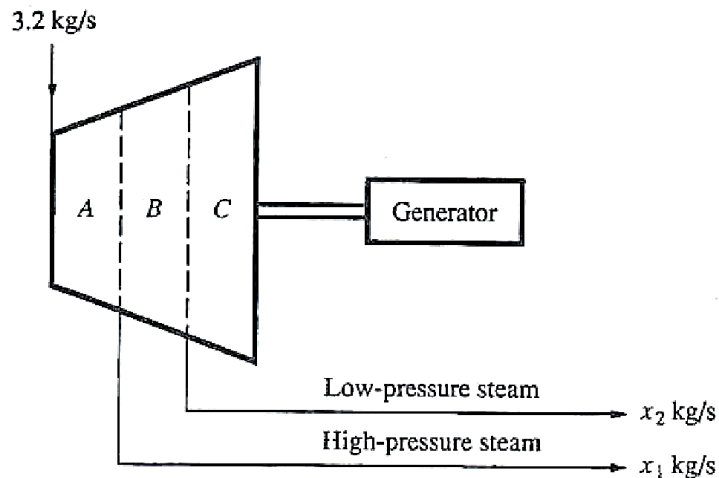
} inequality constraints

cf) Lagrange method is applicable for equality constraints

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12.4 Developing the mathematical statement

Example 12.1 : Objective function of revenue and constraint equations



Revenue rates

Electricity : 0.03 \$/kWh

Low-pressure steam : 1.10 \$/Mg

High-pressure steam : 1.65 \$/Mg

Flow rates w_A, w_B, w_C

$$P_A, \text{kW} = 48w_A$$

$$P_B, \text{kW} = 56w_A$$

$$P_C, \text{kW} = 80w_A \quad (w : \text{kg/s})$$

No less than 0.6kg/s must always flow through section C

Permissible combination of extraction rates : If $x_1 = 0 \rightarrow x_2 \leq 1.8 \text{ kg/s}$

For each x_1 kg extracted 0.25 kg less can be extracted of x_2

The customer of the process steam is primarily interested in total energy and will purchase no more than $4x_1 + 3x_2 \leq 9.6$

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12.4 Developing the mathematical statement

(Solution)

$$\text{Revenue} = \frac{1.65}{1000}(3600x_1) + \frac{1.10}{1000}(3600x_2) + 0.03(48w_A + 56w_B + 80w_C)$$

$$(w_A = 3.2 \text{ kg/s}, w_B = 3.2 - x_1, w_C = 3.2 - x_1 - x_2)$$

$$= 17.66 + 1.86x_1 + 1.56x_2$$

$$\text{Maximize } y = 1.86x_1 + 1.56x_2$$

$$\text{Constraints } x_1 + x_2 \leq 2.6$$

$$x_1 + 4x_2 \leq 7.2$$

$$4x_1 + 3x_2 \leq 9.6$$

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12.5 Geometric Visualization of the Linear-Programming Problem

(Solution)

