

## Chapter 9. SEARCH METHODS

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# Chapter 9. SEARCH METHODS

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## 9.1 Overview of search methods

- The major effort in the **optimization** was determining the values of the independent variables that provide the optimum.
- Search methods generally fall into categories;
  - ┌ elimination
  - └ hill-climbing
    - no one systematic procedure
    - ultimate approach if other optimization methods fail

# Chapter 9. SEARCH METHODS

## 9.1 Overview of search methods

Single variable

- a. Exhaustive
- b. Efficient

Multivariable, unconstrained

- a. Lattice
- b. Univariate
- c. Steepest ascent

Multivariable, constrained

- a. Penalty functions
- b. Search along a constraint

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## 9.2 Interval of uncertainty

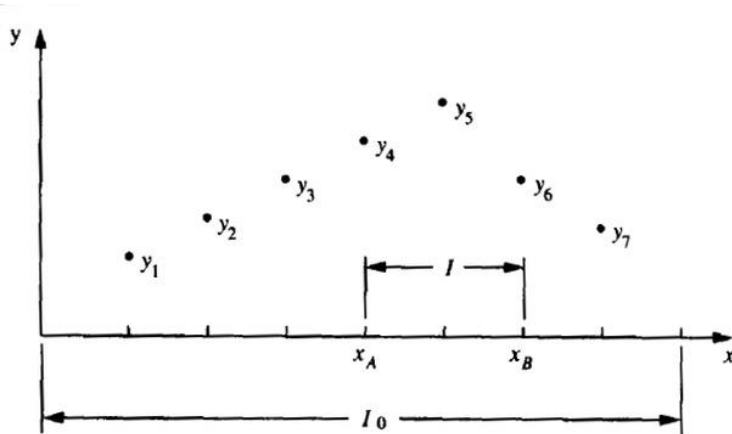
- In search methods, the **precise point** at which the optimum occurs **will never be known**
- The best that can be achieved is to specify the **interval of uncertainty**

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## 9.3 Exhaustive search (linear search)

- The exhaustive search is most widely used
- Interval of interest is uniformly divided by (number of observation + 1)

$I_0$



number of observation:  $n = 7$

devided interval:  $\frac{1}{n+1} I_0$

Maximum lies :  $y(x_A) < y_{\max} < y(x_B)$

Interval of uncertainty:  $I = \frac{2}{(n+1)} I_0 = \frac{2}{8} I_0$

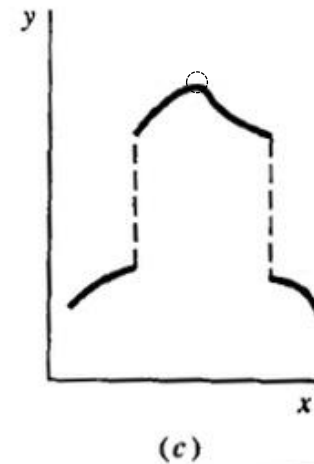
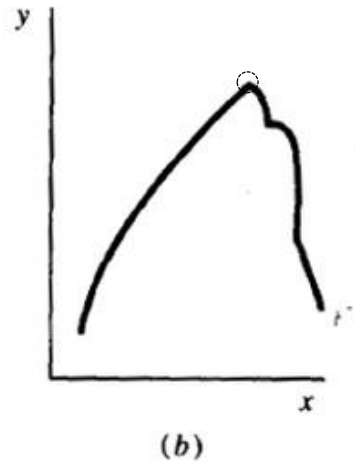
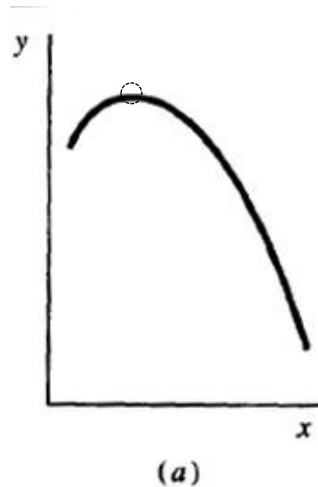
Right next 2 sides of maximum y

