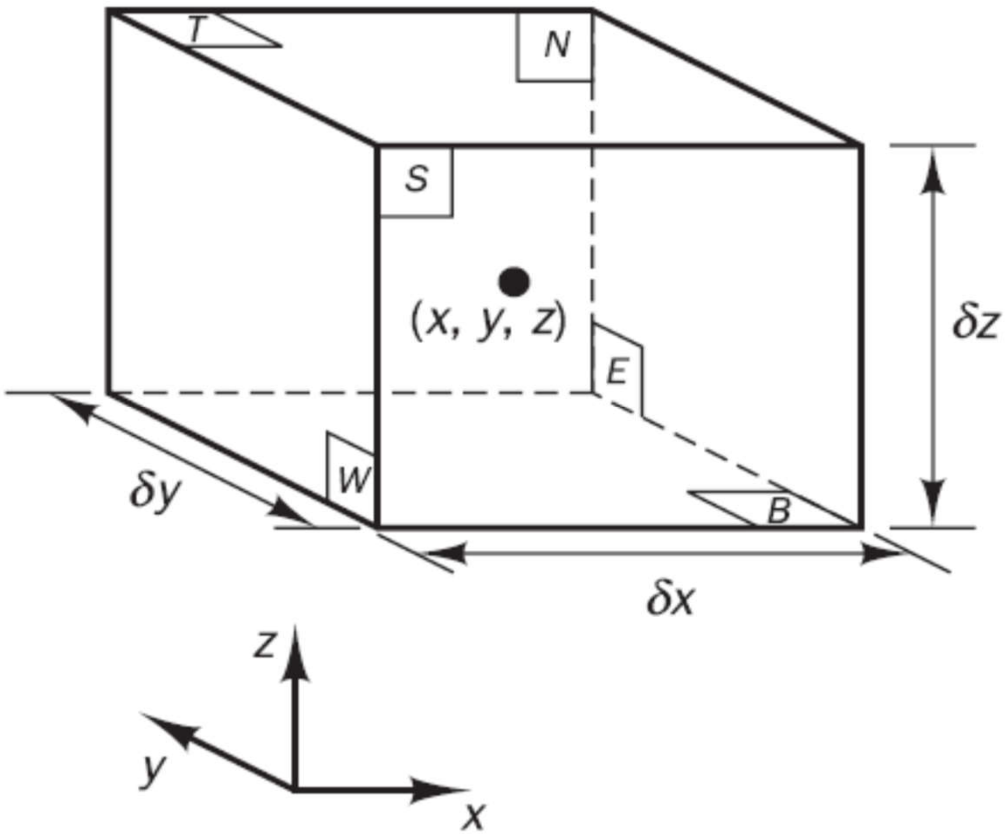
2.1 Governing equations of fluid flow and heat transfer



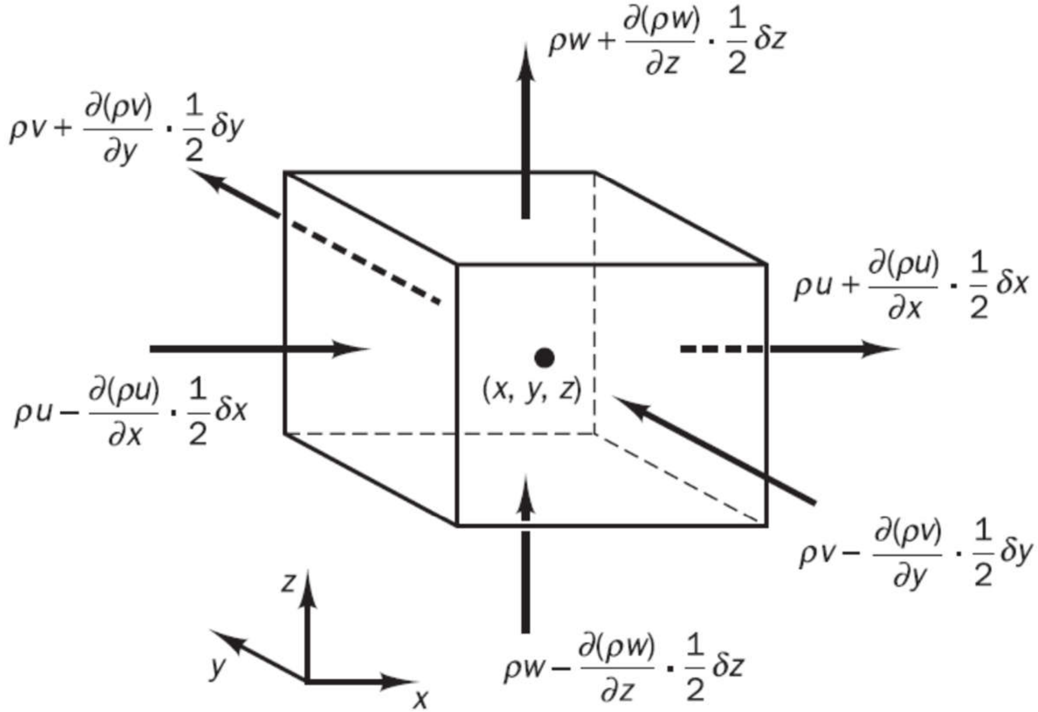
$$p - \frac{\partial p}{\partial x}\frac{1}{2}\delta x \quad \text{and} \quad p + \frac{\partial p}{\partial x}\frac{1}{2}\delta x$$