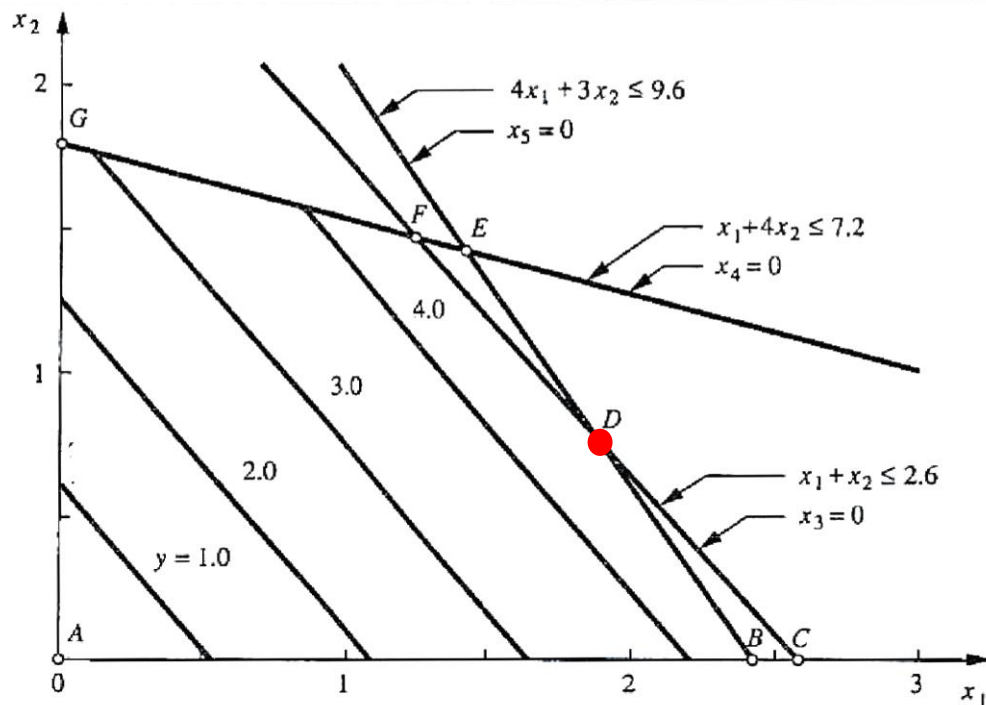


Fig. Constraints and lines of constant profit

- Permitted region : ABDFG
- Optimal point : D
- Optimum solution lies at a corner

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12.6 Introduction of Slack Variables

(Solution)

From Ex 12.1 inequalities can be converted into equalities by introduction of another variable in each equation.

$$x_1 + x_2 + x_3 = 2.6 \qquad x_3 \geq 0$$

$$x_1 + 4x_2 + x_4 = 7.2 \qquad x_4 \geq 0$$

$$4x_1 + 3x_2 + x_5 = 9.6 \qquad x_5 \geq 0$$

Slack variables : x_3, x_4, x_5

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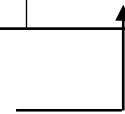
12.7~8 Preparation for the simplex algorithm, Including the objective function in the tableau

(Solution)

objective function : $y - 1.86x_1 - 1.56x_2 = 0$

	x_1	x_2	x_3	x_4	x_5	
	1	1	1			2.6
	1	4		1		7.2
	4	3			1	9.6
	-1.86	-1.56				0

current value of objective function



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12.9 Starting at the origin

(Solution)

Move from one corner to the next corner starting point $x_1=0, x_2=0$

	$x_1=0$	$x_2=0$	x_3	x_4	x_5	
	1	1	1			2.6
	1	4		1		7.2
	4	3			1	9.6
	-1.86	-1.56				0

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12.10 The simplex algorithm

(Solution)

1. Decide the variable

Maximization – largest negative difference coefficient

Minimization – largest positive difference coefficient

2. Determine the controlling constraint

3. Transfer of the controlling constraint

4. For all other boxes

$$\text{New value} = v - wZ$$

old value

coefficient of the variable being programmed

value in the preview controlling equation

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12.11 Solution of Example 12.1

Table 1	$x_1=0$	$x_2=0$	x_3	x_4	x_5	
2.6/1=2.6	1	1	1			2.6
7.2/1=7.2	1	4		1		7.2
9.6/4=2.4	4	3			1	9.6
controlling constraint (smallest)	-1.86	-1.56				0

Step 1 : largest negative x_1 should be programmed (increased from zero)

Step 2 : How much x_1 can be increased?

$$x_1 = 0 \rightarrow x_1 \neq 0$$

$$x_2 = 0 \rightarrow x_2 = 0$$

$$x_5 \neq 0 \rightarrow x_1 = 0 \quad \leftarrow x_1 \text{ increases until } x_5 \text{ becomes zero.}$$

(In Fig. 12 – 2, A → B)

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12.11 Solution of Example 12.1

Step 3 :

	x_1	$x_2=0$	x_3	x_4	$x_5=0$	
	$1-(1)(1)$	$1-(1)(0.75)$	$1-(1)(0)$	$0-(1)(1)$	$0-(1)(0.25)$	$2.6-(1)(2.4)$
	$1-(1)(1)$	$4-(1)(0.75)$	$0-(1)(0)$	$1-(1)(0)$	$0-(1)(0.25)$	$7.2-(1)(2.4)$
$\div 4$	1	0.75	0	0	0.25	2.4
	$-1.86-(-1.86)(1)$	$-1.56-(-1.86)(0.75)$	$0-(-1.86)(0)$	$0-(-1.86)(0)$	$0-(-1.86)(0.25)$	$0-(-1.86)(2.4)$