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## 9.4 Unimodal functions

- Only one peak (or valley) in the interval of interest

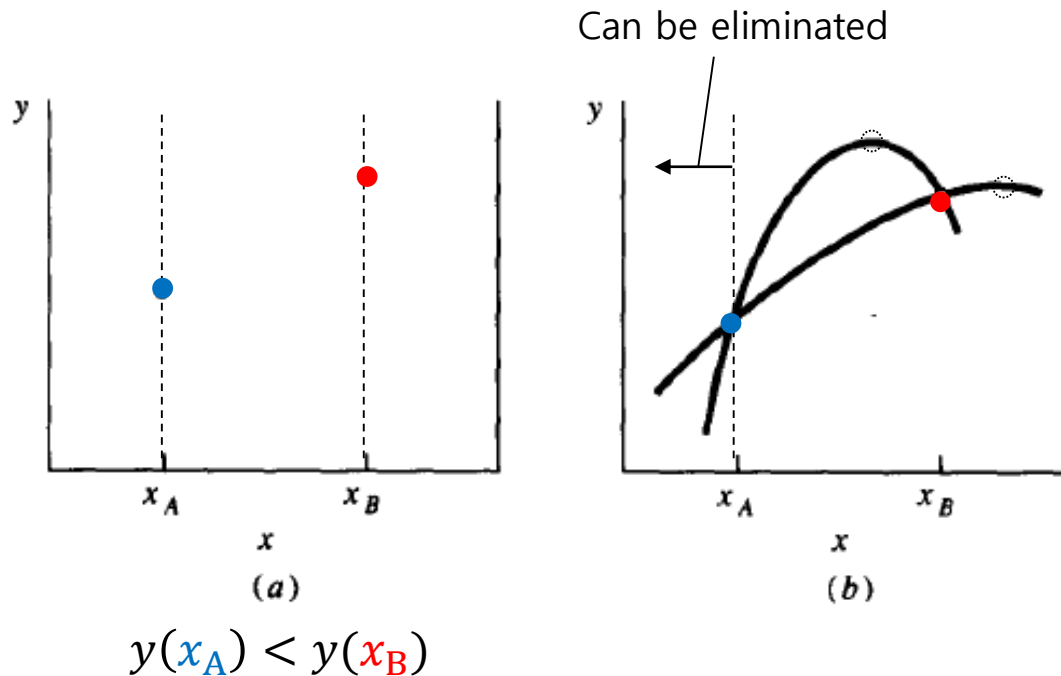
- └ dichotomous search method
- └ Fibonacci search method



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## 9.5 Eliminating a section based on two tests

- It can be eliminated one side at two different position of an unimodal function.



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## 9.6 Dichotomous search

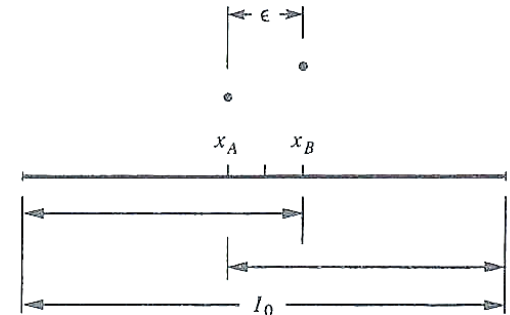
- Searching from the middle of the interval with a range,  $\varepsilon$
- Comparing  $x_A, x_B$ , smaller part of the interval is eliminated

1<sup>st</sup> trial remaining interval : 
$$I = \frac{I_0 + \varepsilon}{2}$$

2<sup>nd</sup> trial remaining interval : 
$$I = \frac{\frac{I_0 + \varepsilon}{2} + \varepsilon}{2} = \frac{I_0}{4} + \left( \varepsilon - \frac{1}{4} \varepsilon \right)$$

3<sup>rd</sup> trial remaining interval : 
$$I = \frac{\frac{\frac{I_0 + \varepsilon}{2} + \varepsilon}{2} + \varepsilon}{2} = \frac{I_0}{8} + \left( \varepsilon - \frac{1}{8} \varepsilon \right)$$

$n$  trial points ( $n=2,4,6,\dots$ ) : 
$$I = \frac{I_0}{2^{n/2}} + \varepsilon \left( 1 - \frac{1}{2^{n/2}} \right)$$



$I$  : interval of uncertainty  
 $I_0$  : interval of interest  
 $\varepsilon$  : space between two points



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## 9.7 Fibonacci search

- What is Fibonacci series?

$$F_1 = 1, \quad F_2 = 1, \quad F_i = F_{i-2} + F_{i-1} \quad (i \geq 2)$$

$$F = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$$

- Fibonacci series in nature

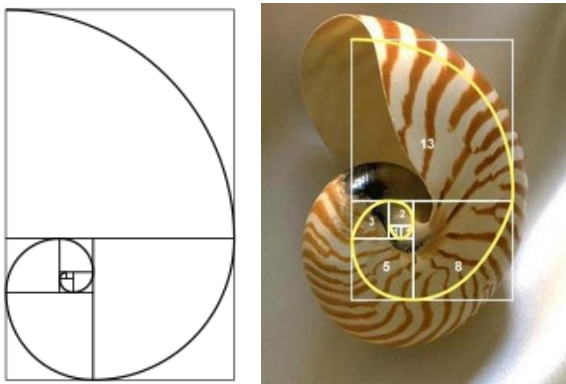


Fig. Fibonacci spiral and shell\*

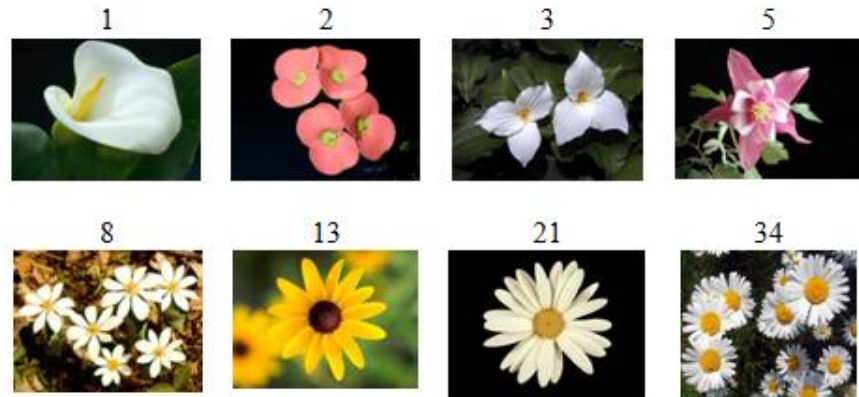


Fig. Number of flower petals and Fibonacci series\*\*

\*[https://en.wikipedia.org/wiki/Golden\\_spiral](https://en.wikipedia.org/wiki/Golden_spiral),20170421

