Chapter 8. Lagrange Multipliers

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Chapter 8. Lagrange Multipliers

8.1 Calculus Methods of optimization

- Differentiable function
- Equality constraints

- Inequaility constraints
- Not continuous function

➡ Using calculus

➡ Using the other methods

8.2 Lagrange Multiplier equations

Function : $y = y(x_1, x_2, \cdots, x_n)$

Constraints :

$$\phi_1(x_1, x_2, \cdots, x_n) = 0$$

$$\phi_2(x_1, x_2, \cdots, x_n) = 0$$

....

$$\phi_m(x_1, x_2, \cdots, x_n) = 0$$

How to optimize the function subject to the constraints?

Chapter 8. Lagrange Multipliers

8.2 Lagrange Multiplier equations

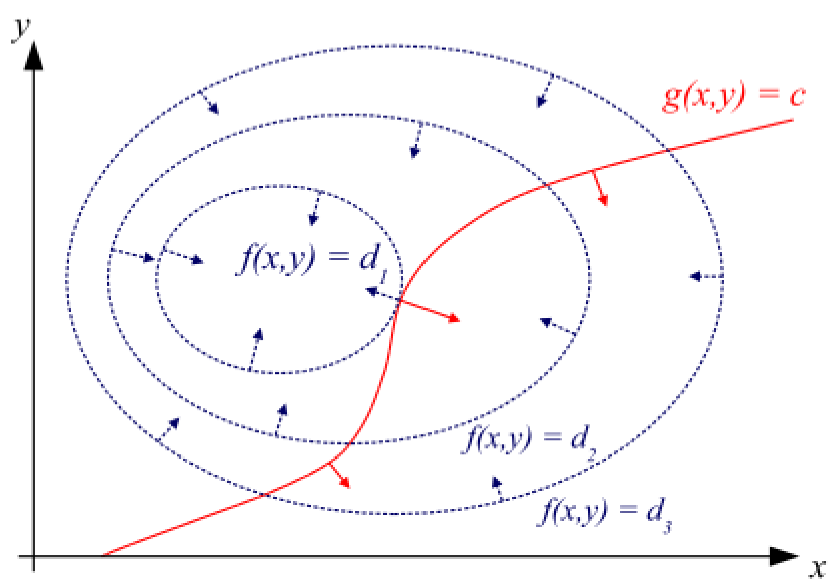
For an optimized point function f(x,y) and g(x,y) are parallel, which means gradient of f(x,y) is tangent to that of g(x,y)

Thus, for a certain constant λ, equation below is established

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

For multiple m constraints

$$\nabla f = \sum_{k=1}^{m} \lambda_m \nabla g_m = 0$$



8.3 The gradient vector

Gradient of scalar

$$\nabla y = \frac{\partial y}{\partial x_1}\hat{\imath}_1 + \frac{\partial y}{\partial x_2}\hat{\imath}_2 + \cdots + \frac{\partial y}{\partial x_n}\hat{\imath}_n$$

$$\nabla \ = gradient\ vector$$

$$\hat{\imath}\ = unit\ vector : have\ direction, unit\ magnitude$$

$$(\hat{\imath}_1,\ \hat{\imath}_2,\ \hat{\imath}_3\ are\ the\ unit\ vector\ in\ the\ x_1, x_2, x_3\ direction)$$

Chapter 8. Lagrange Multipliers

8.4 Further explanation of Lagrange multiplier equations

n scalar equations

$$\hat{\imath}_1: \quad \frac{\partial y}{\partial x_1} - \lambda_1 \frac{\partial \phi_1}{\partial x_1} - \lambda_m \frac{\partial \phi_m}{\partial x_1} = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\hat{\imath}_n: \quad \frac{\partial y}{\partial x_n} - \lambda_1 \frac{\partial \phi_1}{\partial x_n} - \lambda_m \frac{\partial \phi_m}{\partial x_n} = 0$$

+ m constraint equations ➡ m+n simultaneous equations

unknowns $\underbrace{\lambda_1 \cdots \lambda_m}_{m}, \overbrace{x_1^* \cdots x_n^*}^{n}$

at optimum point

if

$m < n$ optimum point can be found

$m = n$ fixed values of x's (no optimization)

8.5 Unconstrained optimization

$y = y(x_1, x_2, \cdots, x_n)$

$$\rightarrow \ \nabla y = 0 \ (\text{no } \phi's) \ \ or \ \frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} = \cdots = \frac{\partial y}{\partial x_n} = 0$$

The state point where the derivatives are zero = **critical point**

⇒ may be max, min, saddle point, ridge, valley

Example 8.1 :