Step 4 :

Chapter 12. LINEAR PROGRAMMING

12.11 Solution of Example 12.1

Table 2	x_1	$x_2=0$	x_3	x_4	$x_5=0$	
$0.2/0.25=0.8$	0	0.25	1	0	-0.25	0.20
$4.8/3.25=1.48$	0	3.25	0	1	-0.25	4.8
$2.4/0.75=3.2$	1	0.75	0	0	0.25	2.4
controlling constraint (smallest)	0	-0.165	0	0	0.465	4.464

Step 1 : largest negative (x_2 is programmed next)

$$\rightarrow x_3 = 0.2, \quad x_4 = 4.8, \quad x_1 = 2.4, \quad y = 4.4464$$

Chapter 12. LINEAR PROGRAMMING

12.11 Solution of Example 12.1

Step 2 : x_2 increases to its limit until x_3 becomes zero

Step 3 :

	x_1	x_2	$x_3=0$	x_4	$x_5=0$	
$\div 0.25$	0	1	4	0	-1	0.8
	$0-(3.25)(0)$	$3.25-(3.25)(1)$	$0-(3.25)(4)$	$1-(3.25)(0)$	$-0.25-(3.25)(-1)$	$4.8-(3.25)(0.8)$
	$1-(0.75)(0)$	$0.75-(0.75)(1)$	$0-(0.75)(4)$	$0-(0.75)(0)$	$0.25-(0.75)(-1)$	$2.4-(0.75)(0.8)$
	$0-(-0.165)(0)$	$-0.165-(-0.165)(1)$	$0-(-0.165)(4)$	$0-(-0.165)(0)$	$0.465-(-0.165)(-1)$	$4.464-(-0.165)(0.8)$

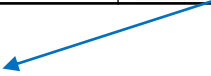
Step 4 :

Chapter 12. LINEAR PROGRAMMING

12.11 Solution of Example 12.1

<u>Table 3</u>	x_1	x_2	$x_3=0$	x_4	$x_5=0$	
	0	1	4	0	-1	0.80
	0	0	-13	1	3	2.2
	1	0	-3	0	1	1.8
	0	0	0.66	0	0.3	4.596

$\therefore x_2 = 0.8, x_1 = 1.8, x_4 = 2.2, y = 4.596$



→ no negative coefficients

→ no further improvement is possible (second constraint has no influence)

Chapter 12. LINEAR PROGRAMMING

12.12 Another Geometric Interpretation of Table Transformation

└ by changing the coordinates so that the current point is always at origin

$$y = 1.86x_1 + 1.56x_2$$

$$\begin{cases} x_1 + x_2 + x_3 = 2.6 \\ x_1 + 4x_2 + x_4 = 7.2 \\ 4x_1 + 3x_2 + x_5 = 9.6 \end{cases}$$

Table 1 x_1, x_2 - physical variables $x_1 = 0, x_2 = 0$ origin

Table 2 x_2, x_5 $x_2 = 0, x_5 = 0$

Table 3 x_3, x_5 $x_3 = 0, x_5 = 0$

Chapter 12. LINEAR PROGRAMMING

12.12 Another Geometric Interpretation of Table Transformation

	<u>Table 1</u>		<u>Table 2</u>
3 rd constraint	$4x_1 + 3x_2 + x_5 = 9.6$	→	$x_1 = -0.75x_2 - 0.25x_5 + 2.4$
1 st constraint	$x_1 + x_2 + x_3 = 2.6$	→	$0.25x_2 + x_3 - 0.25x_5 = 0.2$
2 nd constraint	$x_1 + 4x_2 + x_4 = 7.2$	→	$3.25x_2 + x_4 - 0.25x_5 = 4.8$
Objective function	$y - 1.86x_1 + 1.56x_2 = 0$	→	$y - 0.165x_2 + 0.465x_5 = 4.464$