\*\*<http://developeriq.in/articles/2014,20170421>

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## 9.7 Fibonacci search

- Applying Fibonacci series to search method

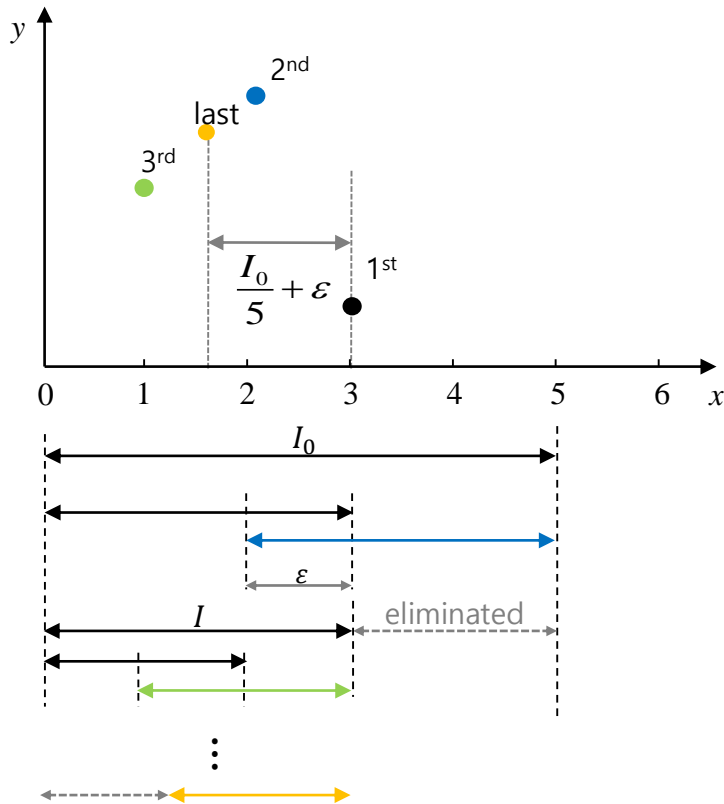
- ① Decide how many observations(n)
- ② Place the first observation in  $I_0$  at a distance of  $I_0 \frac{F_{n-1}}{F_n}$  from both ends
- ③ Place the next observation in the interval of uncertainty at a position that is symmetric to the existing observation
- ④ Interval reduces according to Fibonacci series

$$I_1 = I_0 \frac{F_{n-1}}{F_n} \quad I_2 = I_1 \frac{F_{n-2}}{F_{n-1}} = I_0 \frac{F_{n-2}}{F_n} \quad I_3 = I_2 \frac{F_{n-3}}{F_{n-2}} = I_0 \frac{F_{n-3}}{F_n} \quad \dots$$

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## Example 9.1

Find the maximum of the function  $y = -x^2 + 4x + 2$   
in the interval  $0 < x < 5$



Arbitrarily choose:  $n = 4$ ,  $I_0 = 5$

$$1^{\text{st}}: x_1 = I_0 \frac{F_3}{F_4} = \frac{3}{5} I_0 = 3$$

2<sup>nd</sup>: symmetric  $0 \sim 5 \rightarrow x_2 = 2$   
eliminate  $3 < x < 5$

3<sup>rd</sup>: symmetric  $0 \sim 3 \rightarrow x_3 = 1$   
eliminate  $0 < x < 1$

$\vdots$

Final:  $x = 2 - \epsilon$

Interval of uncertainty

$$2 - \epsilon \leq x \leq 3 \frac{I_0}{5} + \epsilon = \frac{I_0}{F_n} + \epsilon$$

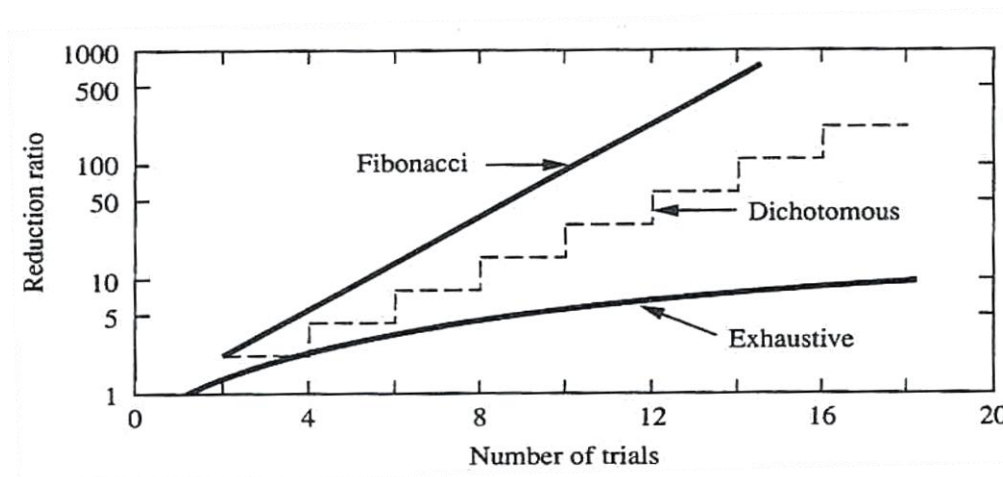
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## 9.8 Comparative effectiveness of search methods