2.1.1 Mass conservation in three dimensions

Rate of increase of mass in fluid element	=	Net rate of flow of mass into fluid element

$$\frac{\partial}{\partial t}(\rho\delta x\delta y\delta z)=\frac{\partial\rho}{\partial t}\delta x\delta y\delta z \qquad (2.1)$$

2.1.1 Mass conservation in three dimensions



$$\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{1}{2}\delta x\right)\delta y\delta z$$

$$+ \left(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{1}{2}\delta y\right)\delta x\delta z$$

$$+ \left(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y - \left(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{1}{2}\delta z\right)\delta x\delta y \qquad (2.2)$$

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2.1.1 Mass conservation in three dimensions

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad (2.3)$$

or in more compact vector notation

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0} \qquad (2.4)$$

$$\operatorname{div} \mathbf{u} = 0 \qquad (2.5)$$

2.1.2 Rates of change following a fluid particle and for a fluid element

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial t} + \mathbf{u}\,.\,\mathrm{grad}\,\phi \qquad (2.7)$$

$$\rho\frac{D\phi}{Dt} = \rho\left(\frac{\partial\phi}{\partial t} + \mathbf{u}\,.\,\mathrm{grad}\,\phi\right) \qquad (2.8)$$

$$\frac{\partial(\rho\phi)}{\partial t} + \mathrm{div}(\rho\phi\mathbf{u}) \qquad (2.9)$$

2.1.2 Rates of change following a fluid particle and for a fluid element

$$\frac{\partial(\rho\phi)}{\partial t} + \operatorname{div}(\rho\phi\mathbf{u}) = \rho\left[\frac{\partial\phi}{\partial t} + \mathbf{u}\,.\,\operatorname{grad}\phi\right] + \phi\left[\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{u})\right]$$

$$= \rho\frac{D\phi}{Dt} \qquad (2.10)$$

Rate of increase of ϕ of fluid element	+	Net rate of flow of ϕ out of fluid element	=	Rate of increase of ϕ for a fluid particle

2.1.2 Rates of change following a fluid particle and for a fluid element

x-momentum	u	$\rho \frac{Du}{Dt}$	$\frac{\partial(\rho u)}{\partial t} + \text{div}(\rho u \mathbf{u})$
y-momentum	v	$\rho \frac{Dv}{Dt}$	$\frac{\partial(\rho v)}{\partial t} + \text{div}(\rho v \mathbf{u})$
z-momentum	w	$\rho \frac{Dw}{Dt}$	$\frac{\partial(\rho w)}{\partial t} + \text{div}(\rho w \mathbf{u})$
energy	E	$\rho \frac{DE}{Dt}$	$\frac{\partial(\rho E)}{\partial t} + \text{div}(\rho E \mathbf{u})$

