Lecture Notes 414.341

선 박 해 양 유 체 역 학

MARINE HYDRODYNAMICS

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4

DYNAMICS

4.1 Forces

Two types of net forces act in flow problems: (1) body forces and (2) surface forces.

4.1.1 Body Forces

- From external source the body force acts throughout volume from afar (e.g., gravity, magnetic attraction). It is convenient to define the net body force as

$$\underline{F}_b(t) = \int_V \underline{F}_B(\underline{x}, t)\, dV \tag{4.1}$$

  where $\underline{F}_B(\underline{x}, t)$ is the body force per unit volume acting at a point $\underline{x}$. For gravity $\underline{F}_B = -\rho g \underline{e}_3$ where $\underline{e}_3$ is the unit vector directed along the upward vertical.

- Often the body force is defined as the body force per unit mass:

$$\underline{F}_b(t) = \int_V \rho \underline{f}(\underline{x}, t)\; dV \tag{4.2}$$

e.g., for gravity $\underline{f}(\underline{x}, t) = -g\underline{e}_3$.

- The torque due to the body force about the spatial point $\underline{x}_0$ is

$$\underline{Q}(t) = \int_V (\underline{x} - \underline{x}_0) \times \underline{F}_B(\underline{x}, t)\; dV \tag{4.3}$$

- We will consider only conservative body forces, for which the body force is derived from a scalar potential

$$\underline{F}_B(\underline{x}, t) = -\nabla\Omega(\underline{x}, t) \tag{4.4}$$

For the body force due to gravity, $\Omega = \rho g x_3$.

4.1.2 Surface Forces

- Internal sources that cancel except at bounding surfaces that have no continuation volume provide an equal but oppositely directed force.

- Surface force is defined in terms of a stress distribution on the bounding surface

$$\underline{F}_S = \oint_S \underline{\tau}(\underline{x}_s, t)\; dS \tag{4.5}$$

where S bounds V, $\underline{\tau}(\underline{x}_s, t)$ is the stress vector at a point $\underline{x}_s$ on the surface S, with three components.

- The torque about a field point $\underline{x}_0$ due to the surface force is

$$\underline{Q}(t) = \oint_S (\underline{x}_s - \underline{x}_0) \times \underline{\tau}(\underline{x}_s, t)\; dS \tag{4.6}$$

4.1.3 Stress and Stress Tensor [1]

- The stress vector $\underline{\tau}(\underline{x}_s, t)$ is associated with a normal vector to the surface upon which it acts in the sense that if the stresses were in local equilibrium (i.e, no acceleration or other surface forces act), the stress on the one side of a surface(the side denoted by the normal) is equal and oppositely directed as that on the other side (see Figures 4.1 and 4.2).

- Let $\underline{n}$ be the normal pointing out of the volume(exterior normal), then $(-\underline{n})$ is pointing into the volume (the interior normal), and $\underline{\tau}_{(\underline{n})} = -\underline{\tau}_{(-\underline{n})}$.

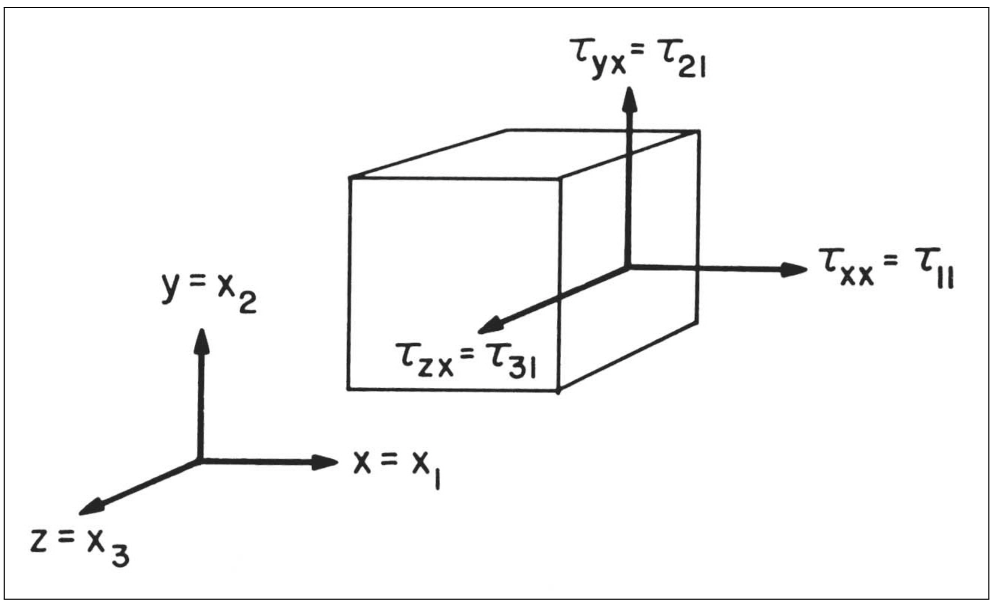


Figure 4.1 Stress tensor in Cartesian coordinates. (From Newman 1977)

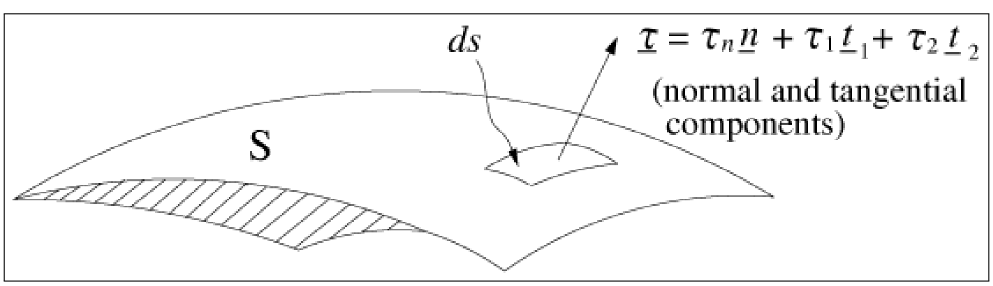


Figure 4.2 Stress vector at surface.

[1] Movie: Stress and Stress Tensor
./mmfm_movies/Stress.mov

- This fact leads one to an expressing for the local stress vector as the dot product of the normal(so the equal and opposite property is satisfied) and a dyadic quantity called the stress tensor of 2nd order

$$\underline{\tau} = \underline{n} \cdot \underline{\underline{\mathbf{T}}} \tag{4.7}$$

where $\underline{\underline{\mathbf{T}}}$ is the stress tensor represented with 3 by 3 elements. [2]

- In three-dimensional Euclidian space, we can write the stress tensor in the dyadic form

$$\underline{\underline{\mathbf{T}}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij}\ \underline{e}_i\ \underline{e}_j \tag{4.8}$$

where $\underline{e}_i$ and $\underline{e}_j$ are unit base vectors and the scalars τ_{ij} are the physical components of the tensor. Our expression has been with Cartesian coordinates but the same concepts apply to other curvilinear coordinate systems just as well, and definition of the stress tensor components in a system compatible with the geometry is desired.

$$\tau_{ij} = \left\{ \begin{matrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{matrix} \right\} = \left\{ \begin{matrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{matrix} \right\} \tag{4.9}$$

- The stress tensor is symmetric, i.e., $\tau_{ij} = \tau_{ji}$, because otherwise there will be an unreasonable motion with an infinite speed due to the resultant unbalanced forces acting on an infinitesimal fluid element. Symmetry means that only 6 components are independent (not fully 9).

4.1.4 Surface Tension

- On an interface surface between two fluids (i.e., stratified fluids), the surface tension should be included to satisfy the continuity of the stress across

[2] Tensors of second order are denoted by using the double underbar $\underline{\underline{\cdot}}$. The stress tensor (of the second order) consists of the magnitude of stress, the direction of stress and the orientation of the surface.

the interface:

$$\underline{n} \cdot (\underline{\underline{\mathbf{T}}}_i - \underline{\underline{\mathbf{T}}}_o) = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \underline{n} \tag{4.10}$$

where σ is called the surface tension(whose unit is given by force per length), R_1 and R_2 are the principal radii of curvature of the interface, $\underline{n}$ is the normal vector at the interface, and the subscripts i and o refer the two fluid sides of the interface. (See Figure **4.3**).

- When the two fluids are stationary, only the pressure terms remain in the above relation. From the force equilibrium in the normal direction for a small element of interface,

$$(p_i - p_o)(R_1 \triangle\phi)(R_2 \triangle\theta) = F_{s_1} \triangle\theta + F_{s_2} \triangle\phi \tag{4.11}$$

where $F_{s_1} = \sigma R_1 \triangle\phi$ and $F_{s_2} = \sigma R_2 \triangle\theta$.

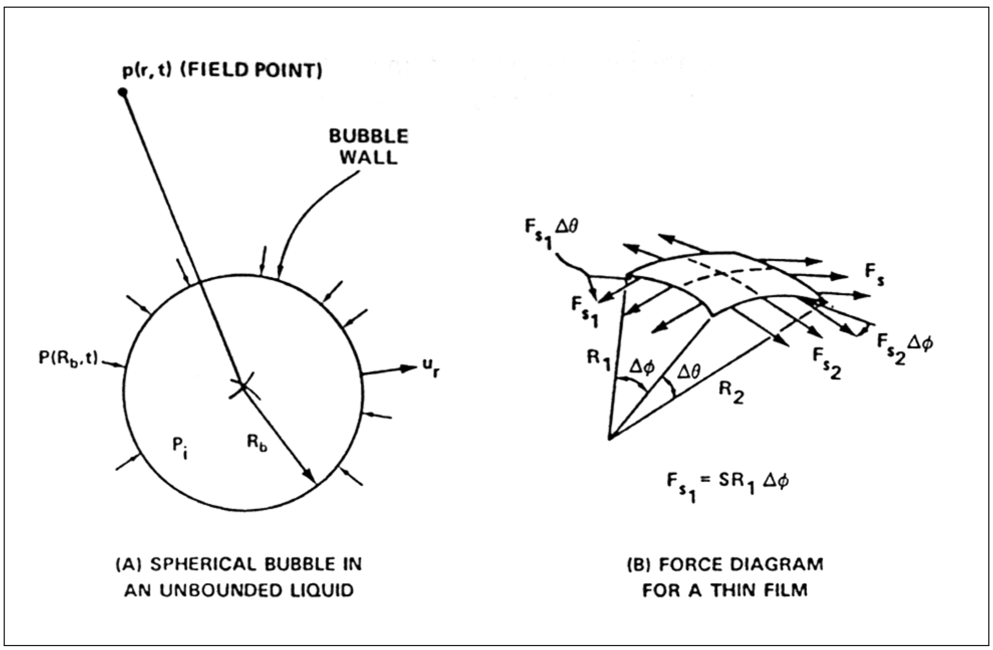


Figure **4.3** Force diagram for a spherical bubble with surface tension.

- Then,

$$p_i - p_o = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \tag{4.12}$$

As a special example, for a stationary spherical droplet(or bubble) of radius R, $p_i - p_o = 2\sigma/R$.

4.2 Equations of Motion: Navier-Stokes Equations

4.2.1 Newton's Second Law for Material Volume

- Newton's second law states that the time rate of change of the linear momentum is equal to the applied forces. This statement is also appropriate for continuum matter. Hence for a moving volume $V(t)$ bounded by a material surface $S(t)$, we have (for detailed derivations, refer to texts dealing with fluid mechanics.)

$$\frac{d}{dt}\int_V \rho\, \underline{q}\, dV = \int_V \underline{F}_B\, dV + \oint_S \underline{n}\cdot \underline{\underline{T}}\, dS \tag{4.13}$$

In the tensor notation,

$$\frac{d}{dt}\int_V \rho\, q_i\, dV = \int_V F_{B_i}\, dV + \oint_S \tau_{ij}\, n_j\, dS \tag{4.14}$$

Divergence theorem for the surface integral

$$\int_V \nabla\cdot \underline{A}\, dV = \oint_S \underline{A}\cdot \underline{n}\, dS \tag{4.15}$$

Conservation laws of momentum for the fluid

$$\frac{d}{dt}\int_V \rho\, q_i\, dV = \int_V \left[\frac{\partial \tau_{ij}}{\partial x_j} + F_{B_i}\right] dV \tag{4.16}$$

By the transport theorem, we have

$$\int_V \left[\frac{\partial(\rho\, q_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho\, q_i\, q_j)\right] dV = \int_V \left[\frac{\partial \tau_{ij}}{\partial x_j} + F_{B_i}\right] dV, \tag{4.17}$$

or, in a form of differential equations,

$$\frac{\partial(\rho\, q_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho\, q_i\, q_j) = \frac{\partial \tau_{ij}}{\partial x_j} + F_{B_i} \tag{4.18}$$

and thus, with aid of the continuity equation,

$$\boxed{\rho\left[\frac{\partial q_i}{\partial t}+q_j\frac{\partial q_i}{\partial x_j}\right]=\frac{\partial \tau_{ij}}{\partial x_j}+F_{B_i}} \tag{4.19}$$

4.2.2 Surface Forces for Differential Control Volume

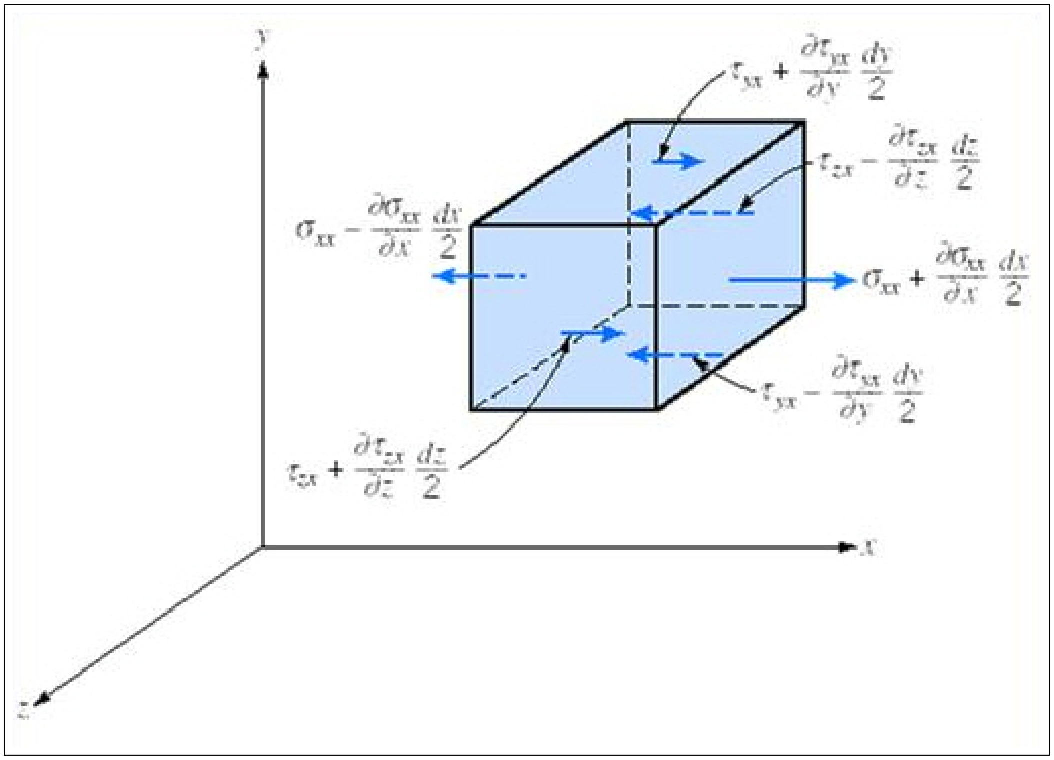


Figure 4.4 Stresses in the x-direction on a differential control volume in Cartesian coordinates. (From Fox, McDonald & Pritchard 2004)

- Surface forces acting on a fluid particle:

$$dF_{S_x} = \left[\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right]dx\,dy\,dz \tag{4.20}$$

$$dF_{S_y} = \left[\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right]dx\,dy\,dz \tag{4.21}$$

$$dF_{S_z} = \left[\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}\right]dx\,dy\,dz \tag{4.22}$$

These components correspond to $\dfrac{\partial \tau_{ij}}{\partial x_j}\,dx\,dy\,dz$.

4.2.3 Stress and Strain Rate in a Newtonian Fluid

- A fluid is defined as a material that cannot be in stationary equilibrium with applied shear stress.
  A Newtonian fluid has a resistance to shear deformation that is linearly proportional to the rate of deformation(i.e., proportional to the gradient of velocity), while an ideal(perfect) fluid has no resistance to shear deformation. [3]

- Four motions of a fluid particle element are possible: (1) translation, (2) rotation, (3) squeeze motion (angular defomation), and (4) volumetric change (linear deformation). [4]
  Among them, (3) and (4) cause stress in a fluid, while (1) and (2) repre-

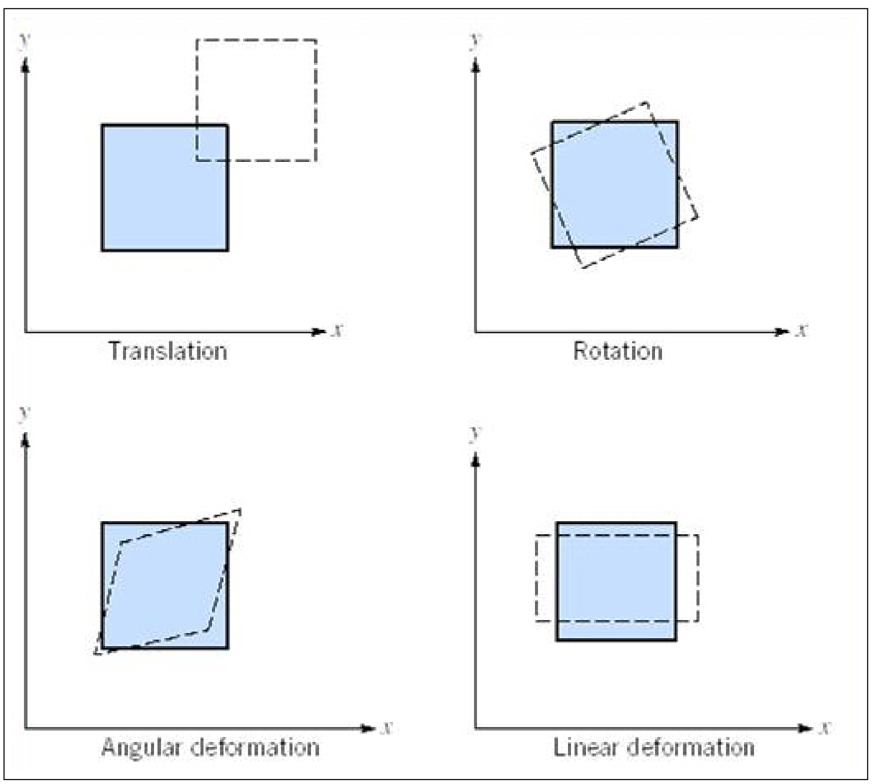


Figure 4.5 Motion of a fluid particle. (From Fox, McDonald & Pritchard 2004)

sent only rigid body motion for which there will be no stress developed.

[3]In solid mechanics, this linearity can be applied to elastic solids that follow Hooke's law.