Chapter 12. LINEAR PROGRAMMING

12.12 Another Geometric Interpretation of Table Transformation

	<u>Table 2</u>		<u>Table 3</u>
1 st constraint	$0.25x_2 + x_3 - 0.25x_5 = 0.2$	→	$x_2 = -4x_3 + x_5 + 0.8$
2 nd constraint	$3.25x_2 + x_4 - 0.25x_5 = 4.8$	→	$-13x_3 + x_4 + 3x_5 = 2.2$
3 rd constraint	$x_1 = -0.75x_2 - 0.25x_5 + 2.4$	→	$x_1 - 3x_3 + x_5 = 1.8$
Objective function	$y - 0.165x_2 + 0.465x_5 = 4.464$	→	$y + 0.66x_3 + 0.3x_5 = 4.596$

Chapter 12. LINEAR PROGRAMMING

12.12 Another Geometric Interpretation of Table Transformation

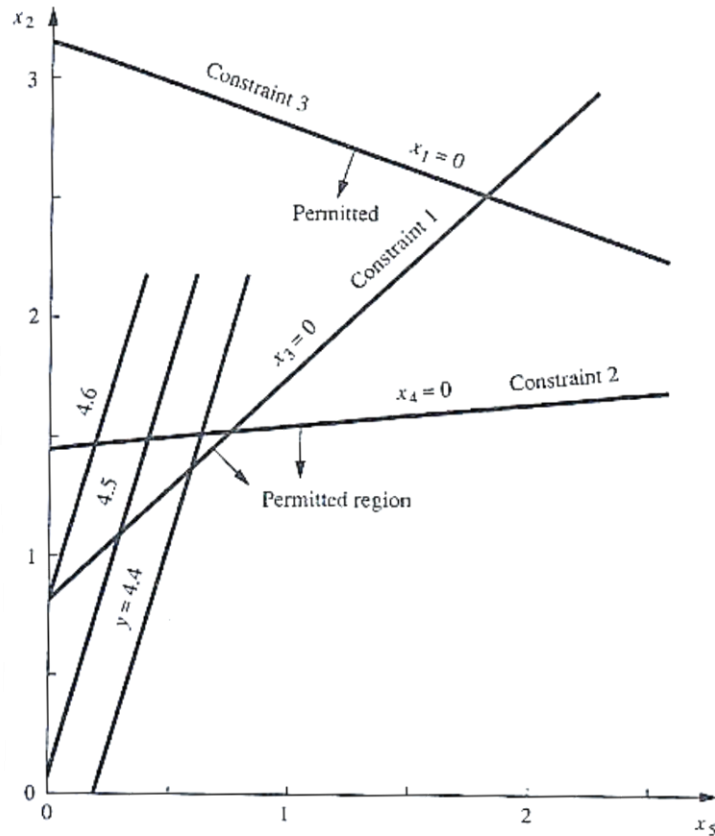


FIGURE 12-3
Tableau 2 expressed on x_5x_2 coordinates.

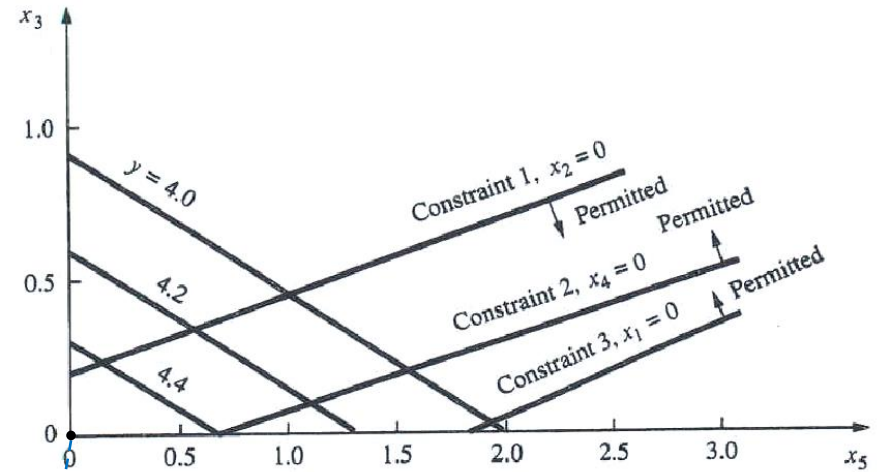


FIGURE 12-4
Tableau 3 expressed on x_5x_3 coordinates.