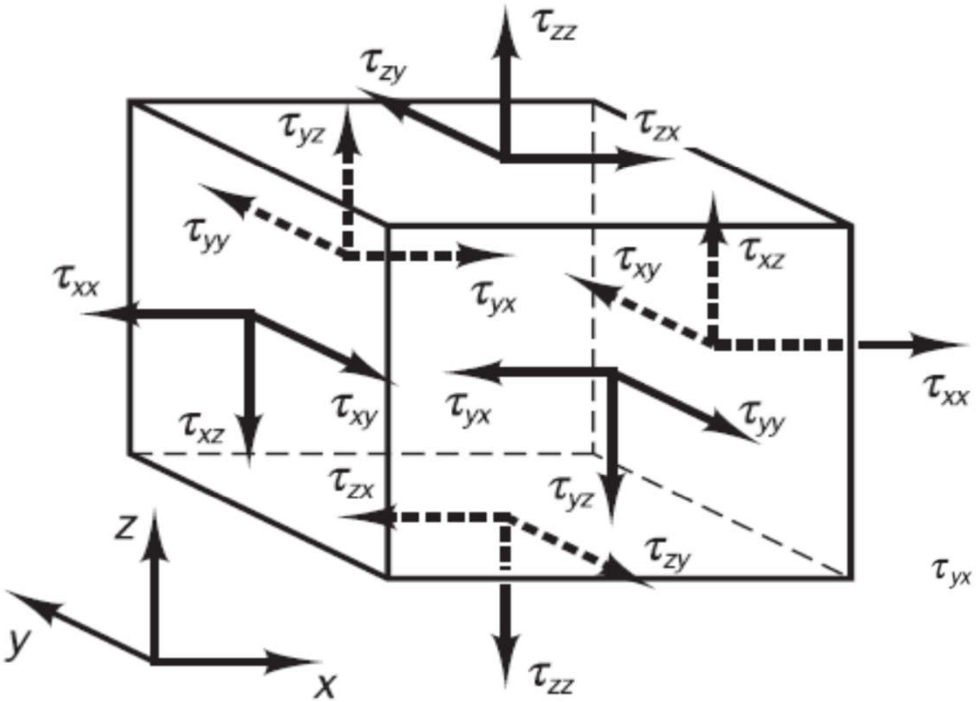
2.1.3 Momentum equation in three dimensions

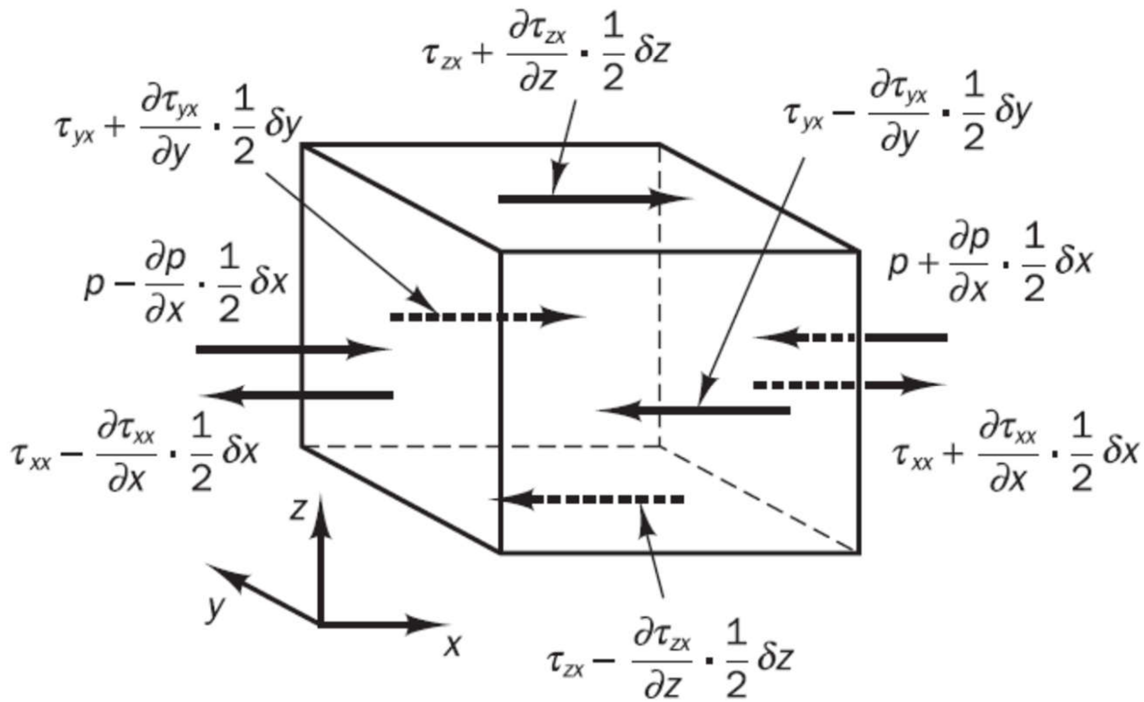
Rate of increase of momentum of fluid particle	=	Sum of forces on fluid particle