[4] Movie: Motion of fluid particle (Boundary layer)
./mmfm_movies/1_02032.mov ./mmfm_movies/1_02030.mov

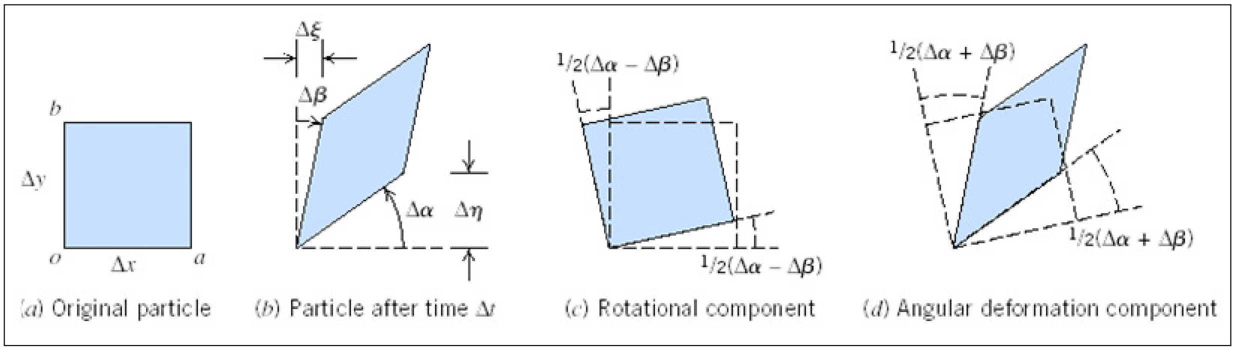


Figure 4.6 Deformation of fluid element in 2-D flows. (From Fox, McDonald & Pritchard 2004)

- Rigid body motion without deformation: Translation + Rotation

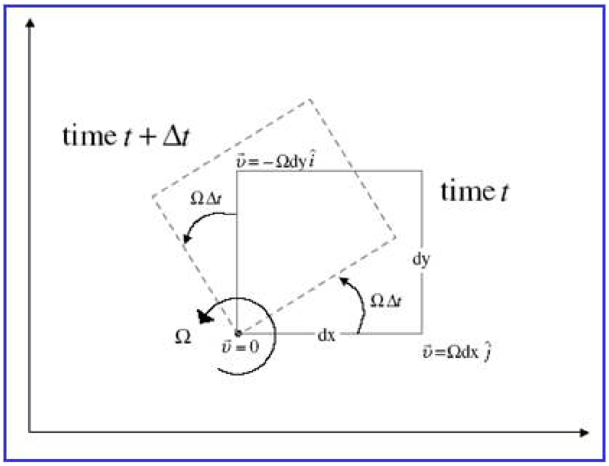


Figure 4.7 Rotational motion of a fluid particle. (From MIT website 2004)

$$\underline{q} = \frac{d\underline{x}_0}{dt} + \underline{\Omega} \times (\underline{x} - \underline{x}_0) \tag{4.23}$$

– Fluid Translation: Acceleration of a Fluid Particle in a Velocity Field

$$\underline{a}_p = \frac{D\underline{q}}{Dt} = \frac{\partial \underline{q}}{\partial t} + \underline{q} \cdot \nabla \underline{q} \tag{4.24}$$

– Fluid Rotation: Relationship between rotation and vorticity [5]

$$\underline{\Omega} = \frac{1}{2} \nabla \times \underline{q} \tag{4.25}$$

[5] Movie: Motion of fluid particle (Rotation)
./mmfm_movies/tire.mov ./mmfm_movies/3_14035.mov

- Squeeze motion (angular deformation):

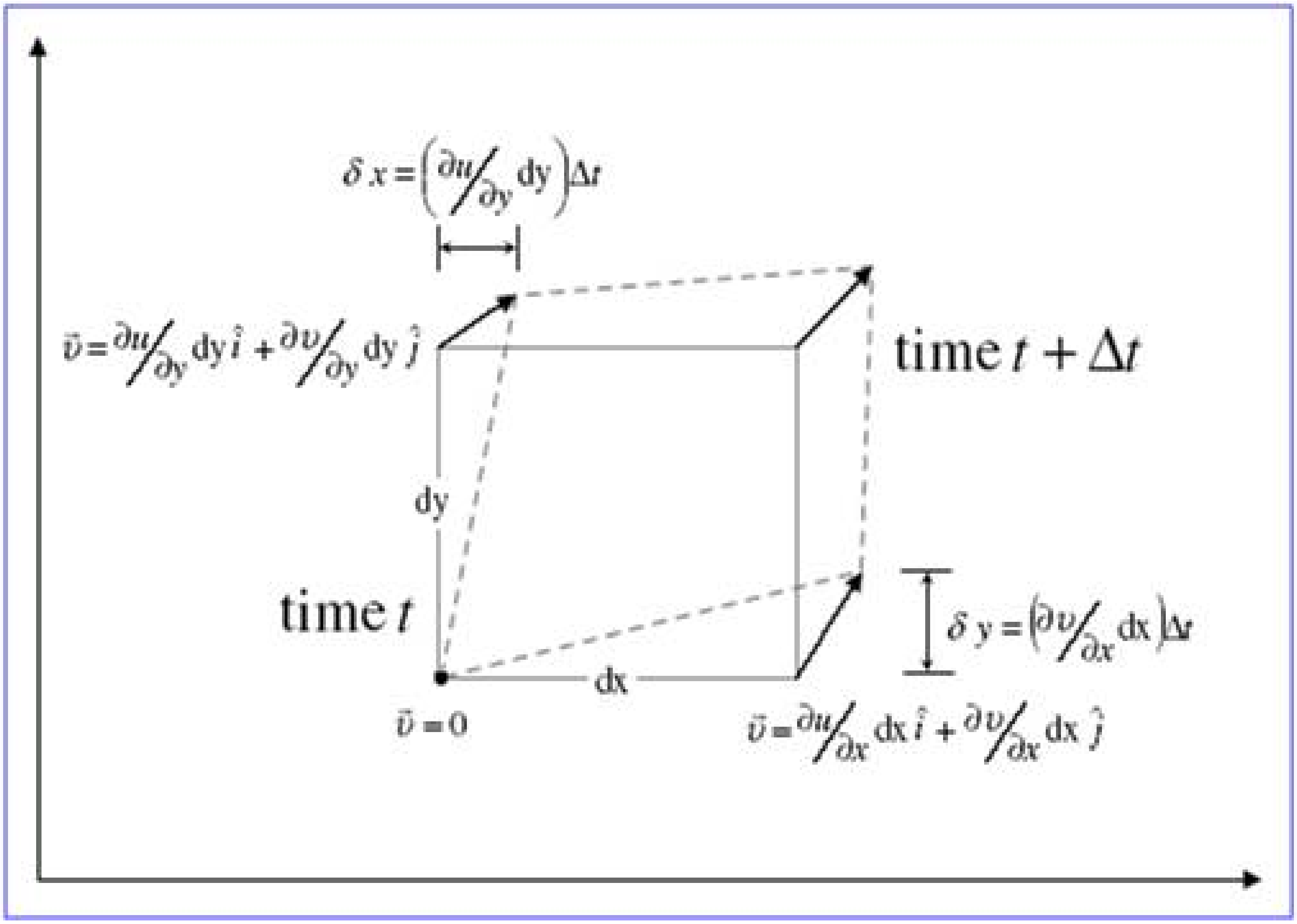


Figure 4.8 Squeeze motion of a fluid particle. (From MIT website 2004)

$$\text{Rate of angular deformation in } xy \text{ plane} = \lim_{\triangle t \to 0} \frac{\left(\frac{\partial v}{\partial x}\frac{\triangle x}{\triangle x}\triangle t + \frac{\partial u}{\partial y}\frac{\triangle y}{\triangle y}\triangle t\right)}{\triangle t} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{4.26}$$

$$\text{Rate of angular deformation in } yz \text{ plane} = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{4.27}$$

$$\text{Rate of angular deformation in } zx \text{ plane} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{4.28}$$

- Volumetric change (linear deformation) [6]

$$\text{Volume dilation rate} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \underline{q} \tag{4.29}$$

  - It vanishes for incompressible fluid.
  - This velocity vector is called solenoidal.

- If there were no deformation(also including rigid body motion), then only a static pressure acts normal to the surface of the volume of interest. The stress vector is simply $\underline{\tau} = -p\ \underline{n}$. Thus the stress tensor $\underline{\underline{\mathbf{T}}} = -p\ \underline{\underline{\mathbf{I}}} = -p\ \delta_{ij}$. The pressure diagram of a fluid is shown in Figure **4.9** .

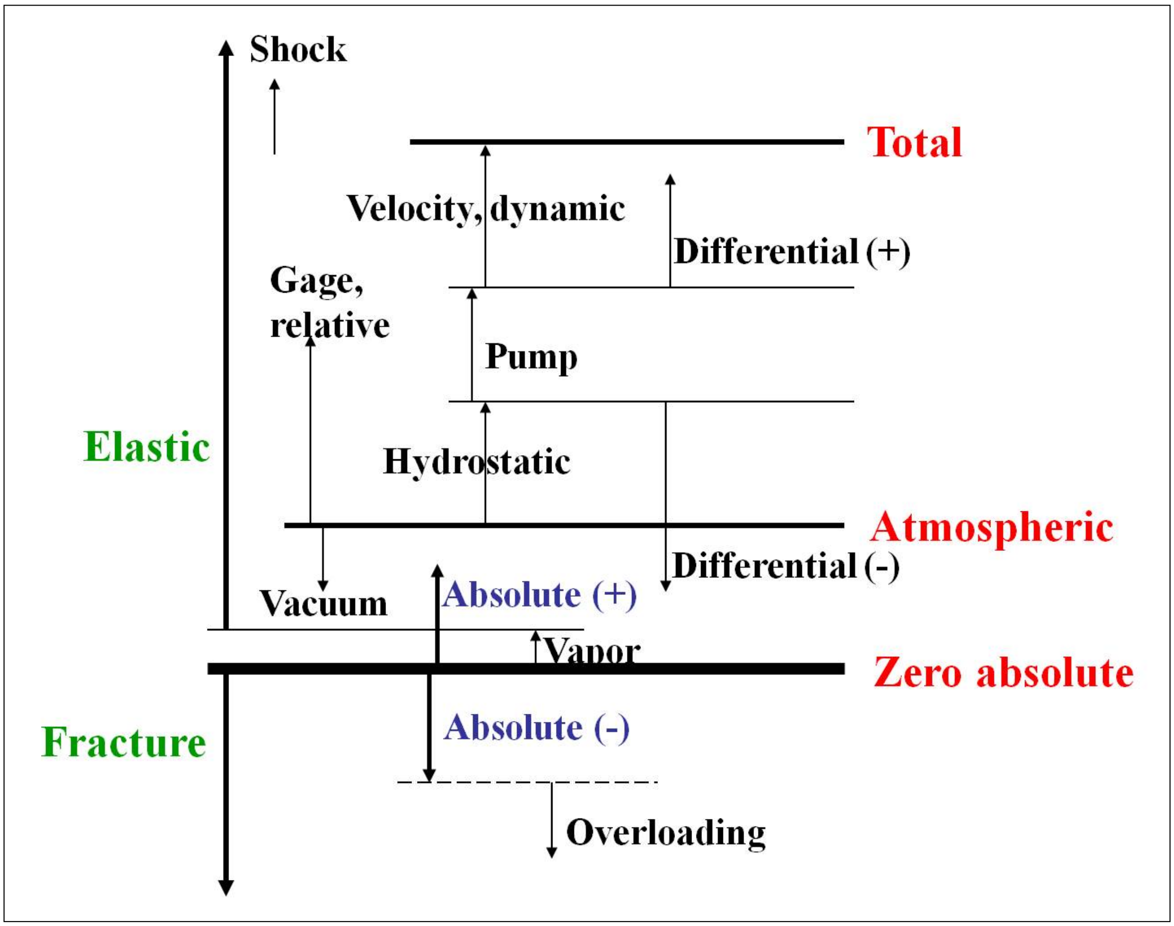


Figure 4.9 Pressure diagram of a fluid.

[6] Movie: Motion of fluid particle (Volumetric change) ./mmfm_movies/3_14040.mov

- Combining the above statements leads one to, for a Newtonian incompressible fluid, [7]

$$\underline{\underline{T}} \equiv \tau_{ij} = -p\,\underline{\underline{\mathbf{I}}} + \mu\,\left[\nabla \underline{q} + (\nabla \underline{q})^T\right] \tag{4.31}$$

In the tensor notation,

$$\tau_{ij} = -p\,\delta_{ij} + \mu\,\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) \tag{4.32}$$

In the matrix form,

$$\{\tau_{ij}\} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} + \mu \begin{bmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} & 2\dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} & \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} & 2\dfrac{\partial w}{\partial z} \end{bmatrix} \tag{4.33}$$

- The diagonal terms consist of the divergence of the velocity vector and (averaged) static pressures, and hence, for an incompressible fluid, are associated with volumetric changes.

- The off-diagonal terms are associated with the squeeze-like motion.

- The proportionality constant μ is called the viscosity coefficient of the fluid, and the second term is called the viscous stress tensor.
  Most of the common fluids (water, air, oil, etc.) are Newtonian fluids

[7] Stress Tensor for Compressible Fluids

$$\underline{\underline{T}} = -p\,\underline{\underline{\mathbf{I}}} + \mu\,[\nabla \underline{q} + (\nabla \underline{q})^T] + \lambda(\nabla \cdot \underline{q})\,\underline{\underline{\mathbf{I}}} \tag{4.30}$$

where μ is the coefficient of dynamic viscosity and λ is the 'second' coefficient of viscosity (the bulk elasticity).

("Linear" fluids): [8]

$$\tau_{yx} \propto \frac{du}{dy} \Longrightarrow \tau_{yx} = \mu \frac{du}{dy} \tag{4.35}$$

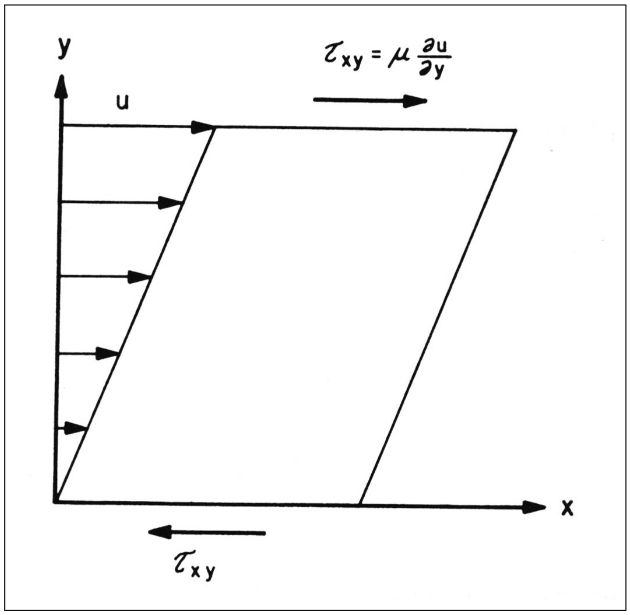


Figure 4.10 Stress and strain for simple shear flow of a Newtonian fluid. (From Newman 1977)

4.2.4 Navier-Stokes Equations for Incompressible Newtonian Fluids

- Divergence of stress tensor in conservation of momentum for a Newtonian incompressible fluid

$$\begin{aligned} \frac{\partial \tau_{ij}}{\partial x_j} &= -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) \\ &= -\frac{\partial p}{\partial x_i} + \mu \left(\frac{\partial^2 q_i}{\partial x_j \partial x_j}\right) \end{aligned} \tag{4.36}$$

[8] Some special fluids (e.g., most biological fluids, toothpaste, some paints, etc.) are non-Newtonian Fluids ("Non-linear" fluids), for which we might approximate the fluid viscosity as, in the locally linearized sense:

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n \Longrightarrow \tau_{yx} = k\left|\frac{du}{dy}\right|^{n-1}\left(\frac{du}{dy}\right) = \mu^* \frac{du}{dy} \tag{4.34}$$

- For incompressible Newtonian fluids, the corresponding differential form becomes the so-called Navier-Stokes equations:

$$\boxed{\rho\left[\frac{\partial \underline{q}}{\partial t}+\underline{q}\cdot\nabla\underline{q}\right]=-\nabla p+\underline{F}_B+\mu\nabla^2\underline{q}} \tag{4.37}$$

In the tensor form,

$$\boxed{\rho\left[\frac{\partial q_i}{\partial t}+q_j\frac{\partial q_i}{\partial x_j}\right]=-\frac{\partial p}{\partial x_i}+F_{B_i}+\mu\left(\frac{\partial^2 q_i}{\partial x_j\partial x_j}\right)} \tag{4.38}$$

- Navier-Stokes equations in Cartesian coordinates:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u+\frac{1}{\rho}F_{B_x} \tag{4.39}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v+\frac{1}{\rho}F_{B_y} \tag{4.40}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w+\frac{1}{\rho}F_{B_z} \tag{4.41}$$

4.2.4.1 Alternate forms of the convective and diffusion terms

Alternate forms of the non-linear convective and the viscous term of the Navier-Stokes equations: [9]

(1) Alternate form of the convective term, $\underline{q}\cdot\nabla\underline{q}$

  (a) Divergence form: $\nabla\cdot(\underline{q}\,\underline{q})=\underline{q}\cdot\nabla\underline{q}+\underline{q}\,(\nabla\cdot\underline{q})$

  (b) Advective/convective form: $\underline{q}\cdot\nabla\underline{q}=\frac{1}{2}\nabla q^2-\underline{q}\times(\nabla\times\underline{q})$

  (c) Rotational form: $\underline{\omega}\times\underline{q}$

  (d) Skew-symmetric (transpose of a matrix equals minus the matrix): $\frac{1}{2}\left[\nabla\cdot(\underline{q}\,\underline{q})+\underline{q}\cdot\nabla\underline{q}\right]=\underline{q}\cdot\nabla\underline{q}+\frac{1}{2}\underline{q}\,(\nabla\cdot\underline{q})$

(2) Alternate form of the viscous term, $\nabla^2\underline{q}$

[9] Gresho, P. M. (1991), "Incompressible fluid dynamics: some fundamental formulation issues", *Annual Review of Fluid Mechanics*, vol. 23, pp. 413–453.