Find minimum value of z for a given constraint

$$z = x^2 + y^2 \qquad\qquad 1 = x^2 + xy + y^2$$

(Solution)

$$\nabla f(x, y) = \lambda \nabla g(x, y) \qquad \cdots\cdots (1)$$

$$g(x, y) - 1 = 0 \qquad \cdots\cdots (2)$$

From (1),

$$2x - \lambda(2x + y) = 0$$
$$2y - \lambda(2y + x) = 0$$

$$y = x \ or \ y = -x$$

Substituting (2) from the result

$$1 = 3x^2 \text{ or } 1 = x^2$$

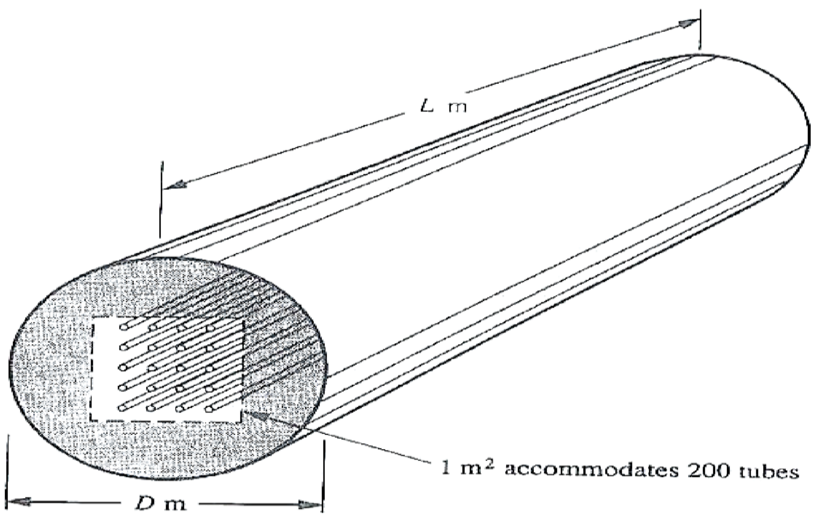
Thus, minimum value z is $\frac{2}{3}$ at $x, y = \left(\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

Example 8.2,3 : Constrained/Unconstrained optimization

A total length of 100 m of tubes must be installed in a shell-and-tube heat exchanger.

$$Total\ cost = 900 + 1100D^{2.5}L + 320DL \qquad (\boldsymbol{a})$$

where L is the length of the heat exchanger and D is the diameter of the shell, both in meters. **200 tubes will fit in a cross-sectional area of 1 m²** in the shell. Determine the D and L of the heat exchanger for **<u>minimum first cost</u>.**



(Solution) – Unconstrained Optimization

Have to put 100m tubes in the shell

$$: \left(\frac{\pi D^2}{4}\,\mathrm{m}^2\right)(200\ \mathrm{tubes/m}^2) * (L[\mathrm{m}]) = 100\ \mathrm{m}$$

$$\Rightarrow 50\pi D^2 L = 100 \qquad (\boldsymbol{b})$$

Substitute (b) into (a)

$$\Rightarrow Total\ cost = 900 + \frac{2200}{\pi} D^{0.5} + \frac{640}{\pi D}$$

$$\frac{d(Total\ cost)}{d(D)} = \frac{1100}{\pi D^{0.5}} - \frac{640}{\pi D^2}$$

$$D^* = 0.7\ \mathrm{m},\ \ L^* = 1.3\ \mathrm{m}, Total\ cost^* = \$1777.45$$

(Solution) – Constrained Optimization

$$Total\ cost = 900 + 1100D^{2.5}L + 320DL \qquad (\boldsymbol{a})$$

$$50\pi D^2 L = 100 \qquad (\boldsymbol{b})$$

$$\nabla \mathrm{y} = [(2.5)(1100)D^{1.5}L + 320L]\widehat{i_1} + (1100D^{2.5} + 320D)\widehat{i_2}$$

$$\nabla \phi = 100\pi DL\widehat{i_1} + 50\pi D^2\widehat{i_2}$$

$$\widehat{i_1}: \qquad 2750D^{1.5}L + 320L - \lambda 100\pi DL = 0 \qquad (\boldsymbol{c})$$