$y = 4.596$ at $x_3 = 0, x_5 = 0$

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12.14 # of variables and # of constraints

n

($m = \#$ of slack variables)

At optimum, n variables are zero (corner)

$m > n$ $m - n$ constraints play no role

$m < n$ $n - m$ variables are zero

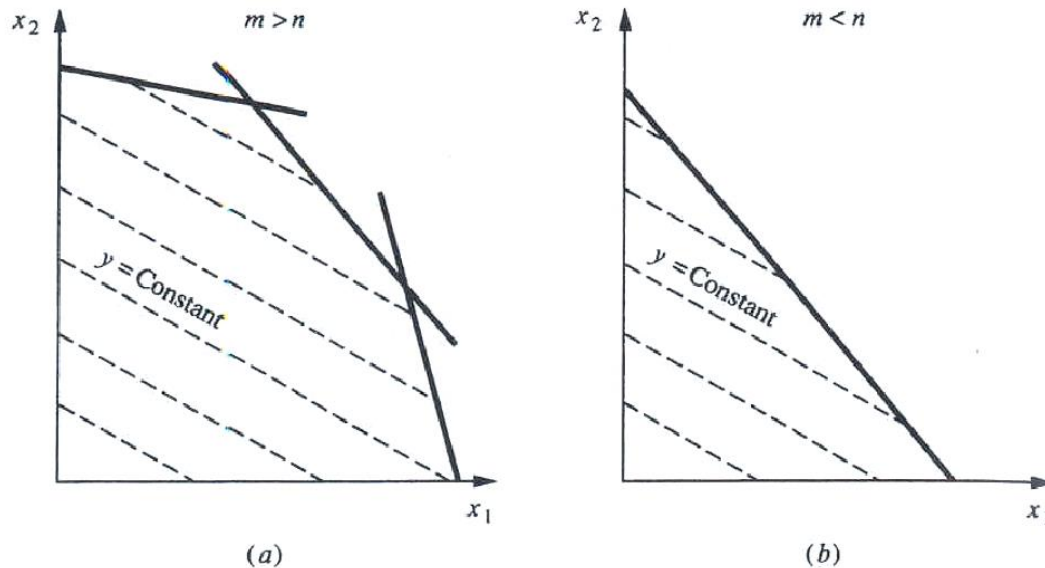


FIGURE 12-5
Relation of number of physical and slack variables.

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12.15 Minimization with greater than constraints

- ✓ Maximization with less than constraints
 - Moving from one corner to another adjacent corner
(start from the origin)
- ✓ Minimization with greater than constraints
 - Locating the first feasible point - difficult
 - introduction of artificial variable (12.16)