(a) Stress-divergence form: $\nabla \cdot \left[(\nabla \underline{q}) + (\nabla \underline{q})^T\right] = \nabla^2 \underline{q} + \nabla(\nabla \cdot \underline{q})$

(b) Div-curl form: $\nabla^2 \underline{q} = \nabla(\nabla \cdot \underline{q}) - \nabla \times (\nabla \times \underline{q})$

(c) Curl form: $-\nabla \times (\nabla \times \underline{q}) = -\nabla \times \underline{\omega}$

4.2.5 Navier-Stokes Equations in a Moving Frame

4.2.5.1 Kinematic description

- Let $\underline{q}(x, y, z, t)$ describe the flow field in a moving coordinate system that is in motion relative to a space fixed system x', y', z'. In general, the

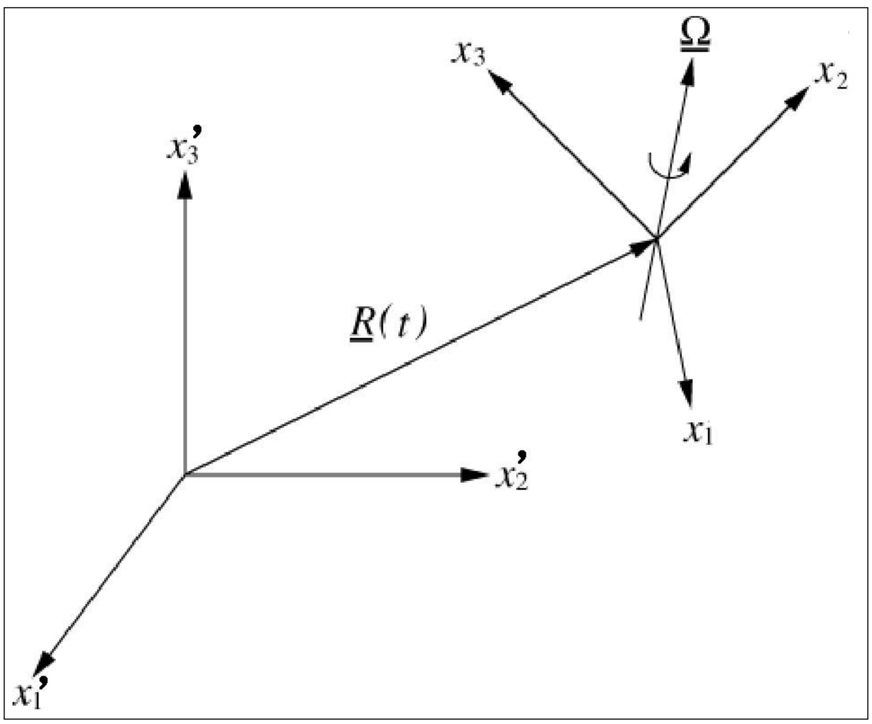


Figure **4.11** Moving coordinate system.

relations between coordinates, velocities, and accelerations are those of Eqs. (A.174–A.177);

$$\underline{x}' = \underline{x} + \underline{R} \tag{4.42}$$

$$\underline{q}' = \underline{q} + \dot{\underline{R}} + \underline{\Omega} \times \underline{x} \tag{4.43}$$

$$\frac{d'\underline{q}}{dt} = \frac{d\underline{q}}{dt} + 2\,\underline{\Omega} \times \underline{q} + \dot{\underline{\Omega}} \times \underline{x} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) + \ddot{\underline{R}} \tag{4.44}$$

- Thus the left-hand side of Eq. (4.44) must be augmented by addition of four new terms, in general, in order to constitute a differential equation for $\underline{q}(x, y, z, t)$.

- On the other hand, space derivatives such as *grad*, *div*, and *curl* are unaffected in form by the transformation of axes. The only change in such terms therefore arise from the process of carrying out these operations on quantities, such as $\underline{q}$, which have additional terms. There are no such terms in the scalar quantity p; hence Eq. (4.44) are altered only by addition of the four new left-hand terms mentioned.

- Before writing down the new equation of motion, let us consider the equation of continuity. Consider the term $\nabla \cdot' \underline{q}$, and let $\nabla \cdot$ denote the divergence operator in the moving system:

$$\nabla' \cdot \underline{q}' = \nabla \cdot \underline{q}' = \nabla \cdot \left(\underline{q} + \underline{\dot{R}} + \underline{\Omega} \times \underline{x} \right) = \nabla \cdot \underline{q} \qquad (4.45)$$

  because the divergences of the last two terms are zero (see Eq. (A.73)).

- It is important to note that the vorticity is not the same in the two systems. Using the primes as before, we see that, according to Eq. (A.74),

$$\nabla' \times \underline{q}' = \nabla \times \underline{q}' = \nabla \times \left(\underline{q} + \underline{\dot{R}} + \underline{\Omega} \times \underline{x} \right) = \nabla \times \underline{q} + 2\,\underline{\Omega} \qquad (4.46)$$

  We find the physical meaning of this result. An important case is one of flow that is irrotational but is viewed in a rotating coordinate system; it appears to be rotational. The case of a rotating propeller frame is typical.

4.2.5.2 Representation of velocity field

- The total velocity, observed in the inertial frame fixed in space is made up of two parts:

$$\underline{q}'_T(\underline{x}', t) = \underline{q}'_o(\underline{x}') + \underline{u}'(\underline{x}', t), \qquad (4.47)$$

  where $\underline{q}'_o(\underline{x}')$ is an onset velocity field that satisfies the continuity equation itself (a steady potential flow when measured in the inertial frame) and $\underline{u}'(\underline{x}', t)$ is the disturbance velocity component to be determined herein.

- In the moving frame the total velocity is the sum of the frame velocity ($\underline{q}_F$) and the fluid velocity ($\underline{q}$) measured by an observer in the moving frame:

$$\underline{q}_T(\underline{x}, t) = \underline{q}_F(\underline{x}, t) + \underline{q}(\underline{x}, t), \tag{4.48}$$

where $\underline{x}$ and $\underline{q}$ are measured relative to the moving frame.

- Equating the two different expressions of Eqs. (4.47) and (4.48), the relative velocity ($\underline{q}$) is expressed by

$$\boxed{\underline{q}(\underline{x}, t) = \underline{q}_o(\underline{x}, t) - \underline{q}_F(\underline{x}, t) + \underline{u}(\underline{x}, t)} \tag{4.49}$$

4.2.5.3 Governing equations

- The relations between coordinates, velocities, and acceleration are, again,

$$\underline{x}' = \underline{x} + \underline{R} \tag{4.50}$$

$$\underline{q}' = \underline{q} + \underline{\Omega} \times \underline{x} + \dot{\underline{R}} \tag{4.51}$$

$$\underline{a}' \equiv \frac{d'^2 \underline{x}'}{dt^2}$$

$$= \underline{a} + 2\,\underline{\Omega} \times \underline{q} + \frac{d\underline{\Omega}}{dt} \times \underline{x} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) + \ddot{\underline{R}} \tag{4.52}$$

- Space derivatives such as grad, div, and curl are unaffected:

$$\nabla' \cdot \underline{q}' = \nabla \cdot \underline{q}' = \nabla \cdot \left\{ \underline{q} + \underline{\Omega} \times \underline{x} + \dot{\underline{R}} \right\} = \nabla \cdot \underline{q} \tag{4.53}$$

- Continuity equation and Navier-Stokes equation:

$$\boxed{\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{q} = 0} \tag{4.54}$$

and

$$\boxed{\frac{D\underline{q}}{Dt} + 2\,\underline{\Omega} \times \underline{q} + \dot{\underline{\Omega}} \times \underline{x} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) + \ddot{\underline{R}} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\underline{F}_B + \nu \nabla^2 \underline{q}} \tag{4.55}$$

4.2.6 Boundary Conditions

- Kinematic (Velocity Continuity) Condition [10]
  - Fluid velocity equals the velocity of the body.
  - Both the normal and tangential velocity components of the fluid and the boundary to be equal.
  - Note: For inviscid flow, only the normal component must be considered.
- Dynamic (Stress Continuity) Condition [11]
  - Stress on one side of the interface between the fluid and the boundary equals the stress on the other side.
  - Both the normal and tangential stress components of the fluid and the boundary to be equal.
  - Note: The surface tension is included, if its effect is significant.

4.3 Example: Low Reynolds Number Flows

- As an example of stress tensors, let us consider low-Reynolds number flow (Stokes flow). [12]
- By introducing a moving frame fixed to the sphere, we can consider equivalently a stream of viscous fluid flows at speed U slowly about a stationary sphere of radius R. Then the relative velocity components for the moving coordinate system are given by

$$\underline{q} = -U\underline{i} + \underline{u}, \tag{4.56}$$

[10] Movie: Kinematic Condition (Solid-Liqud, Liquid-Gas, Liquid-Liquid, Sloshing Stratified Liquds)
./mmfm_movies/438.mov ./mmfm_movies/439.mov
./mmfm_movies/436.mov ./mmfm_movies/76.mov

[11] Movie: Dynamic Condition (Sloshing Stratified Liquds)
./mmfm_movies/454.mov

[12] See, e.g., Brennen, C. E. (1995), *Cavitation and Bubble Dynamics*, Oxford University Press, and Ton Tran-Cong and Blake, J. R. (1984), “General solutions of the Stokes flow equations,” *J. of Mathematical Analysis and Applications*, vol. 92, pp. 72–84.

- In terms of spherical polar coordinates (r, θ, α) where α is the azimuth angle about the axis $\theta = 0$ (see Figure 4.12), the flow is of axi-symmetry and then the continuity equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(q_\theta\, \sin\theta) = 0 \tag{4.57}$$

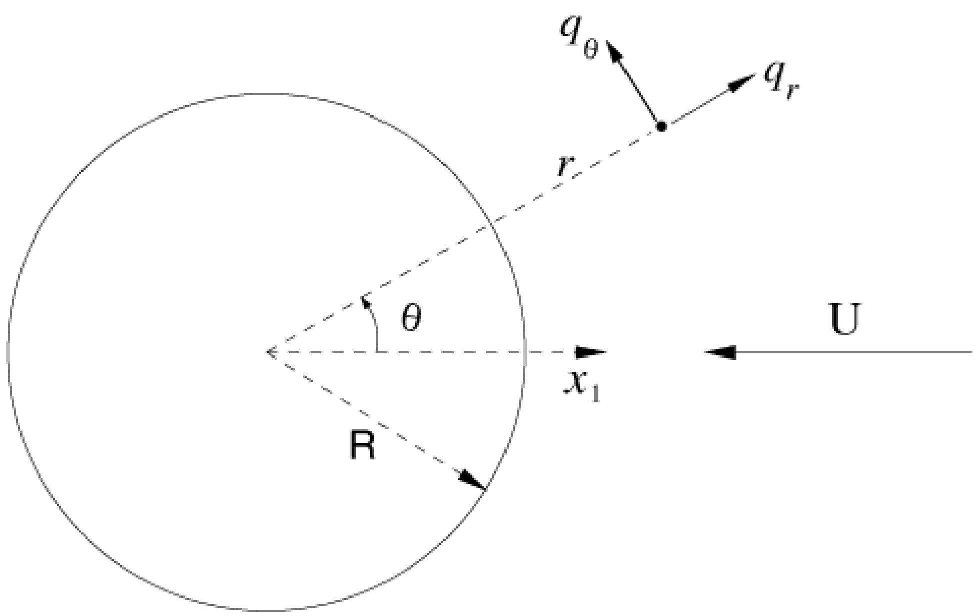


Figure 4.12 Notation for low Reynolds number flow about a spherical body in uniform flow.

4.3.1 Velocity Field: Solution of the Simplified Navier-Stokes Equations

- The governing equations for such a fluid are, reasonably neglecting the inertia terms (and the body force term) of the Navier-Stokes equations due to the very slow motion of the body,

$$\nabla \cdot \underline{u} = 0 \tag{4.58}$$

$$0 = -\nabla p + \mu \nabla^2 \underline{u} \tag{4.59}$$

where $\underline{u}$ is the disturbed velocity about a sphere moving in x_1-axis direction in otherwise fluid at rest. The second equation above physically implies that the pressure gradient balances the viscous force.

- The general solution to these equations is given in a form of

$$\underline{u} = \nabla(\underline{r} \cdot \underline{B} + B_o) - 2\underline{B} \quad \text{and} \quad p = 2\mu(\nabla \cdot \underline{B}) \tag{4.60}$$

where $\underline{B}$ and B_0 should satisfy the following conditions, respectively,

$$\nabla^2 \underline{B} = 0, \quad \nabla^2 B_o = 0 \tag{4.61}$$

- For Reynolds number values of $O(1)$, the solution for the disturbed velocity field is known to be, by setting $B_x = -3UR/4r$, $B_o = UR^3x/4r^3$, $B_y = B_z = 0$ where U is the moving speed and R is the radius of a sphere,

$$\underline{u} = \left(\frac{3R}{4r} + \frac{R^3}{4r^3}\right) U\underline{i} - \left(-\frac{3Rx}{4r^3} + \frac{3R^3x}{4r^5}\right) U\underline{r} \tag{4.62}$$

where $x = r\cos\theta$, $\underline{i} = \underline{e}_r \cos\theta - \underline{e}_\theta \sin\theta$, $\underline{r} = r\,\underline{e}_r$ (see Figure **4.12**). Namely,

$$q_r = -U\cos\theta - 2\left(\frac{C}{r^3} + \frac{D}{r}\right)\cos\theta \tag{4.63}$$

$$q_\theta = U\sin\theta - \left(\frac{C}{r^3} - \frac{D}{r}\right)\sin\theta \tag{4.64}$$

$$q_\alpha = 0 \tag{4.65}$$

where $C = UR^3/4$ and $D = -3UR/4$.

- The vorticity is

$$\underline{\omega} = \omega\,\underline{e}_\alpha = \left(\frac{1}{r}\frac{\partial(r\,q_\theta)}{\partial r} - \frac{1}{r}\frac{\partial q_r}{\partial\theta}\right)\underline{e}_\alpha = \frac{3}{2}UR\,\frac{\sin\theta}{r^2}\,\underline{e}_\alpha \tag{4.66}$$

- The pressure can be obtained from the momentum equation $\nabla p = \mu\nabla^2\underline{q}$, i.e., $\nabla p = -\mu\nabla\times\underline{\omega}$, using Eqs. (A.136) and (4.66)

$$\frac{\partial p}{\partial r} = -\frac{3\mu\,UR\cos\theta}{r^3}, \quad \frac{1}{r}\frac{\partial p}{\partial\theta} = -\frac{3\mu\,UR\sin\theta}{2r^3}, \tag{4.67}$$

- Integrating with respect to either r or θ, the solution for pressure is known to be

$$p = p_0 + \frac{3}{2}\mu\, RU\, \frac{\cos\theta}{r^2} \tag{4.68}$$

where p_0 is a reference pressure at infinity.

- Now, we define the stream function $\underline{\Psi} = (0, 0, \psi/r\sin\theta)$ to satisfy the continuity equation automatically, such that $\underline{q} = \nabla\times\underline{\Psi} = \nabla\times\left(\frac{\psi\,\underline{e}_\alpha}{r\sin\theta}\right)$, i.e.,

$$q_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\; q_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{4.69}$$

  The stream function would be of a form

$$\psi = \sin^2\theta\left(\frac{C}{r} + Dr + Er^2 + Fr^4\right) \tag{4.70}$$

- Applying the boundary conditions on the sphere surface ($q_r = q_\theta = 0$) and at infinity ($\psi_\infty = -Ur^2\sin^2\theta/2$), we may set $C = UR^3/4$, $D = -3\,UR/4$, $E = -U/2$, and $F = 0$.

- The first term and the third terms represent the inviscid flow past a sphere, while the second term corresponds to the viscous correction.

4.3.2 Stress Tensor and Drag

- From these expressions, the stress tensor is related to rate of strain tensor in a spherical coordinate system. Only 6 components are expressed as:

$$\tau_{rr} = -p + 2\mu \frac{\partial q_r}{\partial r} \tag{4.71}$$

$$\tau_{r\theta} = \mu \left(\frac{1}{r}\frac{\partial q_r}{\partial \theta} + \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r} \right) \tag{4.72}$$

$$\tau_{r\alpha} = \mu \left(\frac{\partial q_\alpha}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial q_r}{\partial \alpha} - \frac{q_\alpha}{r} \right) \tag{4.73}$$

$$\tau_{\theta\theta} = -p + 2\mu \left(\frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r} \right) \tag{4.74}$$

$$\tau_{\theta\alpha} = \mu \left(\frac{1}{r}\frac{\partial q_\alpha}{\partial \theta} - \frac{q_\alpha}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial \alpha} \right) \tag{4.75}$$

$$\tau_{\alpha\alpha} = -p + 2\mu \left(\frac{1}{r\sin\theta}\frac{\partial q_\alpha}{\partial \alpha} + \frac{q_r}{r} + \frac{q_\theta}{r}\cot\theta \right) \tag{4.76}$$

- With the solution for the velocity field, the values of these components are evaluated on the surface of the sphere, i.e., on $r = R$:

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{\alpha\alpha} = -p(R,\theta) \tag{4.77}$$

$$\tau_{r\theta} = \frac{3U}{2R}\,\mu\,\sin\theta, \qquad \tau_{r\alpha} = \tau_{\theta\alpha} = 0 \tag{4.78}$$

- On the surface of the sphere, the normal vector $\underline{n} = \underline{e}_r$(pointing into the fluid from the surface) is taken to find forces acting on the sphere by the fluid). Then, the surface stresses become $\underline{\tau} = \tau_{rr}\ \underline{e}_r + \tau_{r\theta}\ \underline{e}_\theta$ where $\underline{e}_\theta = \sin\theta\ \underline{i} + \cos\theta\ \underline{j}$.

- The surface force is composed of two components due to the normal and the tangential stress:

$$\underline{F}_S^{(n)} = \oint_S \tau_{rr}\,\underline{e}_r\,dS = 2\pi\mu\,UR\,\underline{i} \tag{4.79}$$

$$\underline{F}_S^{(t)} = \oint_S \tau_{r\theta}\,\underline{e}_\theta\,dS = 4\pi\mu\,UR\,\underline{i} \tag{4.80}$$

Herein we have used the surface element $dS = 2\pi R^2 \sin\theta \, d\theta$ for the actual integrations.