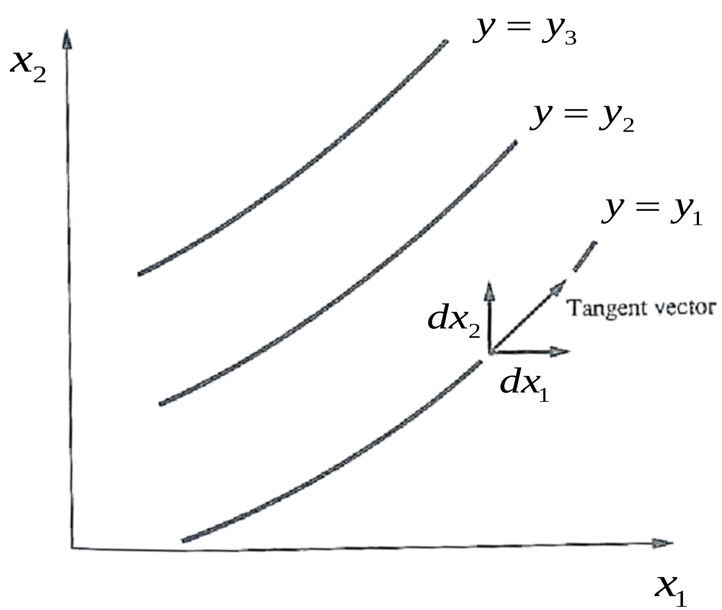
$$\widehat{i_2}: \qquad 1100D^{2.5} + 320D - \lambda 50\pi D^2 = 0 \qquad (\boldsymbol{d})$$

Using (b), (c), (d), find D^*, L^*, λ

Result is the **same** as unconstrained optimization

8.7 Gradient vector

- gradient vector is normal to contour line



$$y = y(x_1, x_2)$$

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2$$

Along the line of $y = const.$

$$dy = 0 \rightarrow dx_1 = -dx_2 \frac{\partial y / \partial x_2}{\partial y / \partial x_1}$$

Substitution into arbitrary unit vector

$$\frac{dx_1 \hat{\imath} + dx_2 \hat{\jmath}}{\sqrt{(dx_1)^2 + (dx_2)^2}}$$

8.7 Gradient vector

Tangent vector

$$\hat{T} = \frac{dx_1\hat{\imath} + dx_2\hat{\jmath}}{\sqrt{(dx_1)^2 + (dx_2)^2}} = \frac{\widehat{i_2} - \dfrac{\partial y/\partial x_2}{\partial y/\partial x_1}\widehat{i_1}}{\sqrt{\left(\dfrac{\partial y/\partial x_2}{\partial y/\partial x_1}\right)^2 + 1}} = \frac{-\dfrac{\partial y}{\partial x_2}\widehat{i_1} + \dfrac{\partial y}{\partial x_1}\widehat{i_2}}{\sqrt{\left(\dfrac{\partial y}{\partial x_2}\right)^2 + \left(\dfrac{\partial y}{\partial x_1}\right)^2}}$$

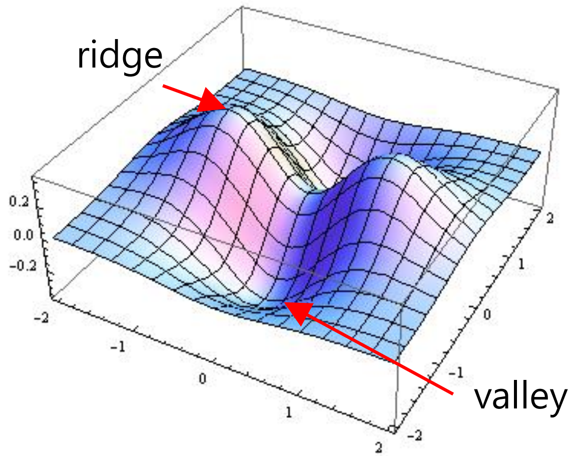
Gradient vector

$$\hat{G} = \frac{\nabla y}{|\nabla y|} = \frac{(\partial y/\partial x_1)\widehat{i_1} + (\partial y/\partial x_2)\widehat{i_2}}{\sqrt{(\partial y/\partial x_1)^2 + (\partial y/\partial x_2)^2}}$$

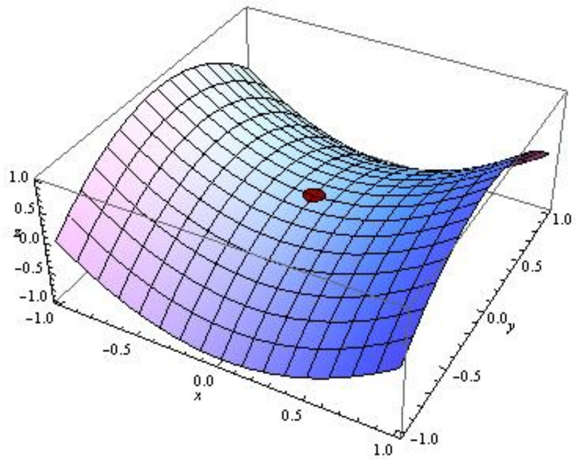
$$\vec{T} \bullet \vec{G} = 0$$

8.9 Test for maximum or minimum

decide whether the point is a maximum, minimum, saddle point, ridge, or valley



ridge & valley point



saddle point

8.9 Test for maximum or minimum

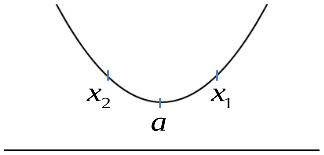
$x = a$: *the minimum is expected to occur*