The **rates of increase of *x*-, *y*- and *z*-momentum** per unit volume of a fluid particle are given by

$$\rho\frac{Du}{Dt} \qquad \rho\frac{Dv}{Dt} \qquad \rho\frac{Dw}{Dt} \tag{2.11}$$

2.1.3 Momentum equation in three dimensions





2.1.3 Momentum equation in three dimensions

On the pair of faces (E, W) we have

$$\left[\left(p-\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)-\left(\tau_{xx}-\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z+\left[-\left(p+\frac{\partial p}{\partial x}\frac{1}{2}\delta x\right)\right.$$

$$\left.+\left(\tau_{xx}+\frac{\partial \tau_{xx}}{\partial x}\frac{1}{2}\delta x\right)\right]\delta y\,\delta z=\left(-\frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}\right)\delta x\,\delta y\,\delta z \qquad (2.12a)$$

The net force in the x-direction on the pair of faces (N, S) is

$$-\left(\tau_{yx}-\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z+\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}\frac{1}{2}\delta y\right)\delta x\,\delta z=\frac{\partial \tau_{yx}}{\partial y}\delta x\,\delta y\,\delta z$$

(2.12b)

2.1.3 Momentum equation in three dimensions

Finally the net force in the x-direction on faces T and B is given by

$$-\left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{1}{2}\delta z\right)\delta x\,\delta y = \frac{\partial \tau_{zx}}{\partial z}\delta x\,\delta y\,\delta z \tag{2.12c}$$

The total force per unit volume on the fluid due to these surface stresses is equal to the sum of (2.12a), (2.12b) and (2.12c) divided by the volume $\delta x\,\delta y\,\delta z$:

$$\frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{2.13}$$

2.1.3 Momentum equation in three dimensions

$$\rho\frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx} \qquad (2.14a)$$

$$\rho\frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p + \tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My} \qquad (2.14b)$$

$$\rho\frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(-p + \tau_{zz})}{\partial z} + S_{Mz} \qquad (2.14c)$$

2.1.3 Momentum equation in three dimensions

$$\rho \frac{Du}{Dt} = \frac{\partial(-p + \tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx} \qquad (2.14a)$$

$$\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p + \tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My} \qquad (2.14b)$$

$$\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(-p + \tau_{zz})}{\partial z} + S_{Mz} \qquad (2.14c)$$

2.1.4 Energy equation in three dimensions

Rate of increase of energy of fluid particle	=	Net rate of heat added to fluid particle	+	Net rate of work done on fluid particle

$$\rho\frac{DE}{Dt} = -\mathrm{div}(p\mathbf{u}) + \left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right] + \mathrm{div}(k\ \mathrm{grad}\ T) + S_E \qquad (2.22)$$

In equation (2.22) we have $E = i + \frac{1}{2}(u^2 + v^2 + w^2)$.

2.2 Equations of state

$$p = p(\rho, T) \quad \text{and} \quad i = i(\rho, T) \tag{2.28}$$

For a **perfect gas** the following, well-known, equations of state are useful:

$$p = \rho R T \quad \text{and} \quad i = C_v T \tag{2.29}$$

2.3 Navier–Stokes equations for a Newtonian fluid

$$s_{xx} = \frac{\partial u}{\partial x} \quad s_{yy} = \frac{\partial v}{\partial y} \quad s_{zz} = \frac{\partial w}{\partial z} \tag{2.30a}$$

$$s_{xy} = s_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad \text{and} \quad s_{xz} = s_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$s_{yz} = s_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{2.30b}$$

2.3 Navier–Stokes equations for a Newtonian fluid

The volumetric deformation is given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\text{div }\mathbf{u} \tag{2.30c}$$

$$\tau_{xx}=2\mu\frac{\partial u}{\partial x}+\lambda\,\text{div }\mathbf{u}\quad \tau_{yy}=2\mu\frac{\partial v}{\partial y}+\lambda\,\text{div }\mathbf{u}\quad \tau_{zz}=2\mu\frac{\partial w}{\partial z}+\lambda\,\text{div }\mathbf{u}$$