- The total drag of the sphere becomes $D = 6\pi\mu UR$ which corresponds to the drag coefficient $C_D = D/(\frac{1}{2}\rho U^2 \pi R^2) = 24/Re$ where Re is the Reynolds number based on the sphere diameter and the speed of the onset flow. [13]

4.4 Simple Problems for Viscous Flows

- Typical Phenomena of Viscous Flow past a Hydrofoil [14]

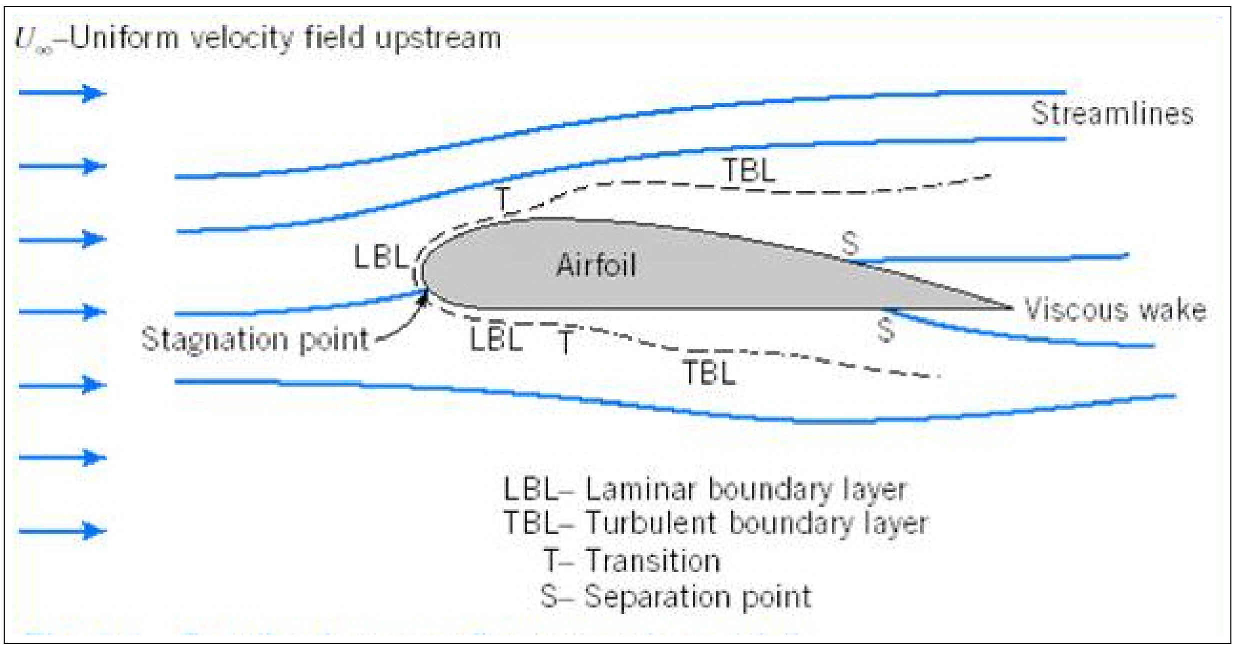


Figure 4.13 Viscous flow around a foil. (From Fox, McDonald & Pritchard 2004)

4.4.1 Flow between Two Parallel Walls (Plane Couette Flow)

- The x-component parallel to the walls is the only nonzero component of velocity vector.

[13] Movie: Low Rn Flow (freely falling sphere in fluid)
./mmfm_movies/351.mov

[14] Movie: Viscous Flows around Foils
./mmfm_movies/2_05015.mov ./mmfm_movies/2_06037.mov

- The flow is independent of the x-coordinate and the z-coordinate.

- Simple System of the Navier-Stokes equations

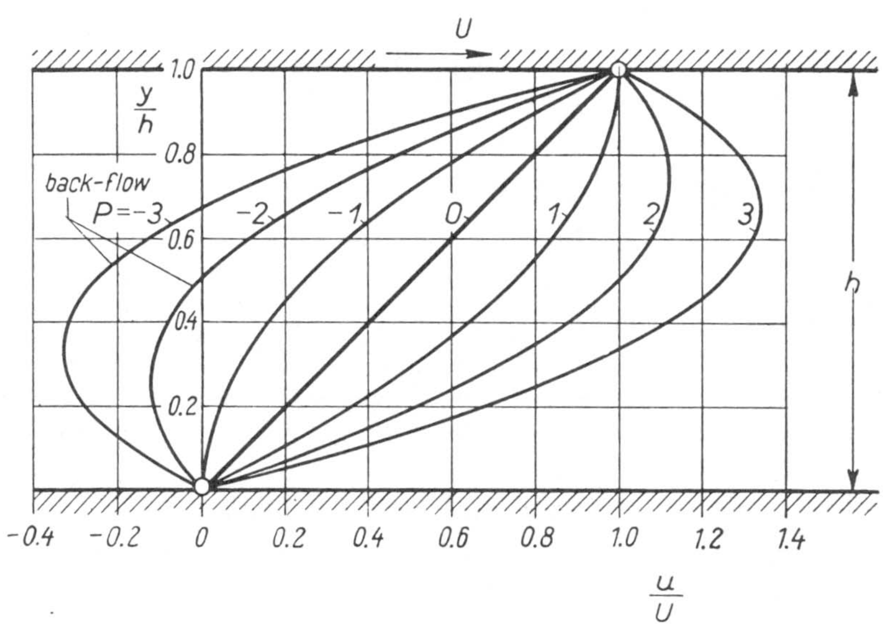
$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2} \tag{4.81}$$

$$\frac{\partial p}{\partial y} = 0 \tag{4.82}$$

$$\frac{\partial p}{\partial z} = 0 \tag{4.83}$$

- Solution

$$u = \frac{1}{2\mu}\frac{dp}{dx}y\,(y-h) + \frac{U}{h}y \tag{4.84}$$



3.3
Couette flow between two parallel walls. The curves are labeled with values of the nondimensional pressure gradient $P = -(h^2/2\mu U)(dp/dx)$. Note the occurrence of a secondary backflow, analogous to separation, when this parameter is less than -1. (From Schlichting 1968)

Figure 4.14 Plane Couette flow between two parallel walls. (From Newman 1977)

4.4.2 Flow through a Pipe(Poiseuille Flow)

- Simple System of the Navier-Stokes equations

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{4.85}$$

- Solution [15]

$$u = -\frac{1}{4\mu}\frac{dp}{dx}\left(r_0^2 - y^2 - z^2\right) = -\frac{1}{4\mu}\frac{dp}{dx}\left(r_0^2 - r^2\right) \tag{4.86}$$

- Volume Rate of Flow (Flux)

$$Q = \int_0^{r_0} u(r)\, 2\pi r\, dr = \frac{\pi r_0^4}{8\mu}\left(-\frac{dp}{dx}\right) \tag{4.87}$$

- Transition with increasing Reynolds numbers [16]

4.4.3 External Flow Past One Flat Plate: Unsteady Motion of a Flat Plate

- External flow problems are relatively complicated and the flow domain is unbounded.
- No slip boundary condition generates the viscous shear action in a thin boundary layer at the body surface at the high Reynolds number.
- Viscous effects are diffusive, having an infinite amount of time to take.
- Governing Equation: Heat (Diffusion) Equation

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \tag{4.88}$$

[15] Movie: Flow through a Pipe (Velocity Profile)
./mmfm_movies/719.mov

[16] Movie: Flow through a Pipe(Laminar, Turbulent)
./mmfm_movies/3596.mov ./mmfm_movies/3597.mov

- Boundary Conditions: No-slip and Far-field Conditions

$$\begin{aligned} u(0,t) &= U(t) \quad \text{on} \quad y=0 && (4.89)\\ u(y,t) &\to 0 \quad \text{as} \quad y\to\infty && (4.90)\end{aligned}$$

- Consider two unsteady problems:

  (1) sinusoidal motion, $U(t) = U_0 \cos(\omega t)$ and

  (2) impulsively started to a constant velocity, $U(t) = U_0$ for $t > 0^+$.

4.4.3.1 Stokes 1st problem: Sinusoidal motion $U(t) = U_0 \cos(\omega t)$

- Complex form of sinusoidal flow

$$u(y,t) = \mathrm{Re}\left[f(y)\, e^{i\omega t}\right] \tag{4.91}$$

- Ordinary differential equation

$$i\omega f = \nu \frac{d^2 f}{dy^2} \tag{4.92}$$

- General solution form

$$f = A\, e^{ky} + B\, e^{-ky} \tag{4.93}$$

  where the characteristic value k

$$k = (i\omega/\nu)^{1/2} = (1+i)\ (\omega/2\nu)^{1/2} \tag{4.94}$$

- Final solution

$$\begin{aligned} u(y,t) &= \mathrm{Re}\left\{U_0 \exp\left(-k\,y + i\,\omega\,t\right)\right\}\\ &= U_0 \exp\left[-\left(\omega/2\nu\right)^{1/2}\, y\right]\, \cos\left[\left(\omega/2\nu\right)^{1/2}\, y - \omega\, t\right] && (4.95)\end{aligned}$$

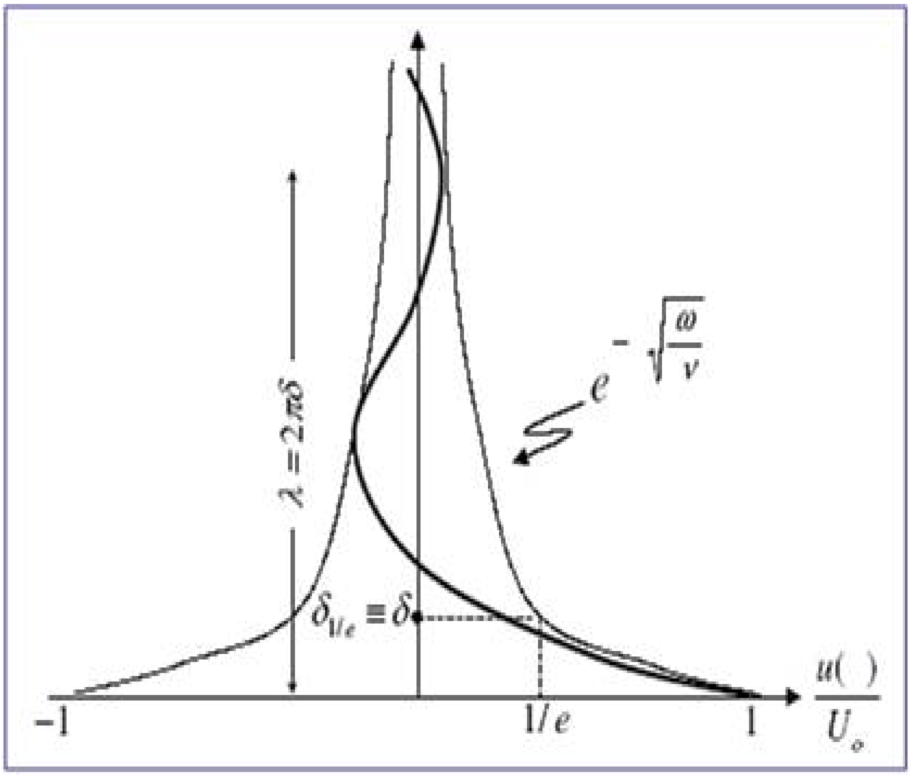


Figure 4.15 Stokes 1st problem: Sinusoidal motion of flat plate. (From MIT website 2004)

- Boundary layer thickness $\delta_{1\%}$ such that $\exp\left[-\left(\omega/2\nu\right)^{1/2}\,\delta_{1\%}\right] = 0.01$: (Example) For fresh water with $\nu = 1.0 \times 10^{-6}$, $\delta_{1\%} = 2.60\ mm$ for $\omega = 2\pi$ (period T = 1 second), and $\delta_{1\%} = 8.22\ mm$ for $\omega = 0.2\pi$ (period T= 10 second).

4.4.3.2 Stokes 2nd problem: Step-function velocity, $U(t) = U_0$ for $t > 0^+$

- Similarity parameter

$$\eta = \frac{y}{2\left(\nu t\right)^{1/2}} \tag{4.96}$$

- Ordinary differential equation

$$\left.\begin{aligned} \eta &= \frac{y}{2\sqrt{\nu t}} \\ \frac{\partial}{\partial t} &= \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -\frac{y}{4t\sqrt{\nu t}}\frac{\partial}{\partial \eta} \\ \frac{\partial^2}{\partial y^2} &= \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2}{\partial \eta^2} = \frac{1}{4\nu t}\frac{\partial^2}{\partial \eta^2} \end{aligned}\right\} \xrightarrow{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}} \underbrace{-2\eta \frac{d\,(u/U_0)}{d\eta} = \frac{d^2\,(u/U_0)}{d\eta^2}}_{2^{\text{nd}}\text{ order ODE}} \tag{4.97}$$

$$\Longrightarrow \boxed{\frac{d^2 f}{d\eta^2} + 2\eta \frac{df}{d\eta} = 0} \tag{4.98}$$

- Final Solution [17]

$$f = \frac{u}{U_0} = 1 - erf(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2}\, d\eta \tag{4.99}$$

where $erf(\eta)$ is the error function.

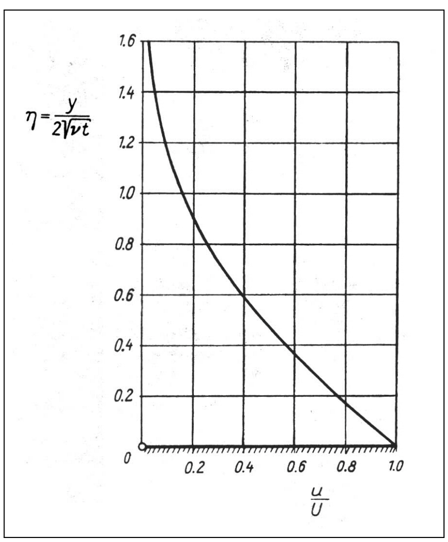


Figure 4.16 Velocity profile near an impulsively started flat plate (Stokes 2nd problem). (From Newman 1977)

[17] Movie: Stokes 2nd problem(Impulsive plate)
./mmfm_movies/534.mov ./mmfm_movies/1_02035_03.mov

- Boundary layer thickness [18]

$$\delta = 3.64\ (\nu t)^{1/2} \tag{4.100}$$

Approximate solution from the original differential equation

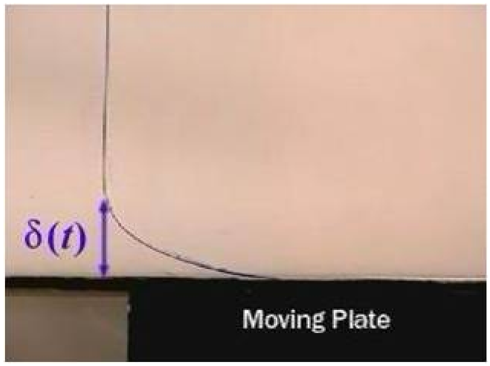


Figure 4.17 Boundary layer thickness in Stokes 2nd problem for an impulsively started flat plate.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \quad \Longrightarrow \quad \frac{U_0}{t} \approx \nu \frac{U_0}{\delta^2} \quad \Longrightarrow \quad \delta \sim \sqrt{\nu t} \tag{4.101}$$

4.5 Boundary Layer Theory

4.5.1 Introduction

- Boundary-Layer Concepts [19]
- Boundary Conditions
  - No-penetration Condition: No fluid can cross a solid boundary.
  - No-slip Condition
    * Viscous force make the fluid ‘stick to the wall (body surface).
    * The velocity of the fluid on wall equals the velocity of the wall.
- Viscous Diffusion Length: $\delta(t)$

[18] Movie: Boundary layer thickness (Moving plate)

Figure 4.18 Variation of boundary layer thickness with Reynolds numbers.

- The boundary layer thickness is a function of Reynolds number ($R_n = Ul/\nu$) and time. [20]
- Fully developed flows: Boundary layers can grow to the point where they are no longer confined to regions close to the wall. [21]

4.5.2 Laminar Boundary Layer: Flat Plate

- Boundary-Layer Thicknesses
- Laminar Flat-Plate Boundary Layer: Exact Solution
- Momentum Integral Equation
- Momentum Equation for Flow with Zero Pressure Gradient
- Pressure Gradients in Boundary-Layer Flow

./mmfm_movies/2_02005.mov

[19] Movie: Boundary layer concept (Velocity profile near plate, Boundary layer thickness on moving plate)
./mmfm_movies/1_02035_04.mov ./mmfm_movies/2_02005.mov

[20] Movie: Boundary layer thickness (Exp. set-up for reverse, Reverse($Rn < 1$), Irreverse($Rn > 1$), Oil reverse)
./mmfm_movies/kinematic_reverse_2.mov ./mmfm_movies/kinematic_reverse.mov
./mmfm_movies/kinematic_irreverse.mov ./mmfm_movies/oil_00010_reverse.mov

[21] Movie: Boundary layer thickness (Low viscosity oil, High viscosity oil, Very high viscosity oil)
./mmfm_movies/oil_00010.mov ./mmfm_movies/oil_00100.mov
./mmfm_movies/oil_10000.mov

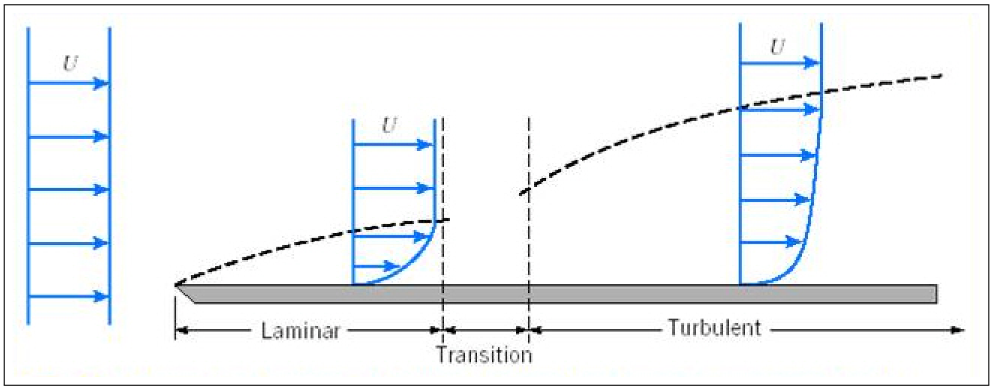


Figure 4.19 Boundary layer on a flat plate. (From Fox, McDonald & Pritchard 2004)

- Continuity and Navier-Stokes Equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.102)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad (4.103)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad (4.104)$$

- Prandtl's Boundary Layer Equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.105)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad (4.106)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (4.107)$$

- Laminar Boundary Layer Equation for a Plate

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.108)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \quad (4.109)$$

- Non-dimensionalize the equations with appropriate variables.

$$x' = x/l, \quad y' = y\,R^{1/2}/l \tag{4.110}$$
$$u' = u/U, \quad v' = v\,R^{1/2}/U \tag{4.111}$$

- Non-dimensional equations

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} \;=\; 0 \tag{4.112}$$
$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} \;=\; \frac{\partial^2 u'}{\partial y'^2} \tag{4.113}$$

with boundary conditions

$$u' = v' = 0 \quad \text{on} \quad y' = 0 \tag{4.114}$$
$$u' \to 1, \quad v' \to 0 \quad \text{for} \quad y' \to \infty \tag{4.115}$$

- With the similarity parameter $\eta = y\sqrt{\dfrac{U}{\nu x}}$, similarity solution form

$$\frac{u}{U} = F(\eta), \qquad v\,(x/U\nu)^{1/2} = G(\eta) \tag{4.116}$$

- Ordinary differential equations

$$-\frac{1}{2}\eta\,\frac{dF}{d\eta} - \frac{dG}{d\eta} \;=\; 0 \tag{4.117}$$
$$-\frac{1}{2}\eta\,F\,\frac{dF}{d\eta} + G\,\frac{dF}{d\eta} - \frac{d^2F}{d\eta^2} \;=\; 0 \tag{4.118}$$

with the boundary conditions

$$F(0) = 0, \quad F(\infty) = 1, \quad G(0) = 0, \quad G(\infty) = 0 \tag{4.119}$$

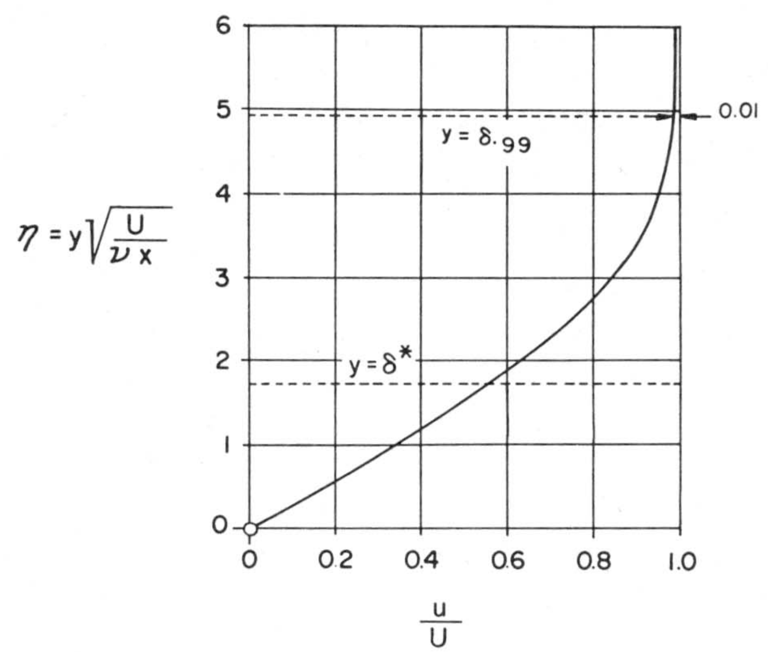
- Stream function $f(\eta)$ such that

$$F = \frac{df}{d\eta}, \quad G = \frac{1}{2}\left(\eta\frac{df}{d\eta} - f\right) \tag{4.120}$$

The ordinary differential equation

$$f\frac{d^2 f}{d\eta^2} + 2\frac{d^3 f}{d\eta^3} = 0 \tag{4.121}$$

with $f = \dfrac{df}{d\eta} = 0$ on $\eta = 0$, $\dfrac{df}{d\eta} = 1$ at $\eta = \infty$.