$$\text{Reduction Ratio (RR)} = \frac{I_0}{I_n}$$

single variable search

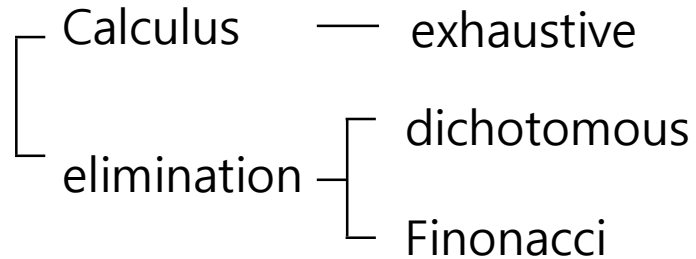
$$= \begin{cases} \frac{n+1}{2} & \text{exhaustive} & \text{O.K.} \\ 2^{\frac{n}{2}} & \text{dichotomous} & \text{good} \\ F_n & \text{Fibonacci} & \text{good} \end{cases}$$



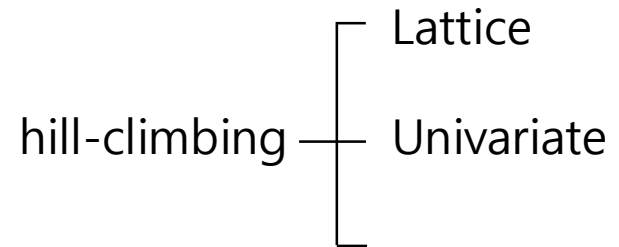
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## 9.10 Multivariable, unconstrained optimization

- Single variable



- Multivariable, unconstrained

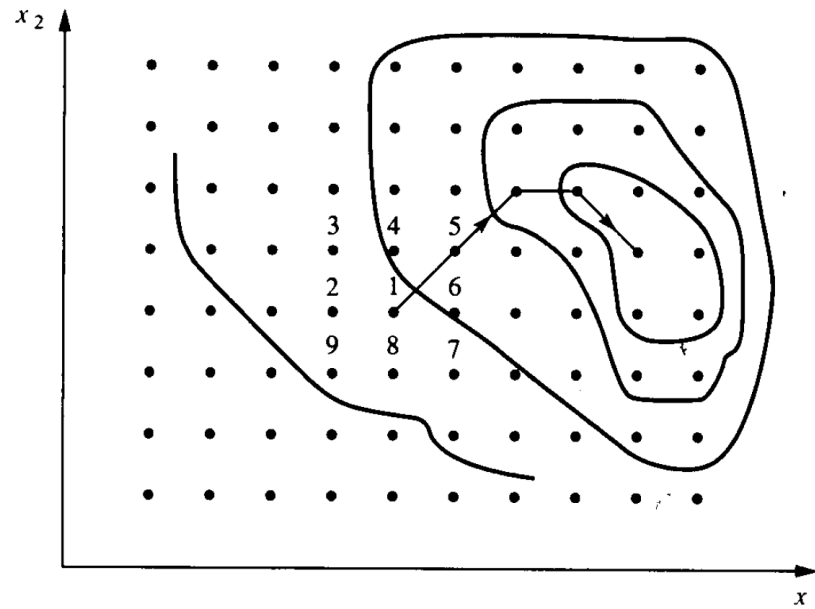


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## 9.11 Lattice search

- Start at on point in the region of interest
- Check a number of points in a grid surrounding the central point
- Move the central point to maximum value of a grid
- If the central point is greater than other surrounding point:

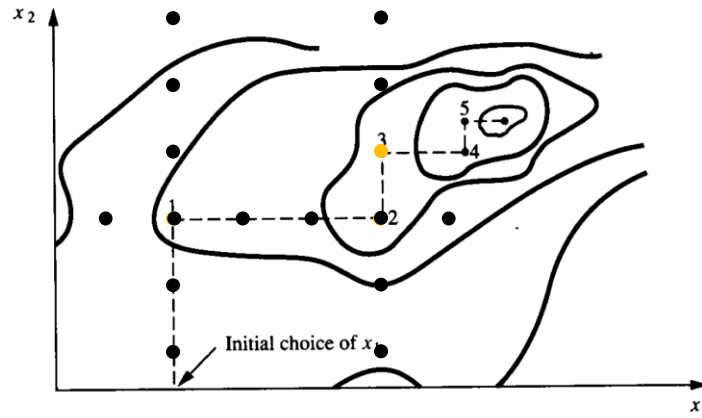
**coarse grid**  
→ **fine grid**



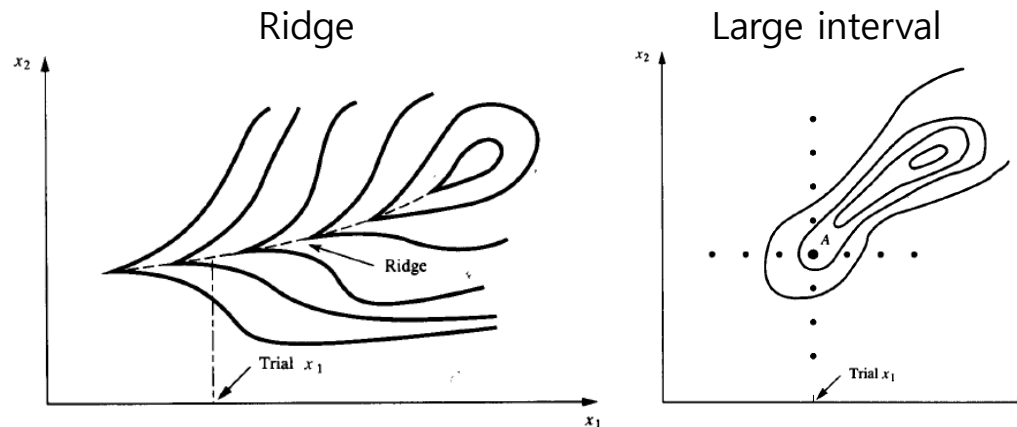
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## 9.12 Univariate search

- Optimization with respect to **one variable** at a time



- Failure occurs



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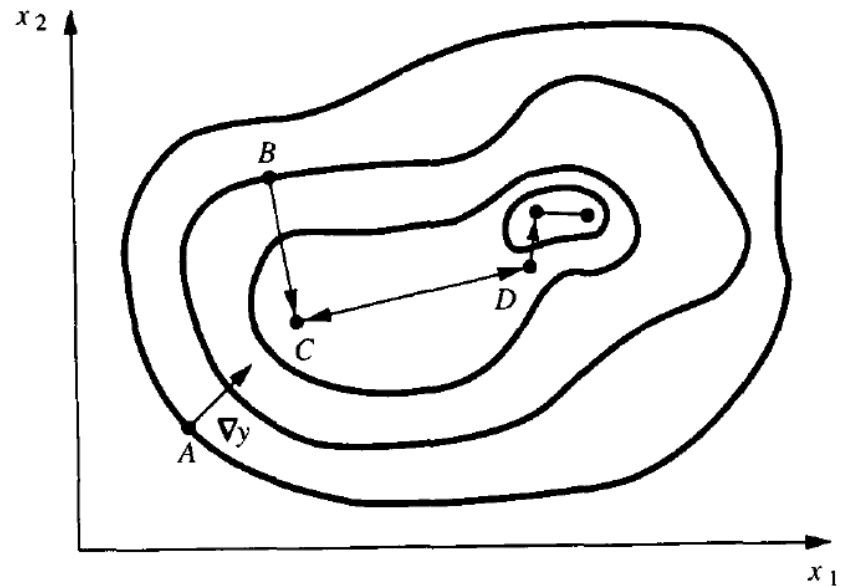
## 9.13 Steepest-ascent method

- Decide in which **direction** to move along the gradient
- Decide how far to move and then move that **distance**

$$\nabla y = \frac{\partial y}{\partial x_1} \hat{i}_1 + \frac{\partial y}{\partial x_2} \hat{i}_2$$

$\hat{i}_1, \hat{i}_2$  : unit vector in the  $x_1$  and  $x_2$

gradient vector (at A) is normal to the contour line (at A)





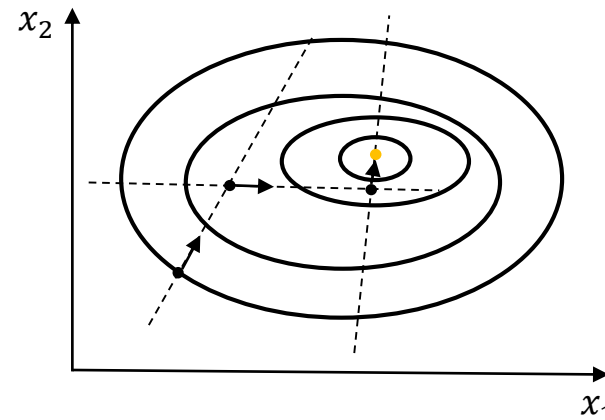
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## 9.13 Steepest-ascent method

- ① trial point as near to the optimum as possible (otherwise, arbitrarily chosen)
- ② Gradient vector is normal to the contour line or surface and therefore indicates the direction of maximum rate of change

$$\frac{\Delta x_1}{\partial y / \partial x_1} = \dots = \frac{\Delta x_n}{\partial y / \partial x_n} \quad \leftarrow \quad \frac{\partial y}{\partial x_1} : \frac{\partial y}{\partial x_2} : \dots : \frac{\partial y}{\partial x_n} = x_1 : x_2 : \dots : x_n$$