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12.16 Artificial variables

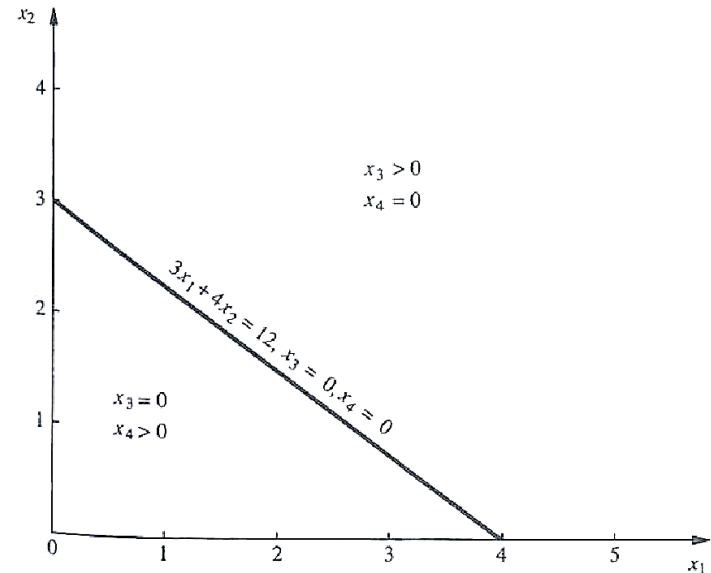
$$3x_1 + 4x_2 \geq 12$$

$$3x_1 + 4x_2 - x_3 = 12 \quad x_3 \geq 0$$

If $x_1 = x_2 = 0$ (origin), it is not realistic

$$3x_1 + 4x_2 - x_3 + x_4 = 12$$

↑ slack variable
↑ artificial variable



Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

Example 12.2 : Minimum value y , magnitudes x_1, x_2
(at the minimum : $y = 6x_1 + 3x_2$)

Subject to the constraints

$$5x_1 + x_2 \geq 10, \quad 9x_1 + 13x_2 \geq 74, \quad x_1 + 3x_2 \geq 9$$

(Solution)

$$\begin{cases} 5x_1 + x_2 - x_3 + x_6 = 10 \\ 9x_1 + 13x_2 - x_4 + x_7 = 74 \\ x_1 + 3x_2 - x_5 + x_8 = 9 \end{cases}$$

$$y = 6x_1 + 3x_2 + Px_6 + Px_7 + Px_8$$

$P \rightarrow$ A numerical value which is extremely large

$x_3, x_4, x_5 \rightarrow$ slack variables

$x_6, x_7, x_8 \rightarrow$ artificial variables

Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

(Solution)

Starting point-origin with all slack variable=0

artificial variable > 0

$$x_6 = 10 - 5x_1 - x_2 + x_3$$

$$x_7 = 74 - 9x_1 - 13x_2 + x_4$$

$$x_8 = 9 - x_1 - 3x_2 + x_5$$

$$y = (6 - 15P)x_1 + (3 - 17P)x_2 + Px_3 + Px_4 + Px_5 + 93P$$

Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

(Solution)

<u>Table 1</u>		x_1	x_2	x_3	x_4	x_5	
10	x_6	5	1	-1	0	0	10
74/13	x_7	9	13	0	-1	0	74
3	x_8	1	3	0	0	-1	9
		15P-6	17P-3	-P	-P	-P	93P

controlling constraint
(smallest)

largest positive
difference coefficient

Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

(Solution)

Table 1

Table 2

$$x_6 = 10 - 5x_1 - x_2 + x_3 \rightarrow x_6 = 7 - \left(\frac{14}{3}\right)x_1 + x_3 - \left(\frac{1}{3}\right)x_5 + \left(\frac{1}{3}\right)x_8$$

$$x_7 = 74 - 9x_1 - 13x_2 + x_4 \rightarrow x_7 = 35 - \left(\frac{14}{3}\right)x_1 + x_4 - \left(\frac{13}{3}\right)x_5 + \left(\frac{13}{3}\right)x_8$$

$$x_8 = 9 - x_1 - 3x_2 + x_5 \rightarrow x_2 = 3 - \left(\frac{1}{3}\right)x_1 + \left(\frac{1}{3}\right)x_5 - \left(\frac{1}{3}\right)x_8$$

$$y = (6 - 15P)x_1 + (3 - 17P)x_2 + Px_3 + Px_4 + Px_5 + 93P$$

$$\rightarrow y = \frac{15 - 28P}{3}x_1 + Px_3 + Px_4 + \frac{3 - 14P}{3}x_5 + \frac{-3 + 17P}{3}x_8 + 42P + 9$$

Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

(Solution)

Table 2		x_1	x_3	x_4	x_5	x_8	
$3/2$	x_6	$14/3$	-1	0	$1/3$	$-1/3$	7
$105/14$	x_7	$14/3$	0	-1	$13/3$	$-13/3$	35
9	x_2	$1/3$	0	0	$-1/3$	$1/3$	3
		$(28P-15)/3$	$-P$	$-P$	$(14P-3)/3$	$(-17P+3)/3$	$42P+9$

controlling constraint
(smallest)

largest positive
difference coefficient

Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

(Solution)