3.5
Velocity distribution in a Blasius laminar boundary layer. The total area under the curve is equal to the area under the dashed line, corresponding to the flux defect and the displacement thickness δ^*.

Figure 4.20 Velocity profile in a Blasius laminar boundary layer. (From Newman 1977)

- Boundary layer thickness definitions [22]
  - Disturbance Thickness, δ

[22] Movie: Boundary layer thickness (Moving plates, Low/high viscosity)
./mmfm_movies/making_timeline2.mov ./mmfm_movies/2_02005.mov
./mmfm_movies/2_03015.mov

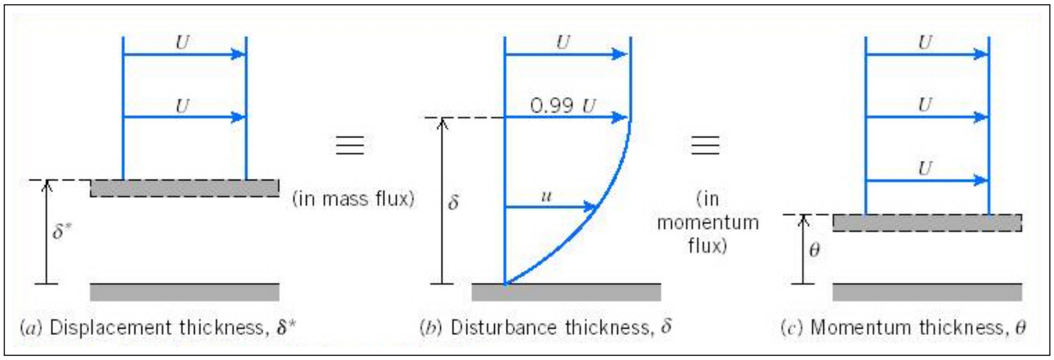


Figure 4.21 Boundary layer thickness definition. (From Fox, McDonald & Pritchard 2004)

– Displacement Thickness, δ^*

$$\delta^* \approx \int_0^{\delta} \left(1 - \frac{u}{U}\right)\, dy \tag{4.122}$$

– Momentum Thickness, θ

$$\theta \approx \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right)\, dy \tag{4.123}$$

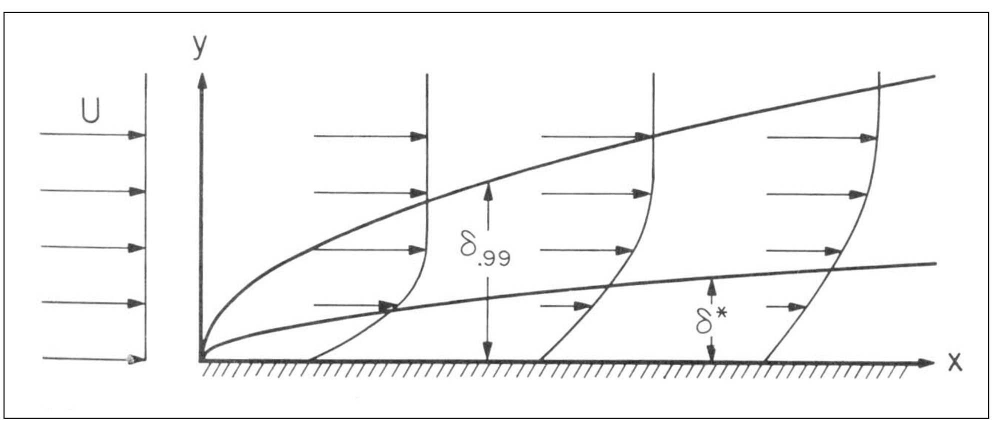


Figure 4.22 Development of laminar boundary layer along a flat plate. The lateral scale is magnified in order to show its development clearly. (From Newman 1977)

- Blasius Solution: Boundary-Layer Thicknesses and Frictional Resistance

– Disturbance Thickness, δ

$$\delta = 4.9\, (\nu\, x/U)^{1/2} \tag{4.124}$$

– Displacement Thickness, δ^*

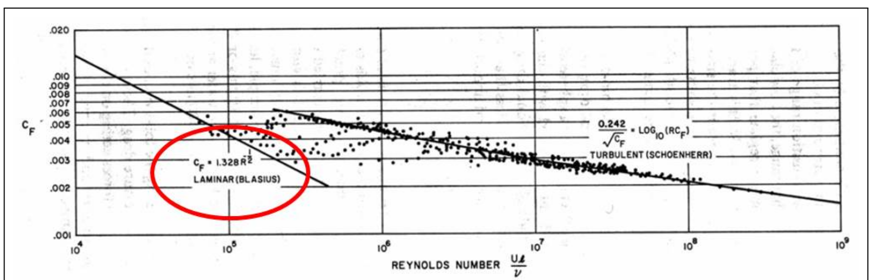
$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right)\, dy = 1.72\,(\nu\, x/U)^{1/2} \tag{4.125}$$

– Momentum Thickness, θ

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)\, dy = 0.664\,(\nu\, x/U)^{1/2} \tag{4.126}$$

– Frictional stress and resistance

$$\begin{aligned}
\tau_{xy} &= \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U\,(U/\nu x)^{1/2} \left(\frac{d^2 f}{d\eta^2}\right)_{\eta=0} \\
&= 0.332\,\rho\, U^2 R_x^{-1/2} && (4.127)\\
D &= b\int_0^l \tau_{xy}\, dx = b\int_0^l \left(0.332\,\rho\, U^2 \left(\frac{Ux}{\nu}\right)^{-1/2}\right) dx \\
&= 0.664 b\left(\mu\,\rho\, l\, U^3\right)^{1/2} && (4.128)\\
C_F &= \frac{D}{\frac{1}{2}\rho\, U^2 (bl)} = 1.328(\nu/Ul)^{1/2} = 1.328 R^{-1/2} && (4.129)
\end{aligned}$$



2.3
Schoenherr's flat-plate frictional drag coefficient, compared with various experimental results and the Blasius boundary-layer theory. Note that transition occurs between $R = 10^5$ and 2×10^6, depending on the plate roughness and ambient turbulence. For references to the experimental data see Todd (1967), figure 3.

Figure 4.23 Flat-plate frictional resistance coefficient: Blasius laminar boundary layer solution. (From Newman 1977)

4.5.3 Laminar Boundary Layer: Steady 2-D Flow

- Interaction between inviscid solution and boundary layer solution [23]
- Von Karman integral relation for boundary layer

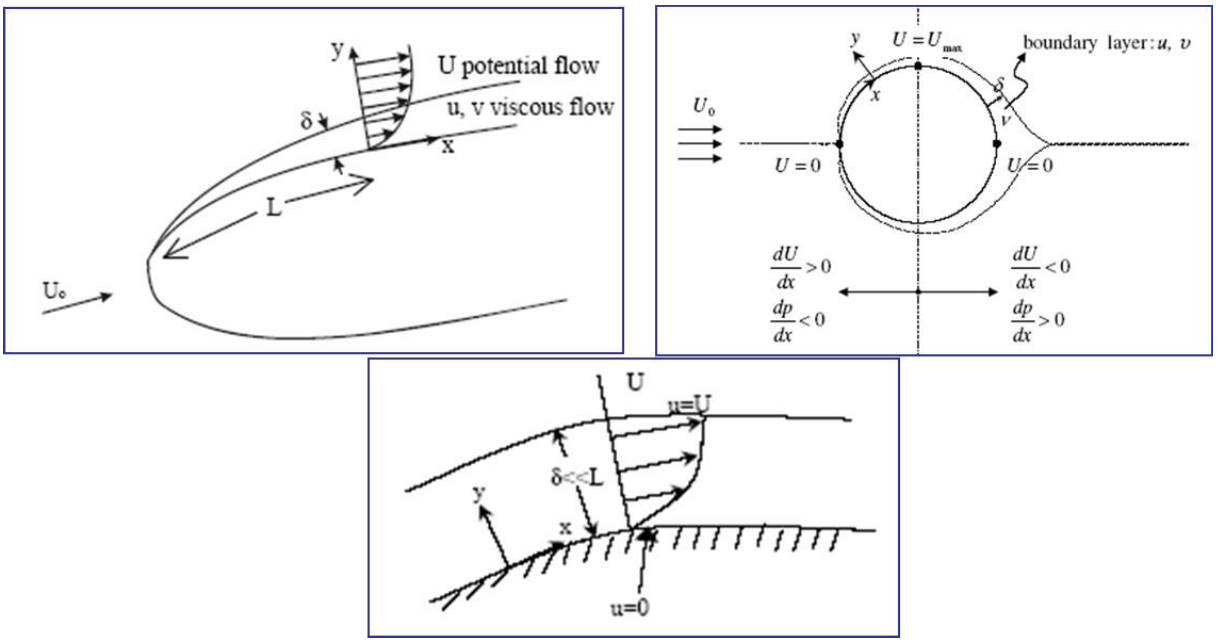


Figure 4.24 Laminar boundary layer: Steady 2-D flow. (From MIT website 2004)

- Pressure Gradients in Boundary-Layer Flow
  - Necessary conditions for flow separation

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0, \quad \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} > 0 \tag{4.130}$$

  - Pressure gradient relation

$$\mu \left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial p}{\partial x} \quad \text{on} \quad y = 0 \tag{4.131}$$

- Boundary Layer Equations

[23] Movie: Velocity profile near leading edge of 2-D foil
./mmfm_movies/2_01005b.mov

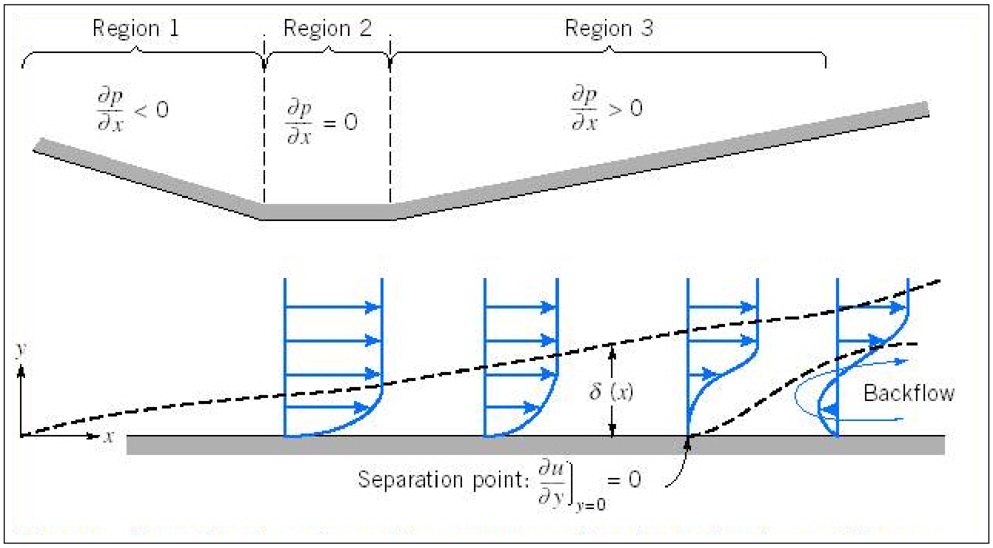
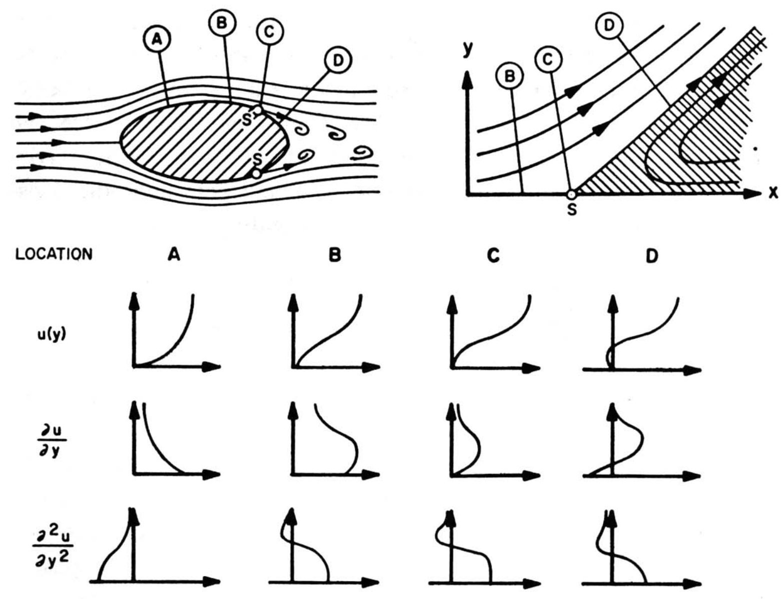


Figure 4.25 Boundary layer flow with pressure gradient. (From Fox, McDonald & Pritchard 2004)



3.7
Flow past a body with separation. The second figure shows the local flow near the separation point, in terms of local coordinates that are tangential (x) and normal (y) to the body at the separation point. The graphs show the different velocity profiles, in the boundary layer, for a favorable pressure gradient on the forebody (A), an unfavorable pressure gradient on the afterbody (B), at the separation point (C), and downstream of the separation point (D). Note the qualitative similarity for each of these velocity profiles with the Couette flows shown in figure 3.3.

Figure 4.26 Flow past a body with separation. (From Newman 1977)

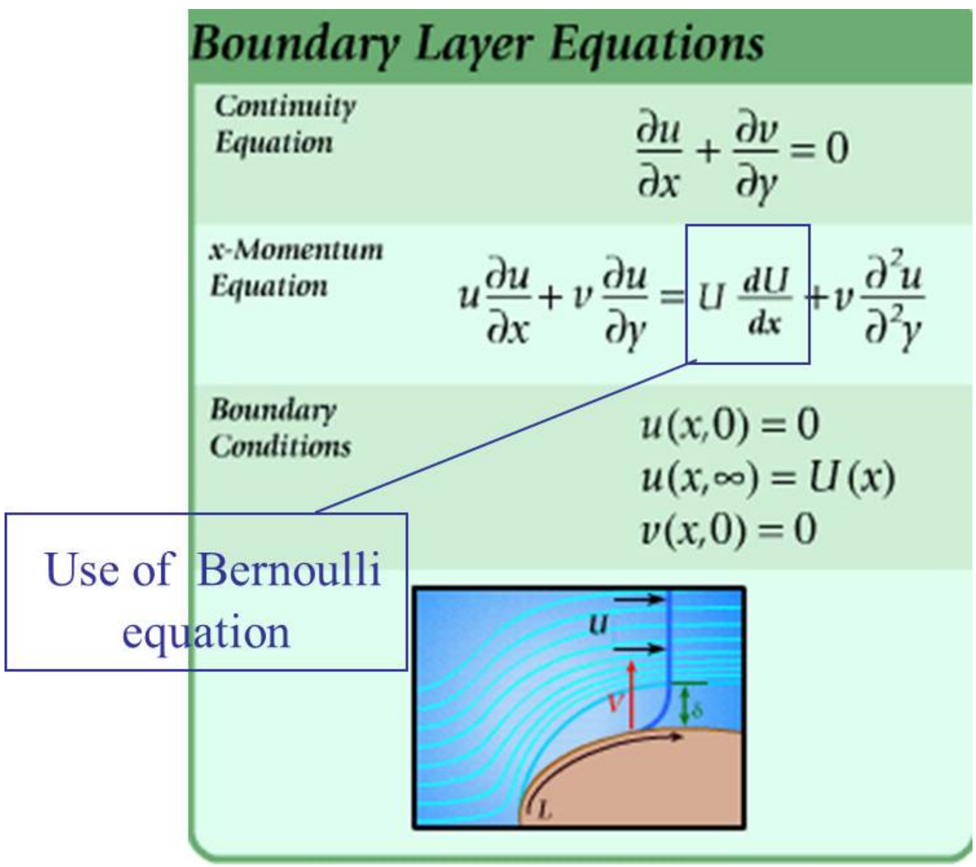


Figure 4.27 Boundary layer equations with pressure gradient.

– Based on continuity and Navier-Stokes equations

– Numerical implementation of boundary layer equations

4.5.3.1 Von Karman integral relation

- Integrate Prandtl boundary layer equation

$$\int_0^\delta \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) dy = -\frac{1}{\rho}\int_0^\delta \frac{\partial p}{\partial x}\, dy + \nu \int_0^\delta \frac{\partial^2 u}{\partial y^2}\, dy \qquad (4.132)$$

- Bernoulli equation at boundary layer edge

$$p = -\frac{1}{2}\rho U^2 + \text{constant}, \qquad \frac{\partial p}{\partial x} = -\rho U \frac{dU}{dx} \qquad (4.133)$$

- Regroup the terms in the integral

$$\int_0^\delta \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx} \right) dy =$$
$$\nu \int_0^\delta \frac{\partial^2 u}{\partial y^2}\, dy = \nu \left\{ \left(\frac{\partial u}{\partial y}\right)_{y=\delta} - \left(\frac{\partial u}{\partial y}\right)_{y=0} \right\} = -\frac{\tau_{xy}}{\rho} \qquad (4.134)$$

- Integrate the continuity equation with boundary condition

$$\int_0^y \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dy = 0, \quad v(x,y) = -\int_0^y \frac{\partial u}{\partial x}\, dy \qquad (4.135)$$

- Substitute the expression for v into the integral expression of the Prandtl boundary layer equation, and integrate by parts

$$
\begin{aligned}
-\tau_{xy}/\rho &= \int_0^\delta \left[u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\left(\int_0^y \frac{\partial u}{\partial x}dy\right) - U\frac{dU}{dx}\right] dy \\
&= \int_0^\delta \left[u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right] dy - U\int_0^\delta \frac{\partial u}{\partial x}dy \\
&= \int_0^\delta \left[2u\frac{\partial u}{\partial x} - U\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right] dy \\
&= -\int_0^\delta \frac{\partial}{\partial x}\left[u(U-u)\right] dy - \int_0^\delta \frac{dU}{dx}(U-u)\, dy \\
&= -\frac{d}{dx}\int_0^\delta u(U-u)\, dy - \frac{dU}{dx}\int_0^\delta (U-u)\, dy \quad (4.136)
\end{aligned}
$$

- Use the definition of boundary layer thicknesses

$$\boxed{\frac{\tau_{xy}}{\rho} = \frac{d}{dx}(U^2\theta) + U\delta^*\frac{dU}{dx}} \qquad (4.137)$$

4.5.3.2 Laminar boundary layers: Closing remarks

- Three-dimensional laminar boundary layers are complicated.
  - Pressure gradient, cross flow component within the boundary layer.
  - Axisymmetric flow is an exceptional case.
- Unsteadiness is another complication, but it is of less practical concern for most design problems.
- Laminar boundary layer up to $Re = 10^5$.
- Growth of boundary layer
  - Boundary layers develop along the foil shown above.
  - The velocity profile changes shape over the curved leading edge.
  - Towards the trailing edge the flow tends to separate as a result of an adverse pressure gradient.