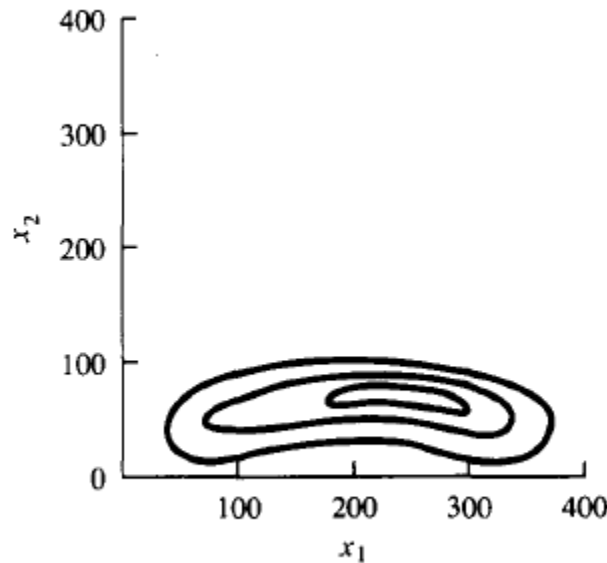
- ③ in the direction of gradient, move until optimum is reached



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## 9.14 Scales of the independent variables

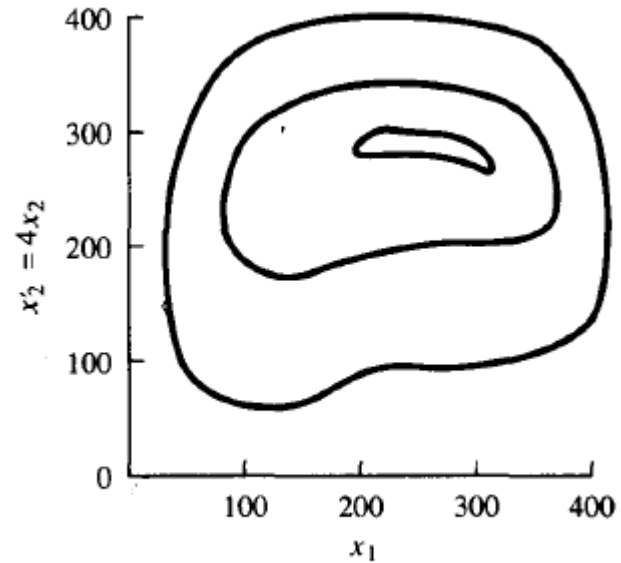
- Contours should be as spherical as possible to accelerate the convergence



(a) Original scale

$$\begin{cases} 0 < x_1 < 400 \\ 0 < x_2 < 100 \end{cases}$$

$$x'_2 = cx_2$$



(b) Revised scale

$$\begin{cases} 0 < x_1 < 400 \\ 0 < x'_2 < 400 \end{cases}$$

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## 9.15 Constrained optimization

- The most frequent and most important ones encountered in the design of thermal systems
    - 1) Conversion to unconstrained by use of penalty functions
    - 2) Searching along the constraint
- equality constraints only

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## 9.16 Penalty functions

$$y = y(x_1, x_2, \dots, x_n) \rightarrow \text{maximum}$$

if minimum

Subject to

$$\phi_1 = y(x_1, x_2, \dots, x_n) = 0$$

$$\vdots$$

$$\phi_m = y(x_1, x_2, \dots, x_n) = 0$$

New unconstrained function

$$Y = y - P_1\phi_1^2 - \dots - P_m\phi_m^2$$

$$Y = y + P_1\phi_1^2 + \dots + P_m\phi_m^2$$

$P_i$  Relative weighting

too high – move very slowly

too small – terminate without satisfying the constraints

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## 9.17 Optimization by searching along a constraint-hemstitching

- Choose a trial point
- Driving toward the constraint(s) (fixed  $x_1$  or  $x_2$ )
- On constraint(s), optimize along the constraint(s) (tangential move)