$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\quad \tau_{xz}=\tau_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$$\tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \tag{2.31}$$

2.3 Navier–Stokes equations for a Newtonian fluid

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu \frac{\partial u}{\partial x} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + S_{Mx} \qquad (2.32a)$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[2\mu \frac{\partial v}{\partial y} + \lambda \operatorname{div} \mathbf{u}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + S_{My} \qquad (2.32b)$$

2.3 Navier–Stokes equations for a Newtonian fluid

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} + \lambda \operatorname{div} \mathbf{u}\right] + S_{Mz} \tag{2.32c}$$

2.3 Navier–Stokes equations for a Newtonian fluid

$$\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}+\lambda\,\mathrm{div}\,\mathbf{u}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$$

$$+\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial x}(\lambda\,\mathrm{div}\,\mathbf{u})\right]$$

$$=\mathrm{div}(\mu\,\mathrm{grad}\,u)+[s_{Mx}]$$

2.3 Navier–Stokes equations for a Newtonian fluid

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mathrm{div}(\mu \ \mathrm{grad}\ u) + S_{Mx} \qquad (2.34a)$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mathrm{div}(\mu \ \mathrm{grad}\ v) + S_{My} \qquad (2.34b)$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \mathrm{div}(\mu \ \mathrm{grad}\ w) + S_{Mz} \qquad (2.34c)$$

2.4 Conservative form of the governing equations of fluid flow

Table 2.1 Governing equations of the flow of a compressible Newtonian fluid

Continuity	$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0$	(2.4)
x-momentum	$\frac{\partial (\rho u)}{\partial t} + \text{div}(\rho u \mathbf{u}) = -\frac{\partial p}{\partial x} + \text{div}(\mu \text{ grad } u) + S_{Mx}$	(2.37a)
y-momentum	$\frac{\partial (\rho v)}{\partial t} + \text{div}(\rho v \mathbf{u}) = -\frac{\partial p}{\partial y} + \text{div}(\mu \text{ grad } v) + S_{My}$	(2.37b)
z-momentum	$\frac{\partial (\rho w)}{\partial t} + \text{div}(\rho w \mathbf{u}) = -\frac{\partial p}{\partial z} + \text{div}(\mu \text{ grad } w) + S_{Mz}$	(2.37c)
Energy	$\frac{\partial (\rho i)}{\partial t} + \text{div}(\rho i \mathbf{u}) = -p \text{ div } \mathbf{u} + \text{div}(k \text{ grad } T) + \Phi + S_i$	(2.38)
Equations of state	$p = p(\rho, T)$ and $i = i(\rho, T)$	(2.28)
	e.g. perfect gas $p = \rho RT$ and $i = C_v T$	(2.29)

2.5 Differential and integral forms of the general transport equations

$$\frac{\partial(\rho\phi)}{\partial t} + \mathrm{div}(\rho\phi\mathbf{u}) = \mathrm{div}(\Gamma\ \mathrm{grad}\ \phi) + S_\phi \tag{2.39}$$

Rate of increase of ϕ of fluid element	+	Net rate of flow of ϕ out of fluid element	=	Rate of increase of ϕ due to diffusion	+	Rate of increase of ϕ due to sources

$$\int_{CV} \frac{\partial(\rho\phi)}{\partial t} dV + \int_{CV} \mathrm{div}(\rho\phi\mathbf{u}) dV = \int_{CV} \mathrm{div}(\Gamma\ \mathrm{grad}\ \phi) dV + \int_{CV} S_\phi dV \tag{2.40}$$

2.5 Differential and integral forms of the general transport equations

$$\int_{CV} \operatorname{div}(\mathbf{a})\,dV = \int_A \mathbf{n}\,.\,\mathbf{a}\,dA \qquad (2.41)$$

$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi\,dV\right) + \int_A \mathbf{n}\,.\,(\rho\phi\mathbf{u})\,dA = \int_A \mathbf{n}\,.\,(\Gamma\,\text{grad}\,\phi)\,dA + \int_{CV} S_\phi\,dV \qquad (2.42)$$