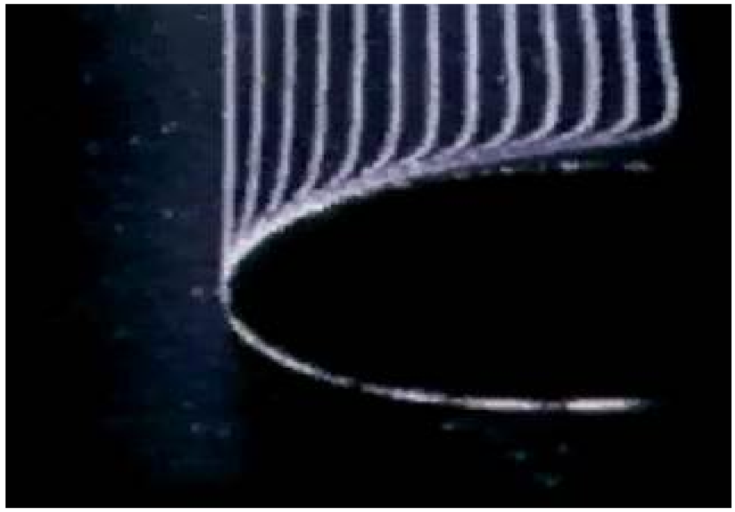


Figure 4.28 Velocity profiles and growth of the boundary layer near the leading edge of a foil.

- Attached and Separated Flows
  - The flow streamlines are redirected around the shape of the body.

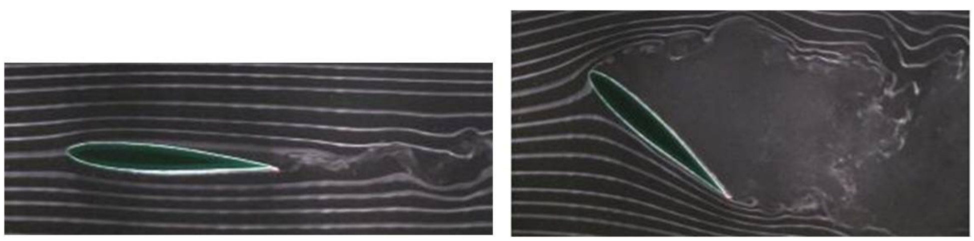


Figure 4.29 Attached and separated flows about a foil.

4.6 Turbulent Flows: General Aspects

4.6.1 Introduction

- Laminar, Transition and Turbulent Flows in Pipe [24]
  - Unlike boundary layer, in fully developed internal flows, Reynolds number (Re) does not change with distance but is fixed by the pipe.

[24] Movie: Turbulent flows
./mmfm_movies/ful_trb_flw_v.mov ./mmfm_movies/turbulent_laminarcombo.mov ./mmfm_movies/Lam-turb.mov ./mmfm_movies/transition2_ovrly.mov ./mmfm_movies/turbulent2_ovrly.mov

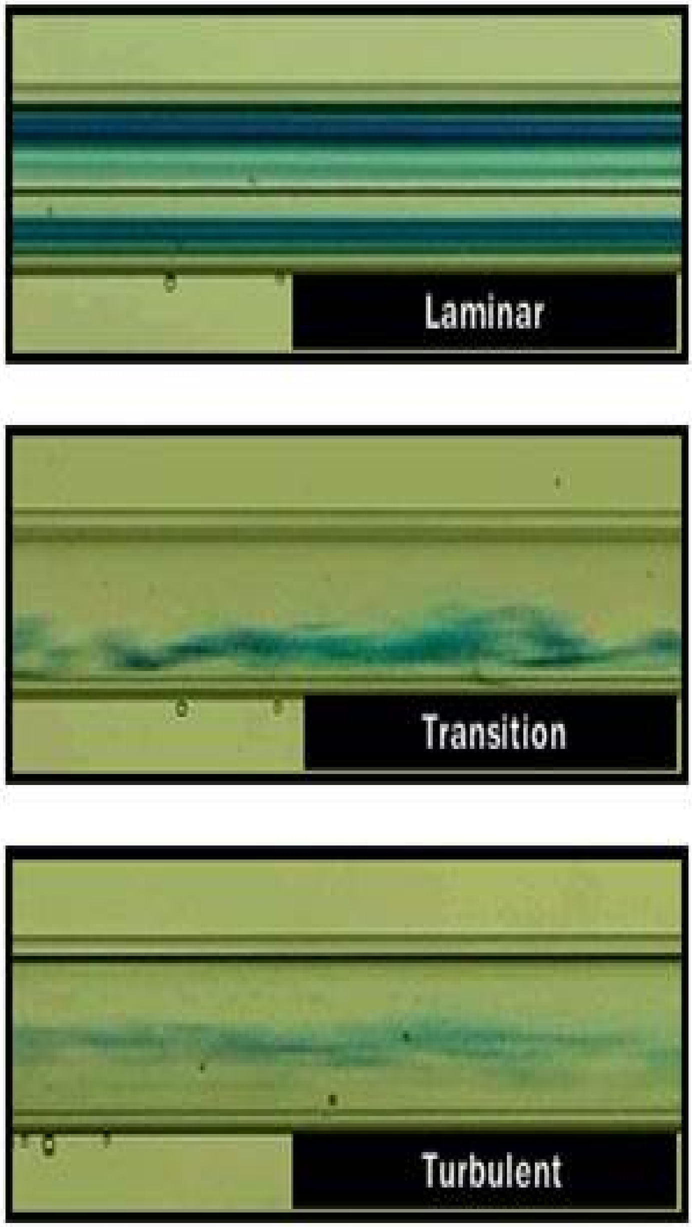


Figure 4.30 Flow visualization of laminar, transition and turbulent flows in pipe.

- The transition from laminar to turbulent flow in a pipe is of considerable historical significance: Re is a key dynamic parameter in fluid mechanics.
- Laminar: At low Reynolds number (Re), the boundary layer velocity profile is smooth and steady.
- Transition: At higher values of Re, the boundary layer becomes unstable.
- Turbulence: Instability leads to the breakdown, or transition of the laminar to turbulent.

- Various Phenomena of Turbulent Flows [25]
  - Jet flow around the exit of a nozzle
  - Vortex near wall, Thermal plume of human body, Dye diffusion, etc
  - Vortex shedding, etc
  - Interaction of vortex pair near wall
  - Turbulent mixing
  - Turbulent diffusion
  - Wake flows for vertical plate

4.6.2 Analysis of Turbulent Flows

- The turbulent boundary layer can be analyzed by recognizing that the velocities and pressures are composed of an average and a fluctuating part. [26]

[25] Movies

Jet flow around the exit of a nozzle: ./mmfm_movies/5448.mov
Vortex near wall, Thermal plume of human body, Dye diffusion, etc: ./mmfm_movies/5454.mov
Vortex shedding, etc: ./mmfm_movies/5457.mov
Interaction of vortex pair near wall: ./mmfm_movies/5104.mov
Turbulent mixing: ./mmfm_movies/4733.mov
Turbulent diffusion: ./mmfm_movies/4732.mov
./mmfm_movies/539.mov
./mmfm_movies/2_07030_diffusion.mov
Wake flows for vertical plate: ./mmfm_movies/607.mov

[26] Movie: Turbulent diffusion
./mmfm_movies/2_07030_diffusion.mov

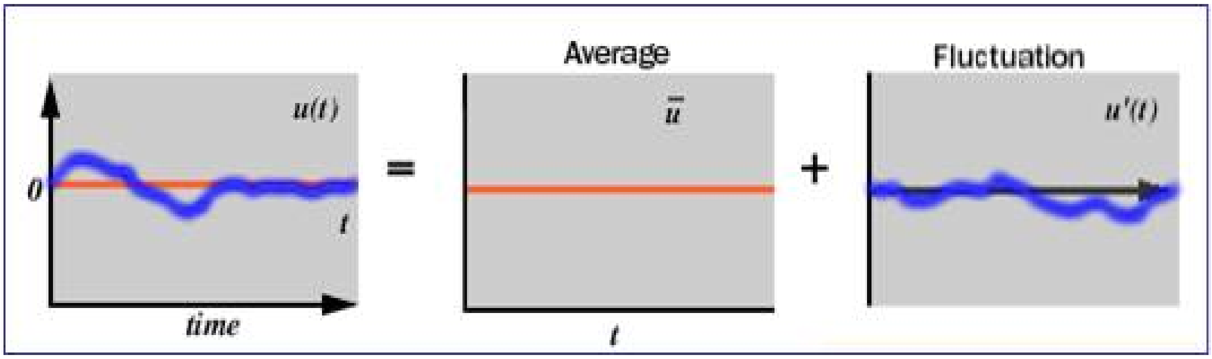


Figure 4.31 Decomposition of turbulent quantities into an average and a fluctuating part.

$$u(x,y,z,t) = \overline{u}(x,y,z,t) + u'(x,y,z,t) \tag{4.138}$$
$$p(x,y,z,t) = \overline{p}(x,y,z,t) + p'(x,y,z,t) \tag{4.139}$$

- Separate velocity vector into mean and fluctuating parts [27]

$$u_i = \overline{u}_i + u_i' \tag{4.140}$$

- Average of the fluctuating parts

$$\overline{u_i'} = 0, \quad \overline{\frac{\partial u_i}{\partial x_j}} = \frac{\partial \overline{u}_i}{\partial x_j}, \quad \overline{\frac{\partial u_i'}{\partial x_j}} = \frac{\partial \overline{u}_i'}{\partial x_j} = 0 \tag{4.141}$$

- Average of the products of the fluctuating parts

$$\overline{u_i\, u_j} = \overline{(\overline{u}_i + u_i')\,\left(\overline{u}_j + u_j'\right)} = \overline{u}_i\, \overline{u}_j + \overline{u_i'\, u_j'} \tag{4.142}$$

- Continuity equation

$$\frac{\partial \overline{u}_i}{\partial x_i} + \frac{\partial u_i'}{\partial x_i} = 0 \tag{4.143}$$

- If averages are taken, continuity equation of the mean velocity and the fluctuating components

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0, \quad \frac{\partial u_i'}{\partial x_i} = 0 \tag{4.144}$$

[27] This separation is based on Taylor's Hypothesis. For details see the article: Smits, McKeon, and Marusic (2011), "High-Reynolds Number Wall Turbulence," *Annual Review of Fluid Mechanics*, vol. 43, pp. 353-375.

- Averaged equations of the Navier-Stokes equations

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + \overline{u'_j \frac{\partial u'_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \nabla^2 \overline{u}_i \tag{4.145}$$

- Contribution of the nonlinear term, by the continuity equation of the mean velocity and the fluctuating components

$$\overline{u'_j \frac{\partial u'_i}{\partial x_j}} = \frac{\partial}{\partial x_j}\left(\overline{u'_i\, u'_j}\right) - \overline{u'_i \frac{\partial u'_j}{\partial x_j}} = \frac{\partial}{\partial x_j}\left(\overline{u'_i\, u'_j}\right) \tag{4.146}$$

- RANSE (Reynolds Averaged Navier-Stokes Equations)

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \nabla^2 \overline{u}_i - \frac{\partial}{\partial x_j}\left(\overline{u'_i\, u'_j}\right) \tag{4.147}$$

- Total stress tensor including Reynolds stress [28]

$$\overline{\tau}_{ij} = -\overline{p}\delta_{ij} + \mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \overline{\rho\, u'_i\, u'_j} \tag{4.148}$$

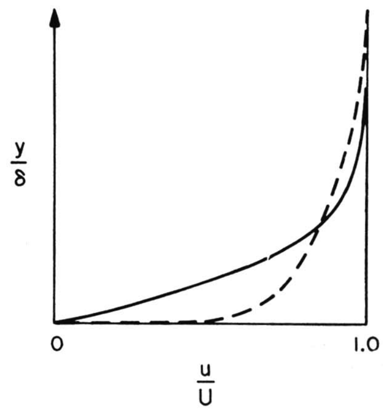
- Increased stress level: Frictional drag is greater in the turbulent regime than is the case for laminar flow.

- Convection of momentum: Velocity profile is more uniform resulting in a larger gradient and shear stress. [29]

- Turbulent velocity profiles

[28] Movie: $\overline{\rho\, u'\, v'} < 0$
./mmfm_movies/5024.mov

[29] Movie: Turbulent diffusion
./mmfm_movies/2_07030_diffusion.mov



3.9
Comparison of laminar (———) and mean turbulent (-----) velocity profiles, for the boundary layer on a flat plate. Since the boundary layer thickness is substantially greater in the turbulent case, the difference in scales should be noted in this comparison.

Figure 4.32 Comparison of laminar and mean turbulent velocity profiles. (From Newman 1977)

4.6.3 Turbulent Boundary Layer on a Flat Plate [30]

- Tangential velocity component is of the form:

$$u = f(x, y, \rho, \mu, U) \tag{4.149}$$

- Replace x and U, respectively, by δ and the friction velocity

$$u_\tau \equiv [\tau_{xy}(x,0)/\rho]^{1/2} \equiv (\tau_0/\rho)^{1/2} \tag{4.150}$$

- Alternative form of u

$$u = f(\delta, y, \rho, \mu, u_\tau) \tag{4.151}$$

$$\frac{u}{u_\tau} = f\left(u_\tau\, y/\nu, y/\delta\right) \tag{4.152}$$

[30] Movie: Turbulent diffusion
./mmfm_movies/5358.mov ./mmfm_movies/4196.mov ./mmfm_movies/4197.mov
./mmfm_movies/4198.mov ./mmfm_movies/3605.mov

- In Viscous Sublayer, $y/\delta \ll 1$: Inner Law or Law of the Wall

$$\frac{u}{u_\tau} \simeq f\left(u_\tau\, y/\nu, 0\right) \equiv f_1\left(u_\tau\, y/\nu\right) \tag{4.153}$$

  – From the boundary condition and the definition of the shear stress

$$\mu \left[\frac{\partial u}{\partial y}\right]_{y=0} = \tau_0 = \mu\, u_\tau \left[\frac{df_1\left(u_\tau\, y/\nu\right)}{dy}\right]_{y=0} = \tau_0 f_1'(0) \tag{4.154}$$

  – Linear behavior of f_1

$$f_1\left(u_\tau\, y/\nu\right) = u_\tau\, y/\nu \tag{4.155}$$

- In Outer Region: Velocity Defect Law

$$\frac{U-u}{u_\tau} = f_2(y/\delta) \tag{4.156}$$

  – Boundary condition: $f_2(1) = 0$

- In Overlap Region

$$f_1\left(u_\tau\, y/\nu\right) = U/u_\tau - f_2(y/\delta) \tag{4.157}$$

  – Matched asymptotic expansions

$$\left(u_\tau\, y/\nu\right) f_1'\left(u_\tau\, y/\nu\right) = -\left(y/\delta\right) f_2'\left(y/\delta\right) \tag{4.158}$$

  – Two variables are independent so that both sides of the equation is equal a constant.

$$f_1'\left(u_\tau\, y/\nu\right) = A\left(\nu/(u_\tau\, y)\right), \quad f_2'\left(y/\delta\right) = -A\left(\delta/y\right) \tag{4.159}$$

  – After integrating,

$$\begin{aligned} f_1 &= A \log_e\left(u_\tau\, y/\nu\right) + C_1 && (4.160)\\ f_2 &= -A \log_e\left(y/\delta\right) + C_2 && (4.161) \end{aligned}$$

– Choosing empirically the constants [31]

$$\frac{u}{u_\tau} = 2.5 \log_e (u_\tau\, y/\nu) + 5.1 \text{ for } u_\tau\, y/\nu > 30 \quad (4.163)$$

$$\frac{U-u}{u_\tau} = -2.5 \log_e (y/\delta) + 2.35 \text{ for } y/\delta < 0.15 \quad (4.164)$$

- Wall Regions and Layers of Turbulent Flows

Table 4.1 Boundary layers of turbulent flows with wall regions

Region	Location	Defining Property
INNER LAYER	$y/\delta < 0.1$	u determined by u_τ and y^+, independent of U and δ
Viscous sublayer	$y^+ < 5$	Reynolds shear stress is negligible compared with the viscous stress
Buffer layer	$5 < y^+ < 30$	Region between he viscous sublayer and the log-law region
Viscous wall region	$y^+ < 50$	Viscous contribution to the shear stress is significant
OUTER LAYER	$y^+ > 50$	Direct effects of viscosity on u are negligible
Log-law region	$y^+ > 30, y/\delta < 0.3$	Log-law holds
Overlap region	$y^+ > 50, y/\delta < 0.1$	Region of overlap between inner and outer layers (at large Reynolds numbers)

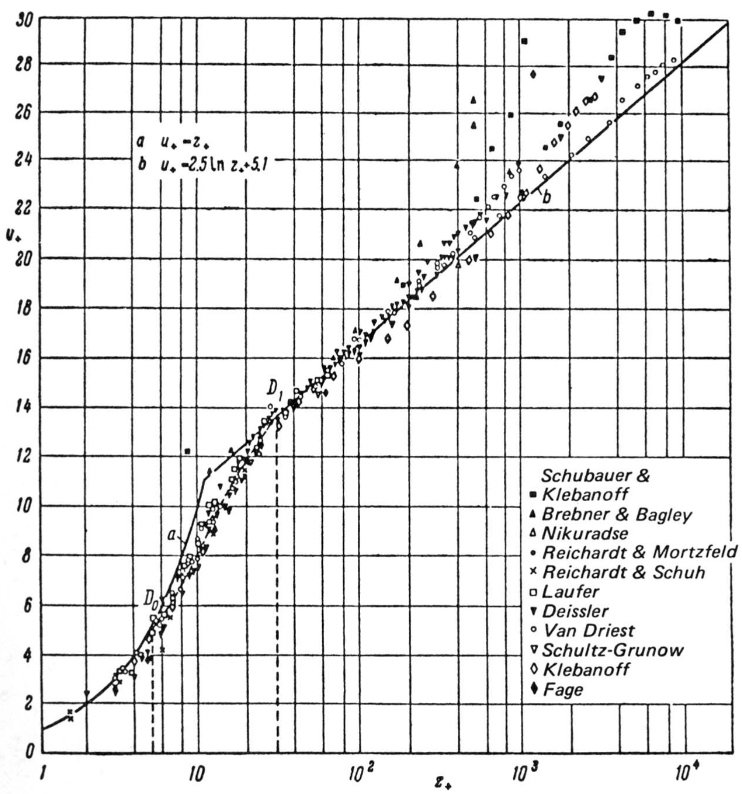
- Relation between the friction velocity and the boundary layer thickness

$$\frac{U}{u_\tau} = A \log_e (u_\tau\, \delta/\nu) + (C_1 + C_2) \quad (4.165)$$

[31] Remarks: Law of Wall/Law of Wake (Source: Smits, McKeon, and Marusic (2011), "High-Reynolds Number Wall Turbulence," *Annual Review of Fluid Mechanics*, vol. 43, pp. 353-375.)
"*In Coles's (1956) description, the velocity profile outside the viscous-dominated near-wall region is described as the sum of a logarithmic part and a wake component, so that the variation of the mean velocity u with distance from the wall y is described by*

$$\frac{u}{u_\tau} = A \log_e (u_\tau\, y/\nu) + C_1 + B\, W\left(\frac{y}{\delta}\right) \quad (4.162)$$

where W is the wake function of y/δ, and B is the wake factor."