## 9.18 Driving toward the constraint(s)

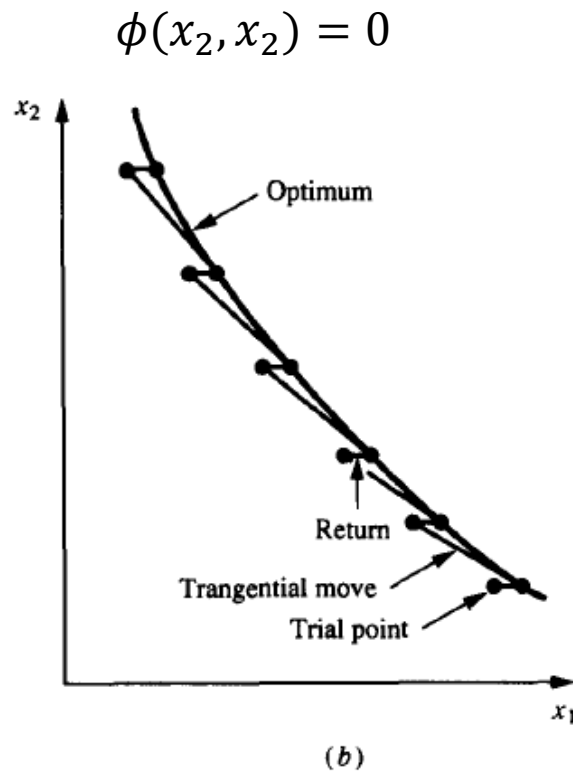
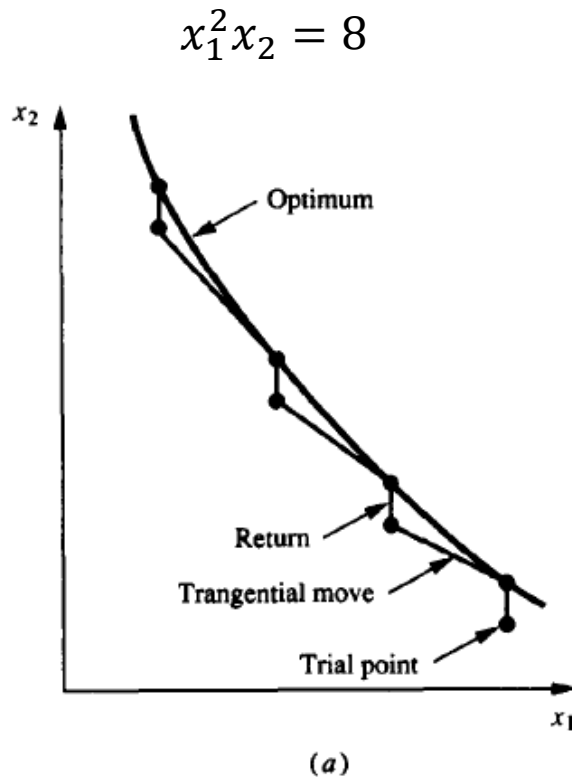
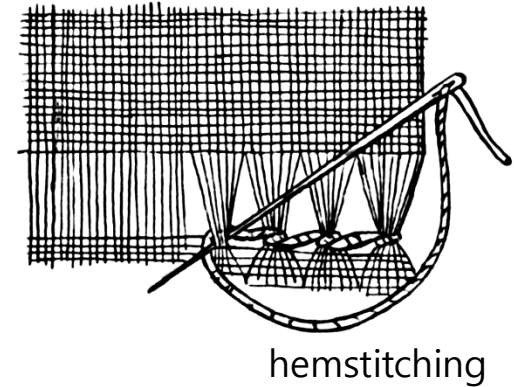
$m < n$                        $m$  : the number of constraints  
                                     $n$  : the number of variables

$n - m$                       : the number of remaining variables which should be solved

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## 9.19 Hemstitching search when $n-m=1$

$$\left. \begin{array}{l} \# \text{ of constraints} = m \\ \# \text{ of variables} = n \end{array} \right\} n - m = 1$$



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## 9.19 Hemstitching search when $n-m=1$

- constraint

$$\phi(x_1, x_2) = 0$$

$$\Delta\phi = \frac{\partial\phi}{\partial x_1} \Delta x_1 + \frac{\partial\phi}{\partial x_2} \Delta x_2 = 0$$

$$\frac{\Delta x_1}{\Delta x_2} = - \frac{\partial\phi / \partial x_1}{\partial\phi / \partial x_2}$$

- objective function

$$\begin{aligned} \Delta y &\approx \frac{\partial y}{\partial x_1} \Delta x_1 + \frac{\partial y}{\partial x_2} \Delta x_2 \\ &= \left( - \frac{\partial y}{\partial x_1} \frac{\partial\phi / \partial x_2}{\partial\phi / \partial x_1} + \frac{\partial y}{\partial x_2} \right) \Delta x_2 = G \Delta x_2 \end{aligned}$$

In minimization,      if  $G > 0$ ,  $\Delta x_2 < 0$   
                                 if  $G < 0$ ,  $\Delta x_2 > 0$

In maximization,      if  $G > 0$ ,  $\Delta x_2 > 0$   
                                 if  $G < 0$ ,  $\Delta x_2 < 0$

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## 9.19 Hemstitching search when $n-m=1$

Three-variable problem where  $n=3$ ,  $m=2$

$$\text{optimize } y = y(x_1, x_2, x_3)$$

$$\text{subject to } \phi_1(x_1, x_2, x_3) = 0$$

$$\phi_2(x_1, x_2, x_3) = 0$$

On the constraints, (tangential move)

$$\Delta\phi_1 = \frac{\partial\phi_1}{\partial x_1}\Delta x_1 + \frac{\partial\phi_1}{\partial x_2}\Delta x_2 + \frac{\partial\phi_1}{\partial x_3}\Delta x_3 = 0$$

$$\Delta\phi_2 = \frac{\partial\phi_2}{\partial x_1}\Delta x_1 + \frac{\partial\phi_2}{\partial x_2}\Delta x_2 + \frac{\partial\phi_2}{\partial x_3}\Delta x_3 = 0$$

$$\begin{aligned}\Delta y &= \frac{\partial y}{\partial x_1}\Delta x_1 + \frac{\partial y}{\partial x_2}\Delta x_2 + \frac{\partial y}{\partial x_3}\Delta x_3 \\ &= G\Delta x_3\end{aligned}$$