Rate of increase of ϕ inside the control volume	+	Net rate of decrease of ϕ due to convection across the control volume boundaries	=	Net rate of increase of ϕ due to diffusion across the control volume boundaries	+	Net rate of creation of ϕ inside the control volume

2.5 Differential and integral forms of the general transport equations

This leads to the integrated form of the steady transport equation:

$$\int_A \mathbf{n} \, . \, (\rho\phi\mathbf{u})\mathrm{d}A = \int_A \mathbf{n} \, . \, (\Gamma \operatorname{grad} \phi)\mathrm{d}A + \int_{CV} S_\phi \mathrm{d}V \tag{2.43}$$

2.5 Differential and integral forms of the general transport equations

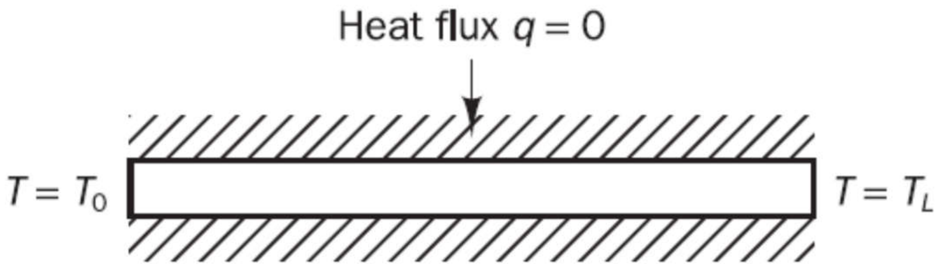
In time-dependent problems it is also necessary to integrate with respect to time t over a small interval Δt from, say, t until $t + \Delta t$. This yields the most general integrated form of the transport equation:

$$\int_{\Delta t} \frac{\partial}{\partial t}\left(\int_{CV} \rho\phi \mathrm{d}V\right)\mathrm{d}t + \int_{\Delta t}\int_{A} \mathbf{n} \,.\, (\rho\phi\mathbf{u})\mathrm{d}A\mathrm{d}t = \int_{\Delta t}\int_{A} \mathbf{n} \,.\, (\Gamma \text{ grad } \phi)\mathrm{d}A\mathrm{d}t + \int_{\Delta t}\int_{CV} S_\phi \mathrm{d}V\mathrm{d}t \quad (2.44)$$

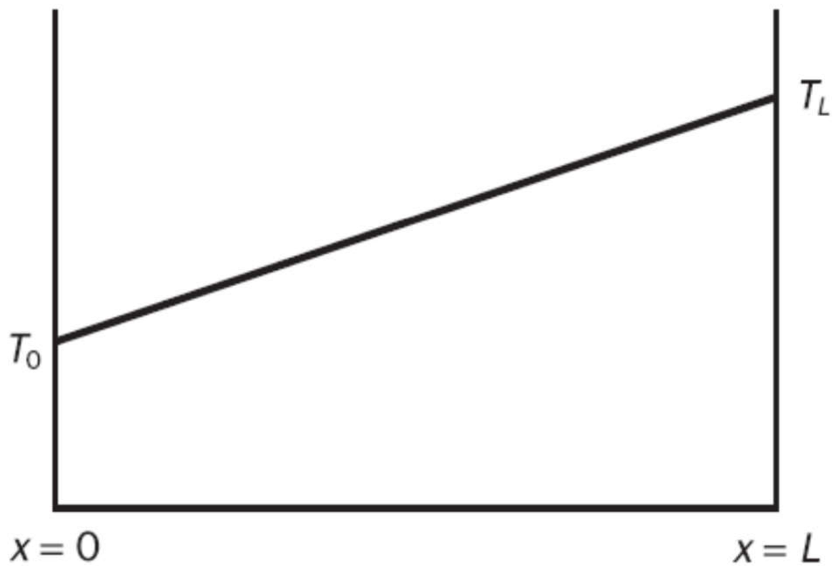
2.6 Classification of physical behaviours