$y(x) = y(a) + \frac{dy}{dx}(x-a) + \frac{1}{2}\frac{d^2y}{dx^2}(x-a)^2 + \cdots$ Taylor series expansion near $x = a$

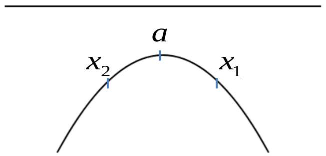
If $\frac{dy}{dx} > 0$ *or* $\frac{dy}{dx} < 0$ *there is no minimum*

$$\rightarrow \frac{dy}{dx} = 0$$

If $\frac{d^2y}{dx^2} > 0$ $\quad y(x_1) > y(a)$ *when* $x_1 > a$; $\quad y(x_2) > y(a)$ *when* $x_2 < a$ $\quad$ minimum



If $\frac{d^2y}{dx^2} < 0$ $\quad y(x_1) < y(a)$ *when* $x_1 > a$; $\quad y(x_2) < y(a)$ *when* $x_2 < a$ $\quad$ maximum



8.9 Test for maximum or minimum

(a_1, a_2): *the minimum of* $y(x_1, x_2)$ *is expected to occur*

$$y(x_1,x_2) = y(a_1,a_2) + \overset{=0}{\boxed{\frac{\partial y}{\partial x_1}}}(x_1 - a_1) + \overset{=0}{\boxed{\frac{\partial y}{\partial x_2}}}(x_2 - a_2) + \frac{1}{2}y''_{11}(x_1 - a_1)^2 +$$

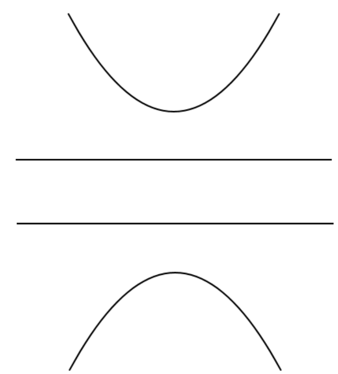
$$y''_{12}(x_1 - a_1)(x_2 - a_2) + \frac{1}{2}y''_{22}(x_2 - a_2)^2 + \cdots$$

$$D = \begin{vmatrix} y''_{11} & y''_{12} \\ y''_{21} & y''_{22} \end{vmatrix}$$

$$= y''_{11}y''_{22} - {y''_{12}}^2$$

$D > 0 \quad and \quad y''_{11} > 0 \quad (y''_{22} > 0)$ min

$D > 0 \quad and \quad y''_{11} < 0 \quad (y''_{22} < 0)$ max



$D < 0 \quad and \quad y''_{11} < 0 \quad (y''_{22} < 0)$

Example 8.4 : Test for maximum or minimum

Optimal values of x_1 and x_2, test max, min

$$y = \frac{x_1^2}{4} + \frac{2}{x_1 x_2} + 4x_2$$

(Solution)

$$\frac{\partial y}{\partial x_1} = \frac{x_1}{2} - \frac{2}{x_1^2 x_2}$$
$$\frac{\partial y}{\partial x_2} = -\frac{2}{x_1 x_2^2} + 4$$

$\Rightarrow$ $x_1^* = 2, x_2^* = \frac{1}{2}$

At x_1 and x_2

$$\frac{\partial^2 y}{\partial x_1^2} = \frac{3}{2}, \frac{\partial^2 y}{\partial x_2^2} = 16, \frac{\partial^2 y}{\partial x_1 \partial x_2} = 2 \quad \rightarrow \begin{vmatrix} \frac{3}{2} & 2 \\ 2 & 16 \end{vmatrix} > 0 \quad and \quad y''_{11} > 0 \quad \text{minimum}$$

8.10 Sensitivity coefficients

- represents the effect on the optimal value of slightly relaxing the constraints
- variation of optimized objective function y^*

In example 8.3, if the total length of the tube increases from 100m to random value 'H', what would be the increase in minimum cost ?

$$50\pi D^2 L = H \ \rightarrow \ D^* = 0.7m, L^* = 0.013H$$

$$Total\ cost^* = 900 + 1100{D^*}^{2.5}L^* + 320D^*L^* = 900 + 8.78H$$

$$SC = \frac{\partial(Total\ cost^*)}{\partial H} = 8.78 = \lambda$$

Extra meter of tube for the heat exchanger would cost additional $8.78