} Eliminate  $\Delta x_1, \Delta x_2$

In minimization,    if  $G > 0$ ,  $\Delta x_3 < 0$   
                             if  $G < 0$ ,  $\Delta x_3 > 0$

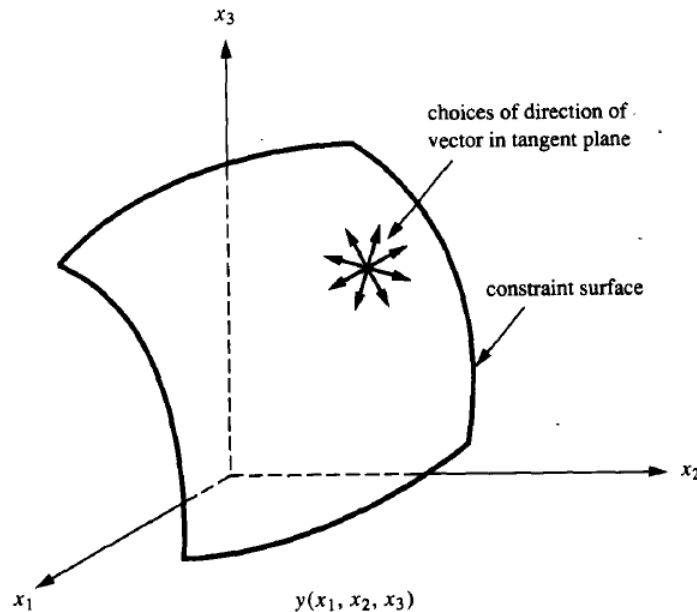
In maximization,    if  $G > 0$ ,  $\Delta x_3 > 0$   
                             if  $G < 0$ ,  $\Delta x_3 < 0$



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## 9.20 Moving tangent to a constraint in three dimensions

$n=3, m=1$



- maximum change of  $y$

$$\Delta y = \frac{\partial y}{\partial x_1} \Delta x_1 + \frac{\partial y}{\partial x_2} \Delta x_2 + \frac{\partial y}{\partial x_3} \Delta x_3$$

- direction (tangent to a constraint)

$$\Delta \phi = \frac{\partial \phi}{\partial x_1} \Delta x_1 + \frac{\partial \phi}{\partial x_2} \Delta x_2 + \frac{\partial \phi}{\partial x_3} \Delta x_3 = 0$$

- distance

$$\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 = r^2 = \text{const.}$$

- maximum

$$\Delta y = ?$$

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## 9.20 Moving tangent to a constraint in three dimensions

Lagrange Multiplier Method

$$\frac{\partial y}{\partial x_1} - \lambda_1(2\Delta x_1) - \lambda_2 \frac{\partial \phi}{\partial x_1} = 0 \quad \dots \textcircled{1}$$

$$\frac{\partial y}{\partial x_2} - \lambda_1(2\Delta x_2) - \lambda_2 \frac{\partial \phi}{\partial x_2} = 0 \quad \dots \textcircled{2}$$

$$\frac{\partial y}{\partial x_3} - \lambda_1(2\Delta x_3) - \lambda_2 \frac{\partial \phi}{\partial x_3} = 0 \quad \dots \textcircled{3}$$

$$\textcircled{1} \times \frac{\partial \phi}{\partial x_1} + \textcircled{2} \times \frac{\partial \phi}{\partial x_2} + \textcircled{3} \times \frac{\partial \phi}{\partial x_3}$$

$$\frac{\partial y}{\partial x_1} \frac{\partial \phi}{\partial x_1} + \frac{\partial y}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \frac{\partial y}{\partial x_3} \frac{\partial \phi}{\partial x_3} - \lambda_2 \left[ \left( \frac{\partial \phi}{\partial x_1} \right)^2 + \left( \frac{\partial \phi}{\partial x_2} \right)^2 + \left( \frac{\partial \phi}{\partial x_3} \right)^2 \right] = 0$$

$$\rightarrow \lambda_2$$

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## 9.20 Moving tangent to a constraint in three dimensions

$$\frac{1}{2\lambda_1} = \frac{\Delta x_1}{\frac{\partial y}{\partial x_1} - \lambda_2 \frac{\partial \phi}{\partial x_1}} = \frac{\Delta x_2}{\frac{\partial y}{\partial x_2} - \lambda_2 \frac{\partial \phi}{\partial x_2}} = \frac{\Delta x_3}{\frac{\partial y}{\partial x_3} - \lambda_2 \frac{\partial \phi}{\partial x_3}}$$

$\Delta x_i$  = step size of on variable in the move

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## 9.21 Summary

- 1. Single variable
  - a. Exhaustive
  - b. Efficient
    - Dichotomous
    - Finonacci
- 2. Multivariable, unconstrained
  - a. Lattice
  - b. Univariate
  - c. Steepest ascent
- 3. Multivariable, constrained
  - a. Penalty functions
  - b. Search along a constraint