Equilibrium problems

Problem specification



Solution

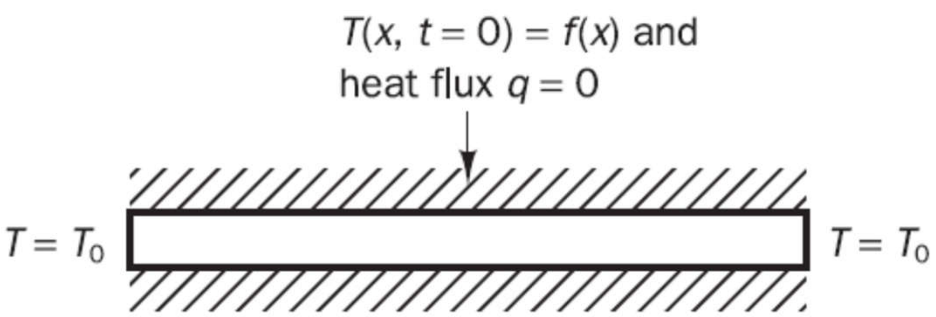


2.6 Classification of physical behaviours

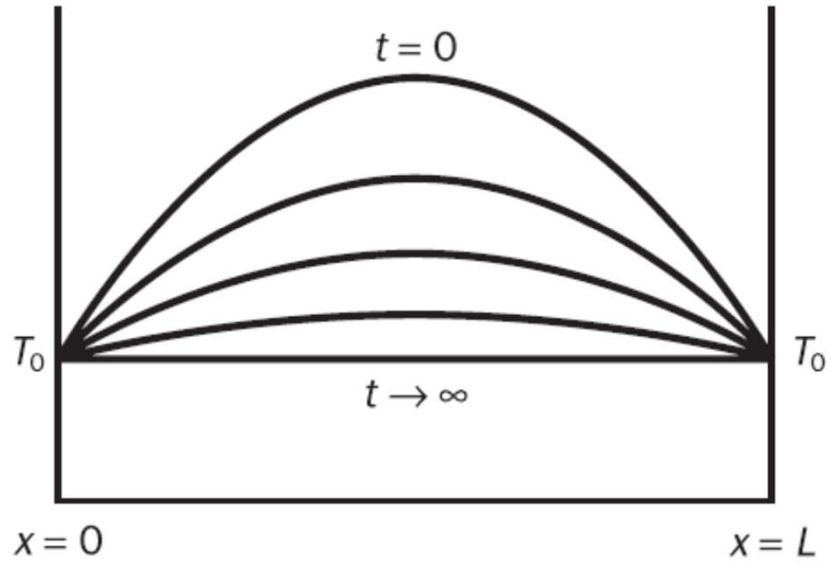
Marching problems

Parabolic equations

Problem specification



Solution

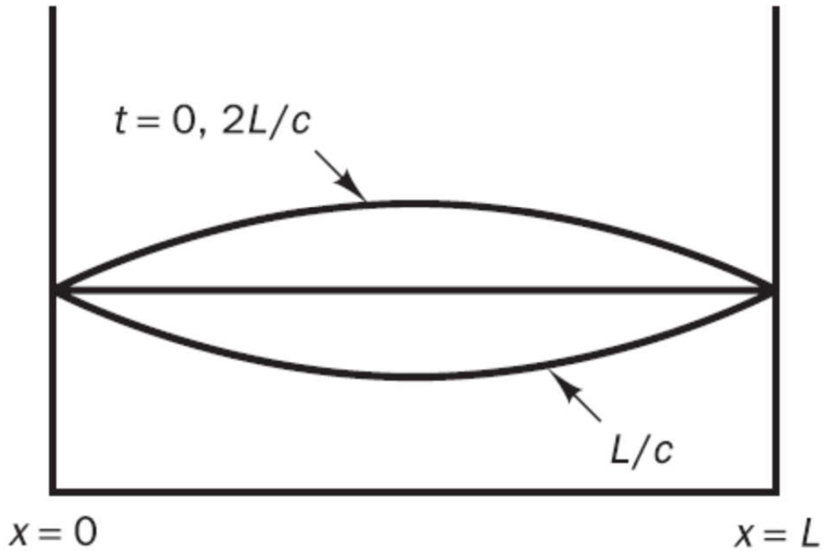


Marching problems

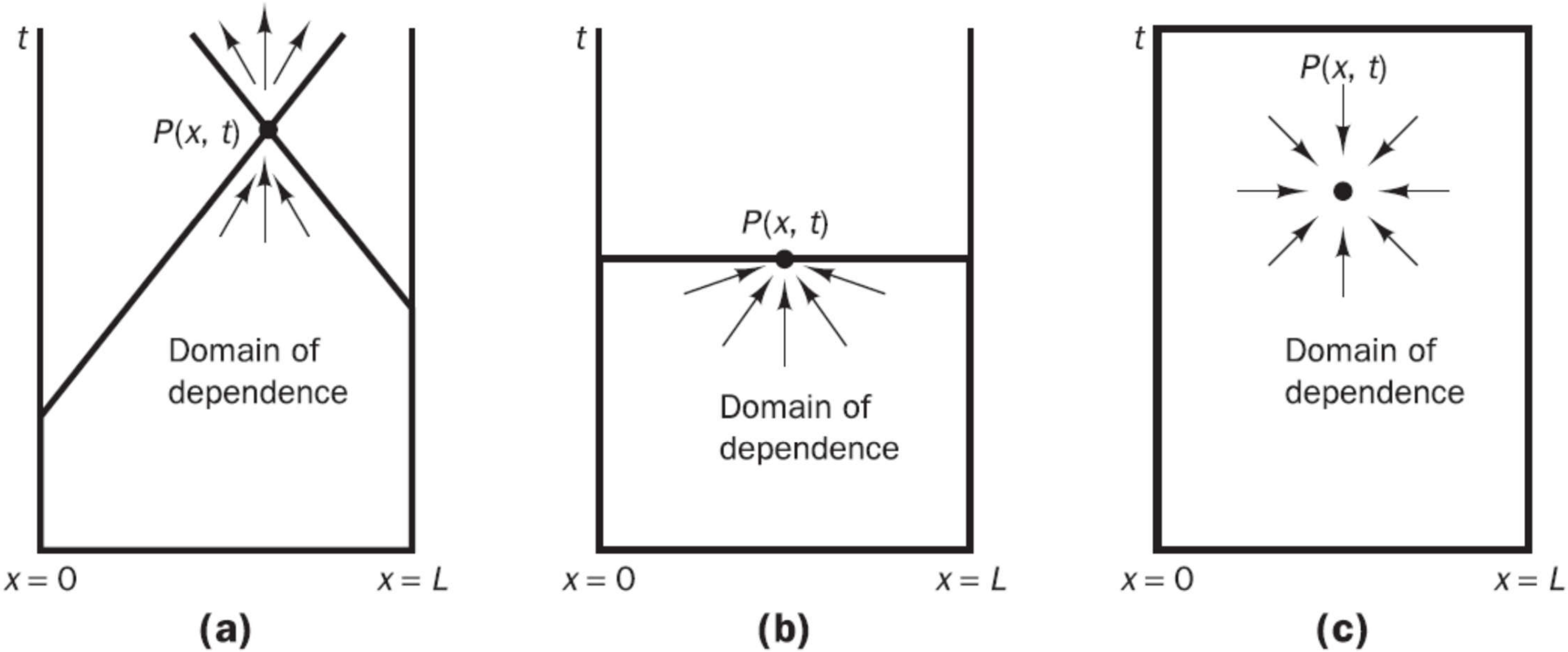
Hyperbolic equations

Problem specification
$y(x, t = 0) = f(x)$ and $\partial y/\partial x(x, t = 0) = 0$

Solution
for first cycle $0 < t < 2L/c$



2.7 The role of characteristics in hyperbolic equations



2.7 The role of characteristics in hyperbolic equations

Table 2.2 Classification of physical behaviours

Problem type	*Equation type*	*Prototype equation*	*Conditions*	*Solution domain*	*Solution smoothness*
Equilibrium problems	Elliptic	$\text{div grad}\ \phi = 0$	Boundary conditions	Closed domain	Always smooth
Marching problems with dissipation	Parabolic	$\frac{\partial \phi}{\partial t} = \alpha\ \text{div grad}\ \phi$	Initial and boundary conditions	Open domain	Always smooth
Marching problems without dissipation	Hyperbolic	$\frac{\partial^2 \phi}{\partial t^2} = c^2\ \text{div grad}\ \phi$	Initial and boundary conditions	Open domain	May be discontinuous

2.9 Classification of fluid flow equations

Table 2.4 Classification of the main categories of fluid flow

	Steady flow	*Unsteady flow*
Viscous flow	Elliptic	Parabolic
Inviscid flow	$M < 1$, elliptic $M > 1$, hyperbolic	Hyperbolic
Thin shear layers	Parabolic	Parabolic