3.10
Universal dimensionless mean velocity profile of turbulent flow close to a smooth wall according to the data of tube-, channel- and boundary-layer measurements. Here $u_+ = u/u_\tau$ and $z_+ = u_\tau y/\nu$. (From Monin and Yaglom 1971)

Figure 4.33 Mean velocity profile of turbulent flow close to a smooth wall. (From Newman 1977)

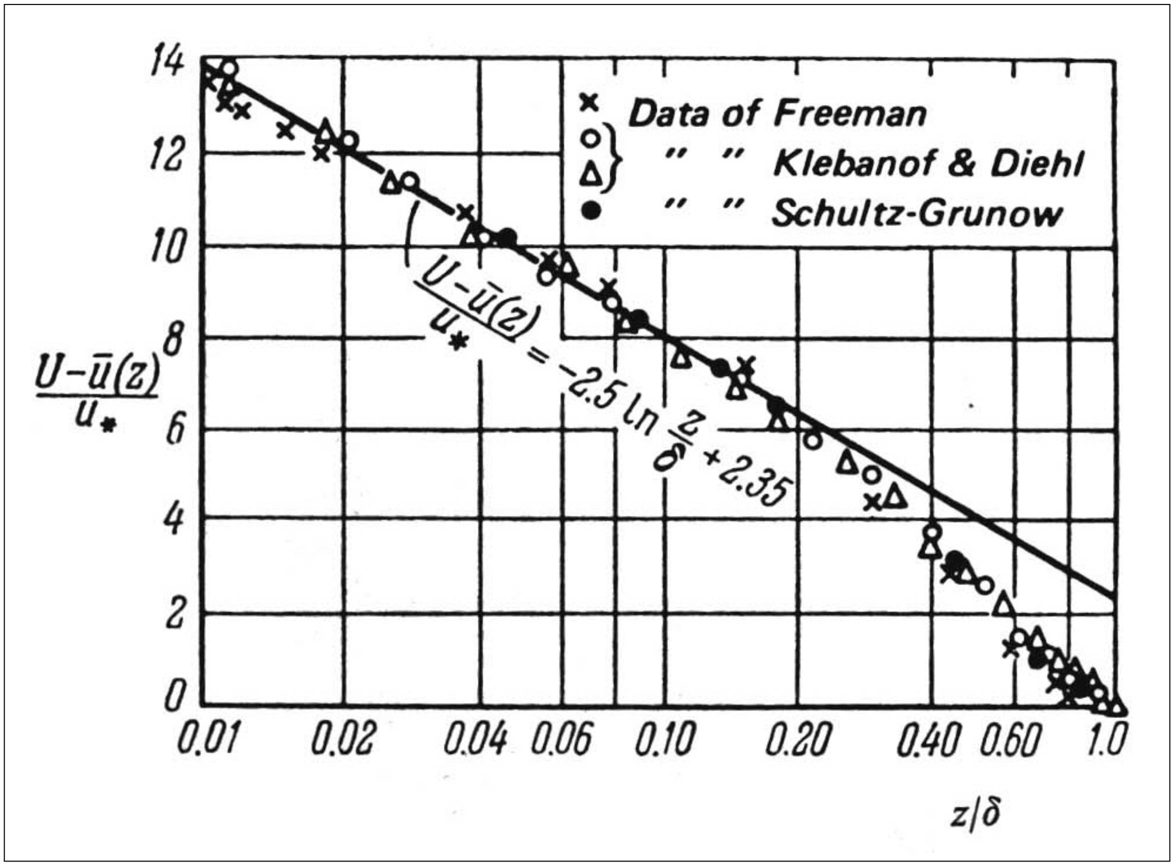


Figure 4.34 Logarithmic form of the velocity defect law for a turbulent boundary layer of a smooth wall. Here $u_* = u_\tau$ and $z = y$. (From Newman 1977)

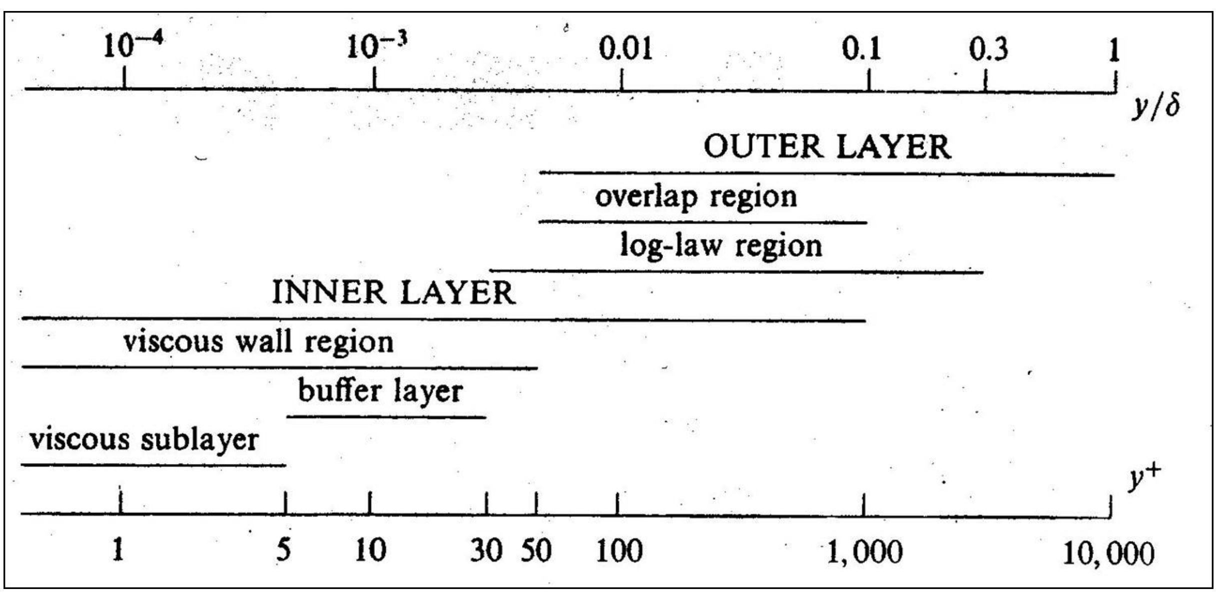


Figure 4.35 Boundary layers of turbulent flows with wall regions. (From Pope 2000)

- Relation between the friction velocity and the shear stress

$$\tau_{xy} = \rho\, U^2 \frac{d\theta}{dx} = \rho\, u_\tau^2 \tag{4.166}$$

- Implicit equation for the local frictional-drag coefficient

$$\frac{1}{\sqrt{c_f}} = 2^{-1/2} A\, \log_e\left(R_x\, c_f\right) + C_3 \tag{4.167}$$

- Total drag coefficient: $C_F = \dfrac{1}{l}\displaystyle\int_0^l c_f(x)\, dx$

$$\frac{1}{\sqrt{C_F}} = 1.79\, \log_e\left(R_l\, C_F\right) = 4.13\, \log_{10}\left(R_l\, C_F\right) \tag{4.168}$$

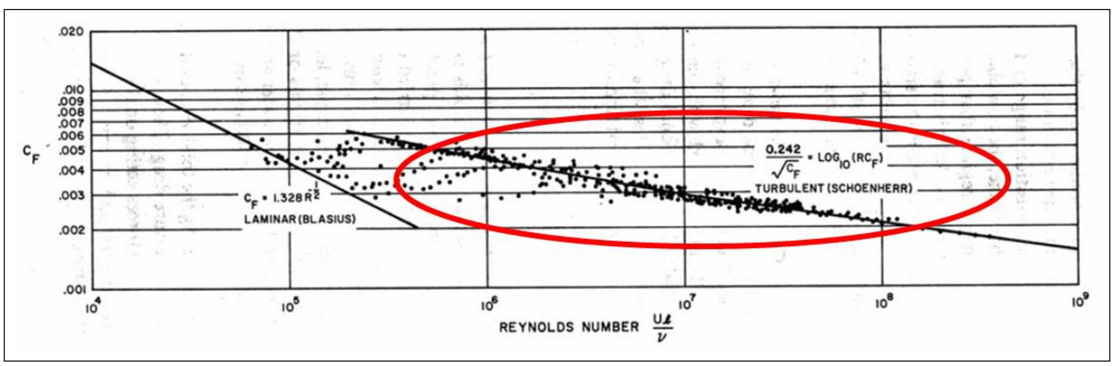


Figure 4.36 Schoenherr's flat plate frictional resistance coefficient (ATTC line). (From Newman 1977)

- The 1/7-Power Approximation
  - Simpler velocity distribution

$$u/u_\tau = 8.7\, (u_\tau\, y/\nu)^{1/7} \tag{4.169}$$

  - Shear stress on the wall, invoking the condition $u = U$ when $y = \delta$:

$$\tau_0 = \rho\, u_\tau^2 = 0.0227\, \rho\, U^2\, \left(U\, \delta/\nu\right)^{-1/4} \tag{4.170}$$

– Momentum thickness

$$\begin{aligned}\theta &= \int_0^\delta \frac{u}{U}\,(1-u/U)\;dy \\ &= \int_0^\delta (y/\delta)^{1/7}\left[1-(y/\delta)^{1/7}\right]\;dy = 0.0972\,\delta \qquad (4.171)\end{aligned}$$

– Differential equation from the von Karman's integral relation

$$0.0972\,\frac{d\delta}{dx} = 0.0227\,(U\,\delta/\nu)^{-1/4} \qquad (4.172)$$

– Integrating this equation

$$\delta^{5/4} = 0.293(\nu/U)^{1/4}\,x + C \qquad (4.173)$$

– Set $C = 0$ and find the final results

$$\begin{aligned}\delta &= 0.373\,x\,R_x^{-1/5} && (4.174)\\ \delta^* &= 0.0467\,x\,R_x^{-1/5} && (4.175)\\ \theta &= 0.0363\,x\,R_x^{-1/5} && (4.176)\\ \tau_0 &= 0.0290\,\rho\,U^2\,R_x^{-1/5} && (4.177)\\ C_F &= 2\,\theta/x = 0.0725\,R_x^{-1/5} && (4.178)\end{aligned}$$

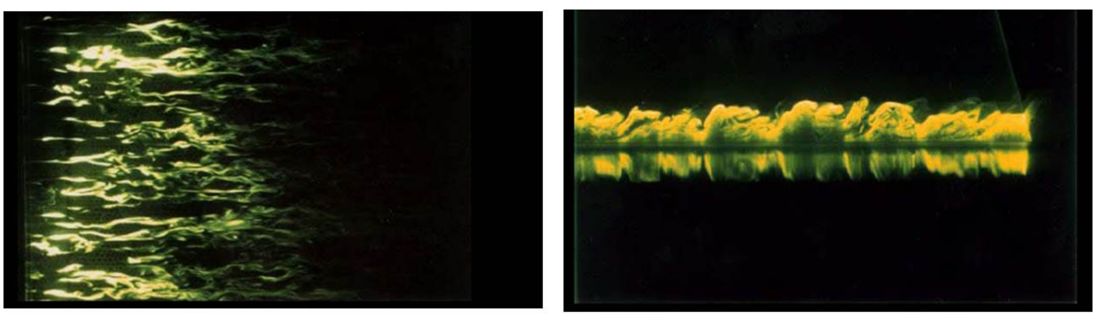
4.6.4 Turbulent Boundary Layers: Closing Remarks

- Comparison between Laminar and Turbulent Boundary Layers

Table 4.2 Comparison between laminar and turbulent boundary layers

Laminar BL(Blasius)	Turbulent BL ($1/7^{th}$ Power law)
$\delta = 4.9\,x\,Re_x^{-1/2} \propto \sqrt{x}$	$\delta = 0.373\,x\,Re_x^{-1/5} \propto x^{4/5}$
$\delta^* = 1.72\,x\,Re_x^{-1/2} \propto \sqrt{x}$	$\delta^* = 0.0467\,x\,Re_x^{-1/5} \propto x^{4/5}$
$\tau_0 = 0.332\,\rho\,U^2\,Re_x^{-1/2}$	$\tau_0 = 0.0290\,\rho\,U^2\,Re_x^{-1/5}$
$D = 0.664\,\rho\,U^2\,b\,l\,Re^{-1/2}$	$D = 0.03625\,\rho\,U^2\,b\,l\,Re^{-1/5}$
$C_F \equiv \dfrac{D}{\frac{1}{2}\rho\,U^2(bl)} = 1.328\,Re^{-1/2}$	$C_F \equiv \dfrac{D}{\frac{1}{2}\rho\,U^2(bl)} = 0.0725\,Re^{-1/5}$

- Eddy Structure of Turbulent Boundary Layer [32] [33]



Top view Side view

Figure 4.37 Eddy Structure of turbulent boundary layers near a flat plate. (From MIT website 2004; Original contributor: Prof. M. Gad-el-Hak, Univ. of Notre Dame) (Note: The pictures are views of the near-wall region of a turbulent boundary layer showing the low-speed streaks. Flow is from left to right and laser induced fluorescence is used to visualize the streaks.)

- Laminar and Turbulent Separation
  - Flow near the wall makes the turbulent boundary layer more resistant to separation.
- Flow Separation for Blunt Bodies

[32] Movie: Eddy structure of turbulent boundary layer simulated by CFD for channel flow
./mmfm_movies/turb_bl.mov

[33] Eddy structure of turbulent boundary layer on wall usually has hairpin vortices. For more details see the article: Smits, McKeon, and Marusic (2011), "High-Reynolds Number Wall Turbulence," *Annual Review of Fluid Mechanics*, vol. 43, pp. 353-375.

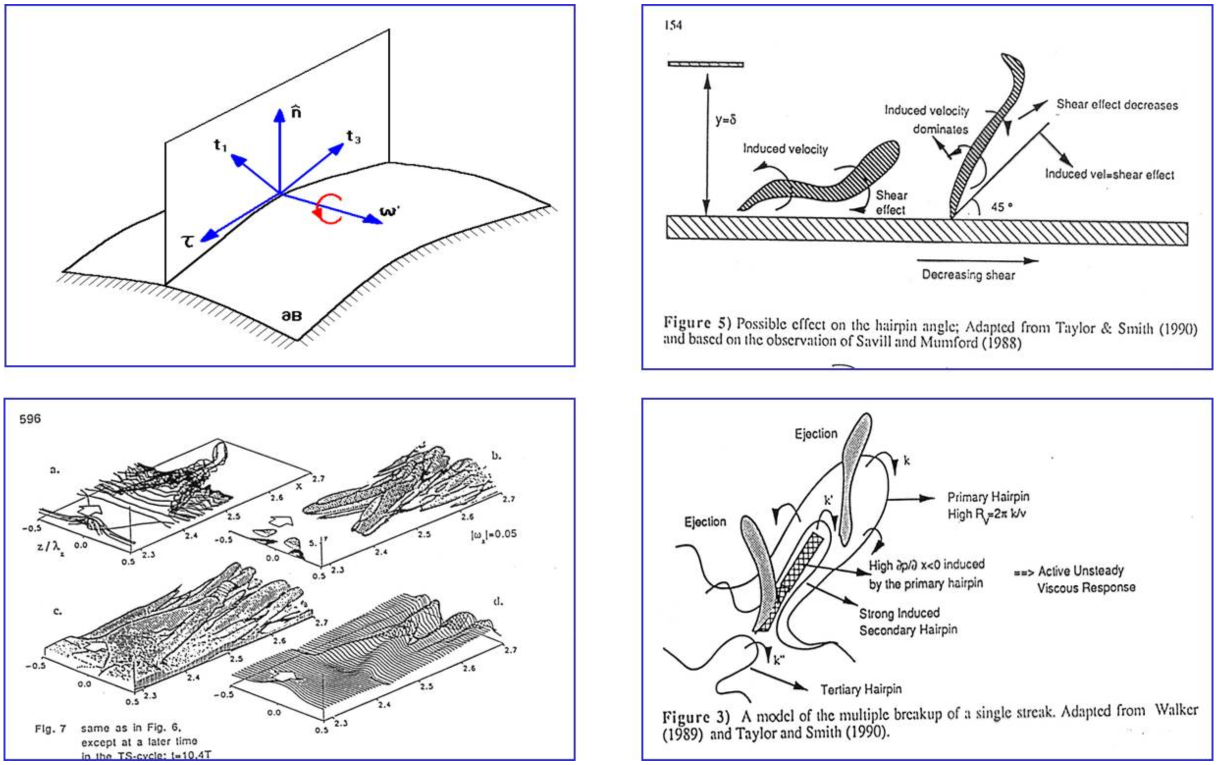


Figure 4.38 Turbulent viscous flows generated on body surface. (From Wu & Wu 1993, and other sources (as shown in caption of sub-figures))

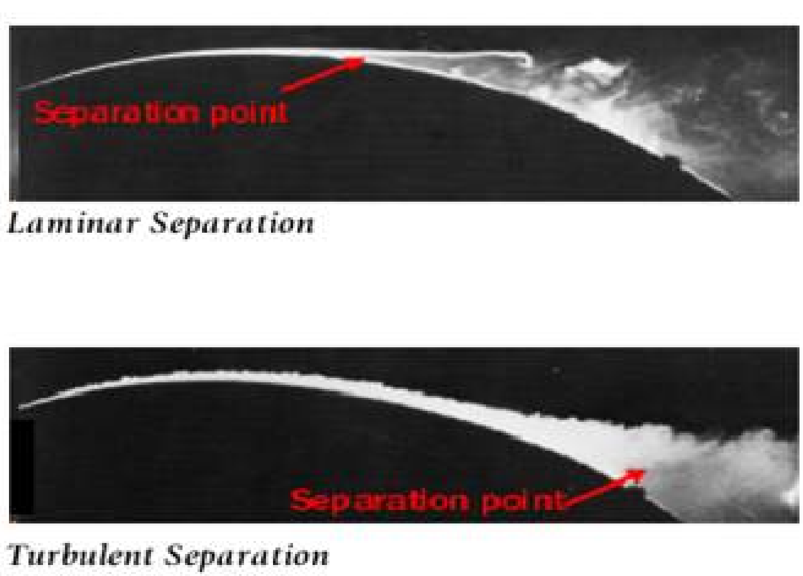


Figure 4.39 Comparison of laminar and turbulent separation on a body surface.