Table 2

Table 3

$$x_6 = 7 - \left(\frac{14}{3}\right)x_1 + x_3 - \left(\frac{1}{3}\right)x_5 + \left(\frac{1}{3}\right)x_8 \rightarrow x_1 = \frac{3}{2} + \left(\frac{3}{14}\right)x_3 - \left(\frac{1}{14}\right)x_5 - \left(\frac{3}{14}\right)x_6 + \left(\frac{1}{14}\right)x_8$$

$$x_7 = 35 - \left(\frac{14}{3}\right)x_1 + x_4 - \left(\frac{13}{3}\right)x_5 + \left(\frac{13}{3}\right)x_8 \rightarrow x_7 = 28 - x_3 + x_4 - 4x_5 + x_6 + 4x_8$$

$$x_2 = 3 - \left(\frac{1}{3}\right)x_1 + \left(\frac{1}{3}\right)x_5 - \left(\frac{1}{3}\right)x_8 \rightarrow x_2 = \frac{5}{2} - \left(\frac{1}{14}\right)x_3 + \left(\frac{5}{14}\right)x_5 + \left(\frac{1}{14}\right)x_6 - \left(\frac{5}{14}\right)x_8$$

$$y = \frac{15 - 28P}{3}x_1 + Px_3 + Px_4 + \frac{3 - 14P}{3}x_5 + \frac{-3 + 17P}{3}x_5 + 42P + 9$$

$$\rightarrow y = \frac{15 - 14P}{14}x_3 + Px_4 + \frac{9 - 56P}{14}x_5 + \frac{-15 + 28P}{14}x_6 + \frac{-9 + 70P}{14}x_8 + \frac{56P + 33}{30} \quad \frac{2}{34}$$

Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

(Solution)

Table 3		x_3	x_4	x_5	x_6	x_8	
21	x_1	$-3/14$	0	$1/14$	$3/14$	$-1/14$	$3/2$
7	x_7	1	-1	4	-1	-4	28
-7	x_2	$1/14$	0	$-5/14$	$-1/14$	$5/14$	$5/2$
		$(14P-15)/14$	-P	$(56P-9)/14$	$(-28P+15)/14$	$(-70P+9)/14$	$(56P+33)/2$

controlling constraint
(smallest)

largest positive
difference coefficient

Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

(Solution)

Table 3

Table 4

$$x_1 = \frac{3}{2} + \left(\frac{3}{14}\right)x_3 - \left(\frac{1}{14}\right)x_5 - \left(\frac{3}{14}\right)x_6 + \left(\frac{1}{14}\right)x_8 \rightarrow x_1 = 1 + \left(\frac{13}{56}\right)x_3 - \left(\frac{1}{56}\right)x_4 - \left(\frac{13}{56}\right)x_6 + \left(\frac{1}{56}\right)x_7$$

$$x_7 = 28 - x_3 + x_4 - 4x_5 + x_6 + 4x_8 \rightarrow x_5 = 7 - \left(\frac{1}{4}\right)x_3 + \left(\frac{1}{4}\right)x_4 + \left(\frac{1}{4}\right)x_6 - \left(\frac{1}{4}\right)x_7 + x_8$$

$$x_2 = \frac{5}{2} - \left(\frac{1}{14}\right)x_3 + \left(\frac{5}{14}\right)x_5 + \left(\frac{1}{14}\right)x_6 - \left(\frac{5}{14}\right)x_8 \rightarrow x_2 = 5 - \left(\frac{9}{56}\right)x_3 + \left(\frac{5}{56}\right)x_4 + \left(\frac{9}{56}\right)x_6 - \left(\frac{5}{56}\right)x_7$$

$$y = \frac{15 - 14P}{14}x_3 + Px_4 + \frac{9 - 56P}{14}x_5 + \frac{-15 + 28P}{14}x_6 + \frac{-9 + 70P}{14}x_8 + \frac{56P + 33}{2}$$

$$\rightarrow y = \frac{51}{56}x_3 + \frac{9}{56}x_4 + \left(P - \frac{15}{56}\right)x_6 + \left(P - \frac{9}{56}\right)x_7 + Px_8 + 21$$

Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

(Solution)