– Flow over sphere: Effect of Reynolds number

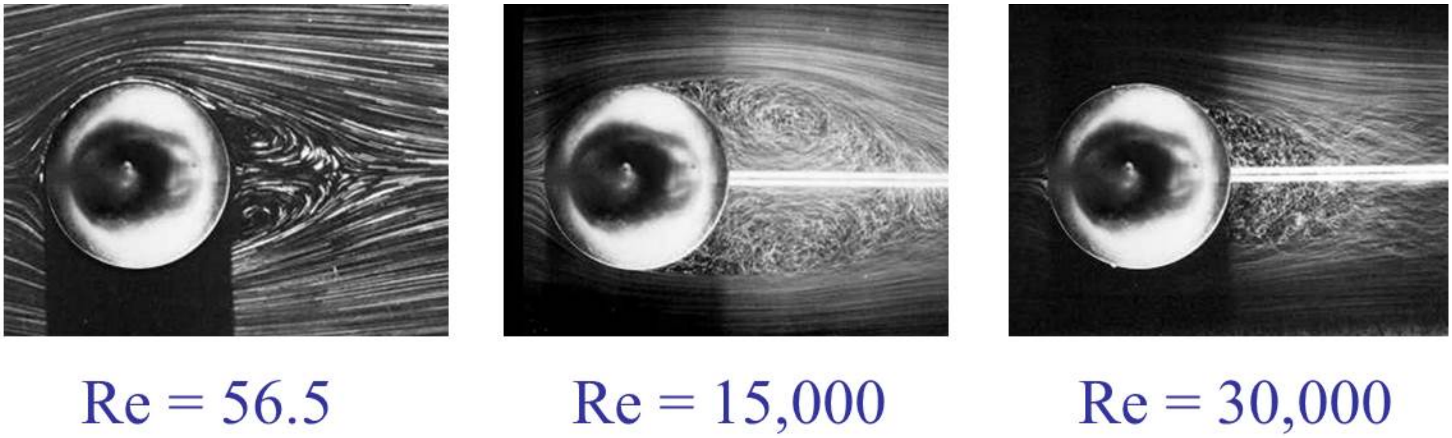


Figure 4.40 Flow over sphere: Effect of Reynolds number.

- Control of Flow Separation

  – Modification of boundary layer flows [34]

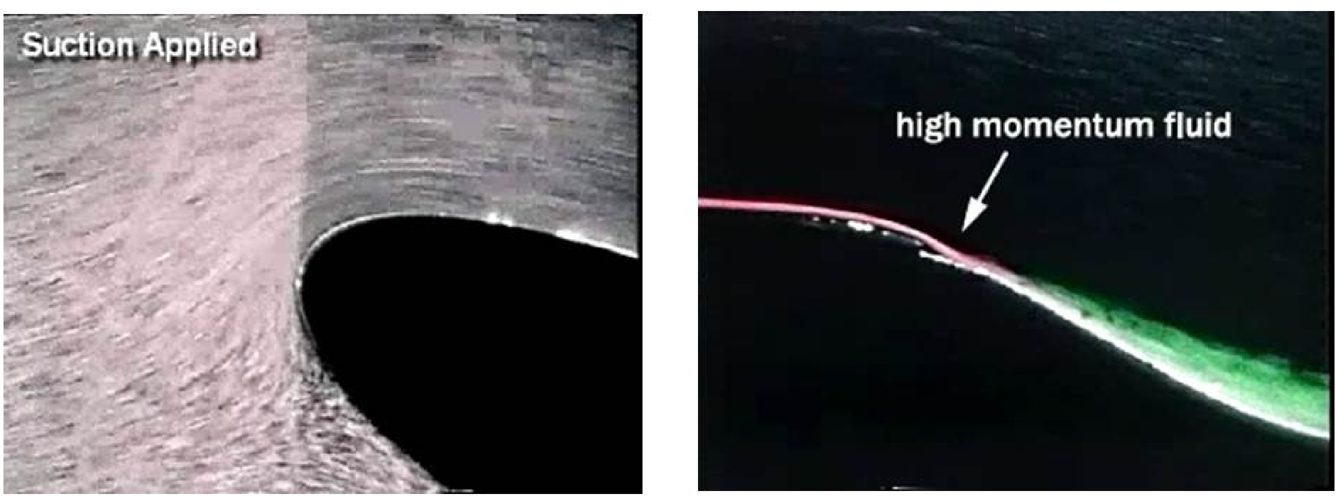


Figure 4.41 Modification of boundary layer flows.

  – Turbulence vs. Laminar: Owing to the higher momentum transport, the turbulent boundary layers are more resistant to separation than laminar boundary layers.

[34] Movie: Modification of boundary layer flows
./mmfm_movies/536.mov ./mmfm_movies/461.mov ./mmfm_movies/164.mov
./mmfm_movies/2_06045.mov

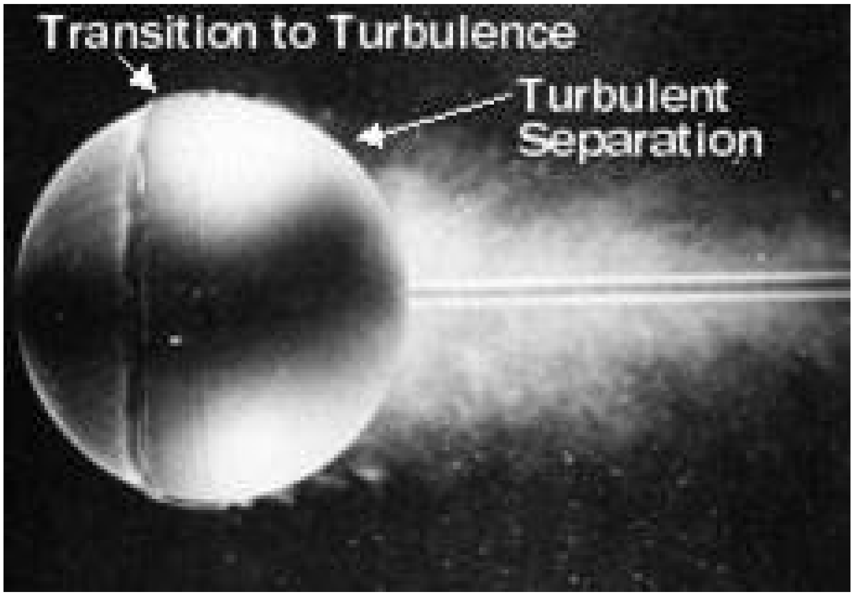


Figure 4.42 Control of flow separation on a sphere by turbulence stimulator.

- Flow Separation for Diverging Channels [35]
  - Effect of Diverging Angle
  - Separation Control: Suction, Rectifying Grids
- Simulation by CFD [36]
- Engineering Problems
  - Flow over bodies: Drag force (Truck cab roof, Sports ball, Ship hull)
  - Effect of vortex shedding on structures: Coupled solid-fluid interaction (Tacoma bridge, Vortex-induced vibration, Bilge keel)
- Turbulent Kinetic Energy Spectrum: Sub-Grid Scale(SGS) stress model and numerical discretization method
- Measurements of Turbulent Flows

[35] Movies: Flow Separation for Diverging Channels ./mmfm_movies/520.mov
Effect of Diverging Angle: ./mmfm_movies/3589.mov ./mmfm_movies/3590.mov ./mmfm_movies/3591.mov
Separation Control (Suction): ./mmfm_movies/149.mov
Separation Control (Rectifying Grids): ./mmfm_movies/3592.mov

[36] Movie: Examples of simulation by CFD(Cylinder, etc.)
./mmfm_movies/105.mov ./mmfm_movies/119.mov ./mmfm_movies/691.mov

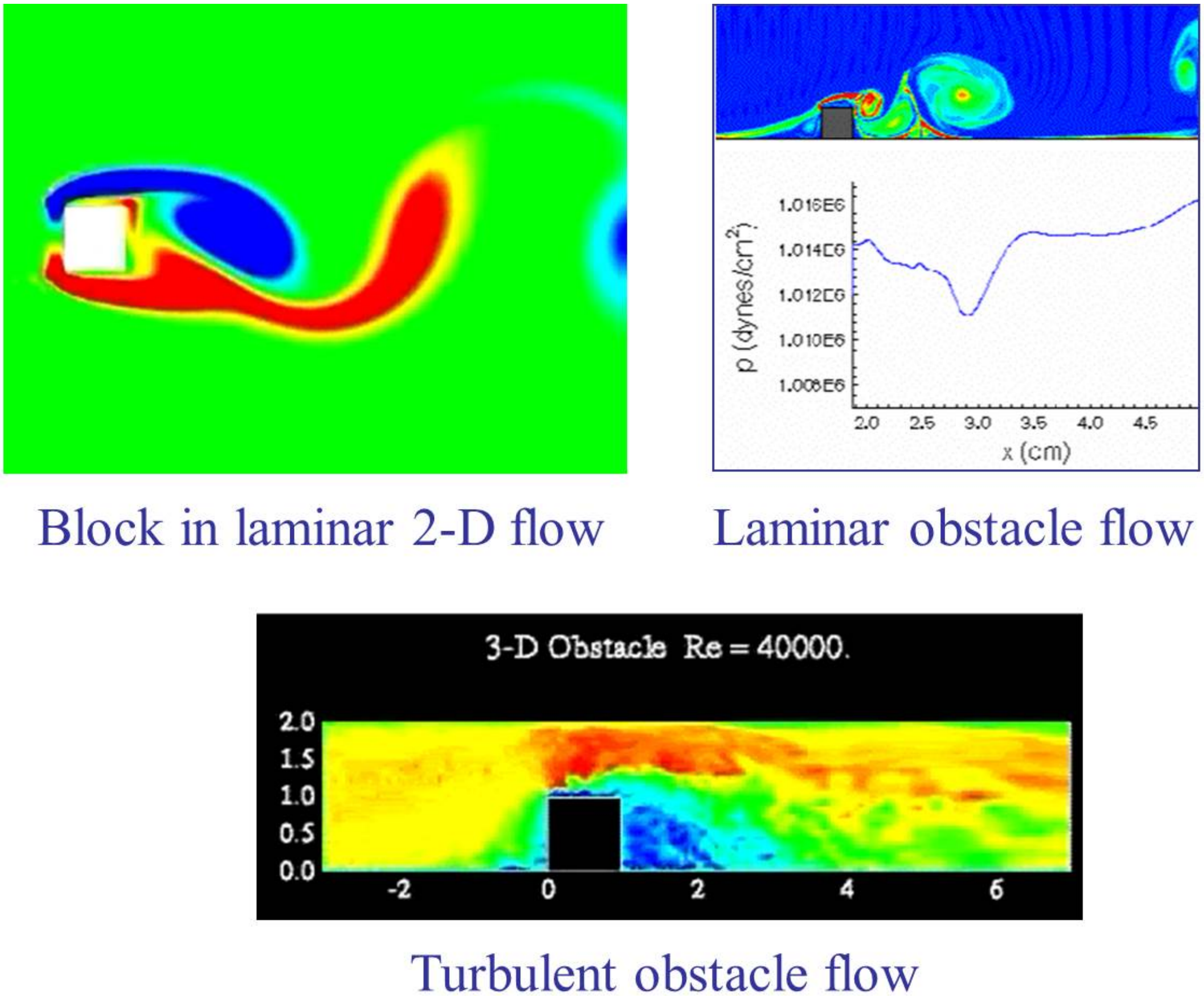


Figure 4.43 Examples of simulation by CFD.

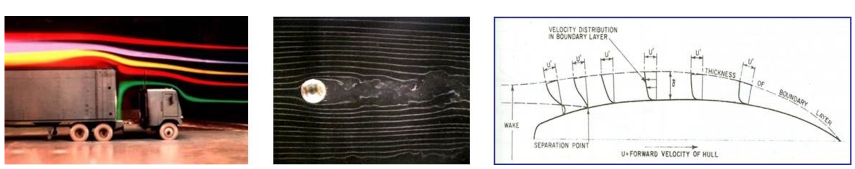


Figure 4.44 Examples of flow over bodies.

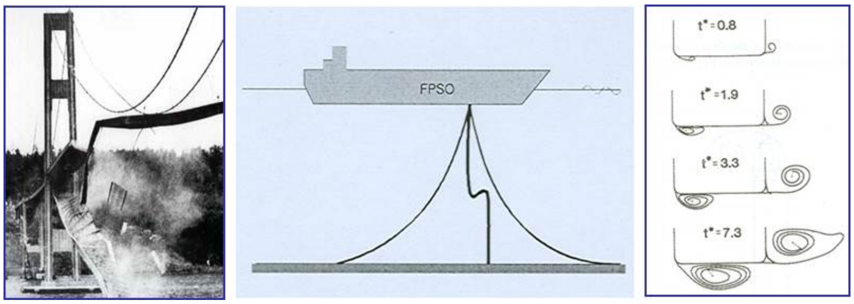


Figure 4.45 Examples of coupled solid-fluid interaction.

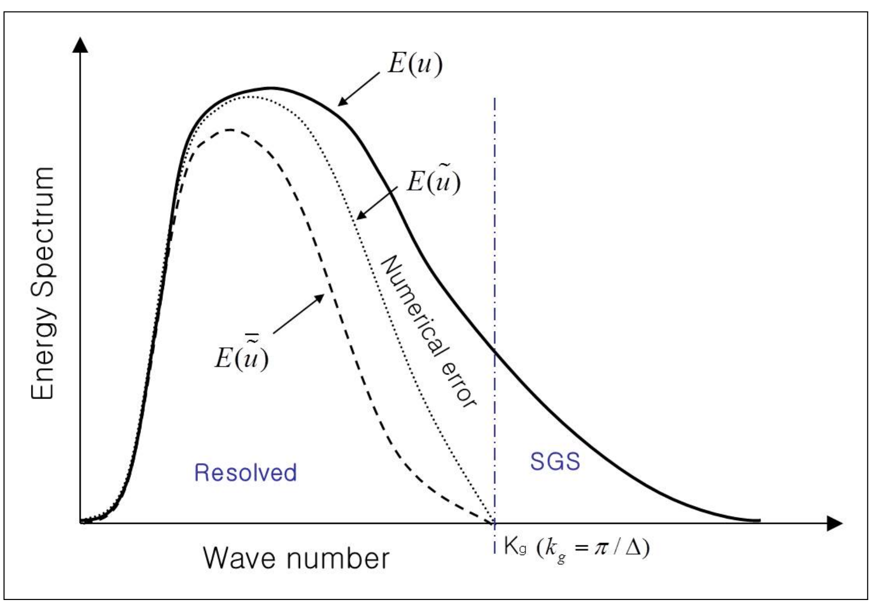


Figure 4.46 Pattern of the turbulent kinetic energy spectrum.

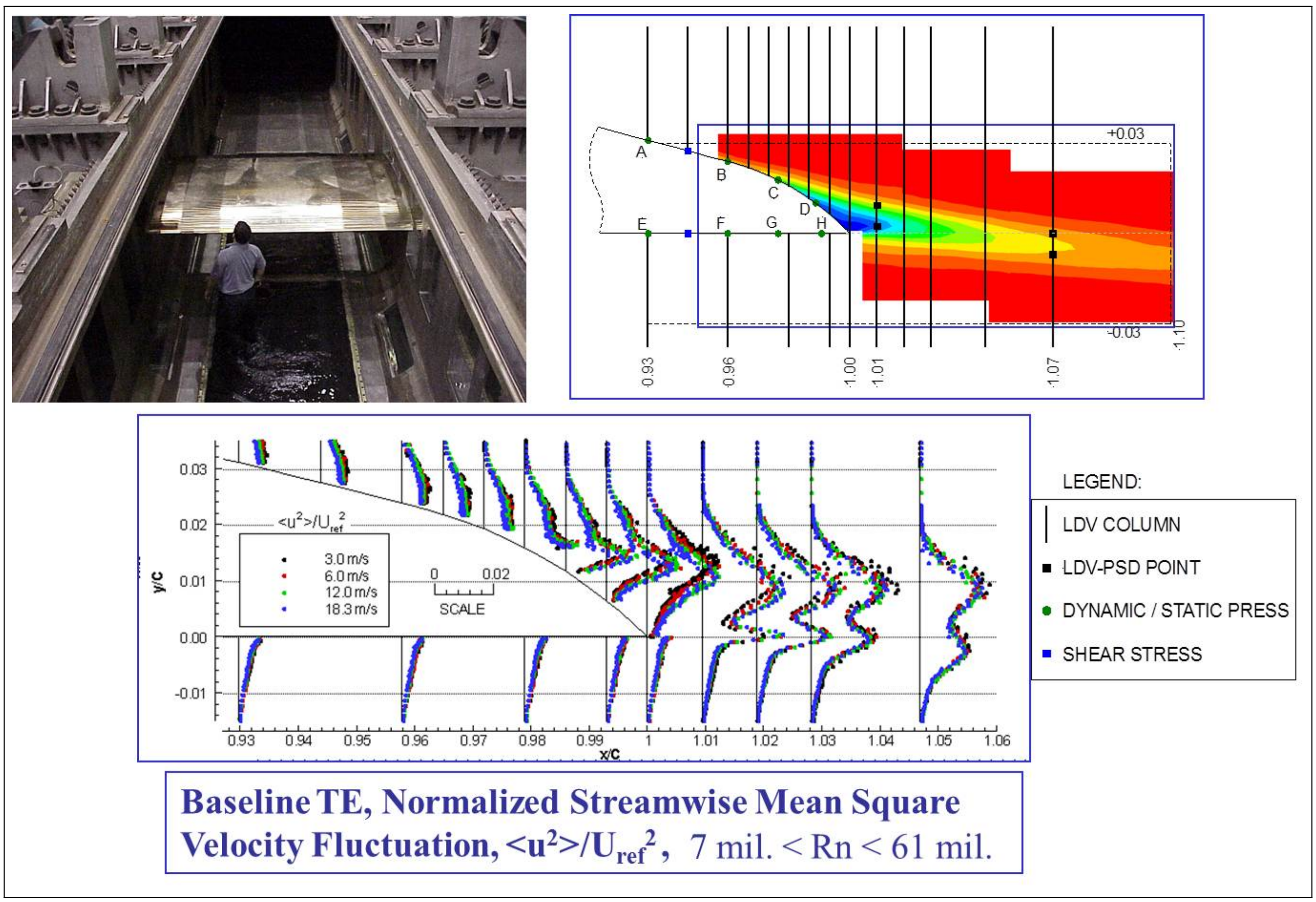


Figure 4.47 Measurements of turbulent flows. (From Kim, Ki-Han 2002)