Table 4	x_3	x_4	x_6	x_7	x_8	
x_1	$-13/56$	$1/56$	$13/56$	$-1/56$	0	1
x_5	$1/4$	$-1/4$	$-1/4$	$1/4$	-1	7
x_2	$9/56$	$-5/56$	$-9/56$	$5/56$	0	5
	$-51/56$	$-9/56$	$-P+51/56$	$-P+9/56$	$-P$	21

$\therefore x_1 = 1, \quad x_5 = 7, \quad x_2 = 5, \quad y = 21$



Chapter 12. LINEAR PROGRAMMING

12.17 Simplex algorithm to minimization problem

(Solution)

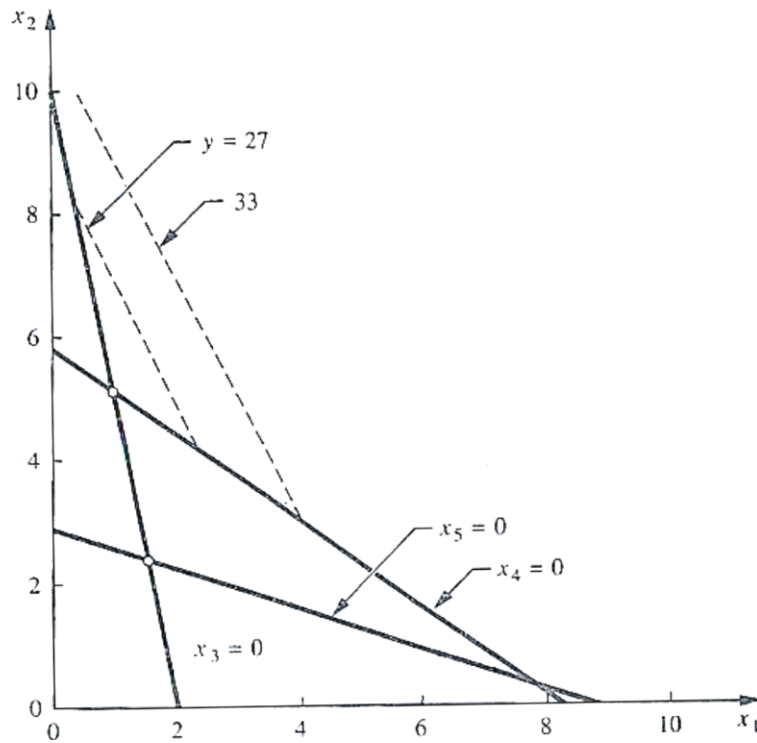


Fig. Minimization

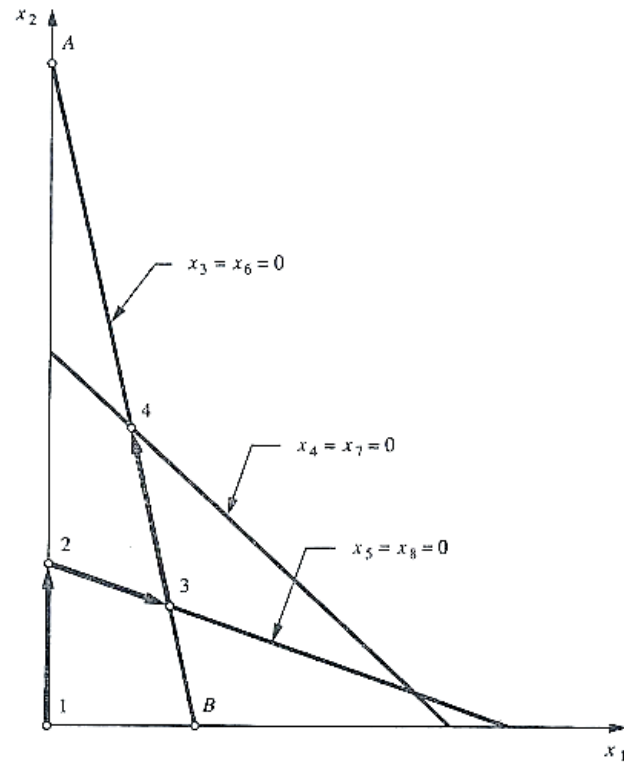


Fig. Points represented by successive tableaux