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- 보 텍 스 방 법 -

COMPUTATIONAL MARINE HYDRODYNAMICS

-VORTEX METHODS-

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Suh, Jung-Chun

서 정 천

Seoul National Univ., Dept. NAOE

서울대학교 공과대학조선해양공학과

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VECTOR ANALYSIS

1.1 Introduction

We will be concerned with both scalar and vector functions, in the form of fields as well as parametric description of curves in space. Because the laws governing physical processes are independent of coordinate systems, vector notation is ideally suited for expressing these laws.

A scalar is often called a zero-order tensor and a vector is a first-order tensor. In addition to them, we shall use even more complicated quantities called tensors. We will use both dyads, or second-order tensors which have a 3×3 matrix form and are described by 9 scalar variables. A special third-order tensor called the alternating tensor ϵ_{ijk} will be frequently used in these notes.

1.1.1 Definition of domain

Some terms related to a domain are defined as follows, but we would not use the mathematical meaning rigorously. [1]

(1) Open ball: Set of points $\underline{x}$ inside ball of radius a centered at the origin, such that $|\underline{x}| < a$.

(2) Closed ball: Set of points such that $|\underline{x}| \leqq a$.

(3) Sphere: Set $|\underline{x}| = a$.

(4) Disk: 2-dimesional concept of the ball.

(5) Circle: 2-dimensional concept of the sphere.

(6) Open set D: For $\underline{x} \in D$, some sufficiently small ball centered at $\underline{x}$ belongs to D.

(7) Boundary B of open set D: For $\underline{x} \notin D$, if every open ball centered at $\underline{x}$ contains a point of D.

(8) Closure of D: Open set D plus boundary B.

(9) Connected: Each pair of points in D can be connected by a curve lying entirely in D.

[1] See, e.g., Stakgold, I. (1979), *Green's Functions and Boundary Value Problems*, John Wiley & Sons Inc. and Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p 44.

(10) Domain: Open connected set, such as an open ball. The union of two disjoint open balls is not a domain.

(11) Simply connected domain: Any all closed curves in domain D can be shrunk into a point in D without leaving D. The curves are called 'reducible'. For example, domain between concentric spheres is simply connected, but donut-shaped domain is not.

(12) Region: Domain plus all or part of boundary. Sometimes, the terms 'domain' and 'region' are used without distinction.

(13) Closed surface: A surface which lies within a bounded region of space and has an inside and an outside. The Klein bottle shown in Figure **1.1** has no inside or outside. Also there are some surfaces that do not have two sides. The Mobius strip is the known example of these surfaces.

(14) Smooth surface: A part of a surface is called 'smooth' if the normal to the surface varies continuously over that part. Some surfaces are made up of a number of subregions which are smooth and are called 'piecewise smooth'.

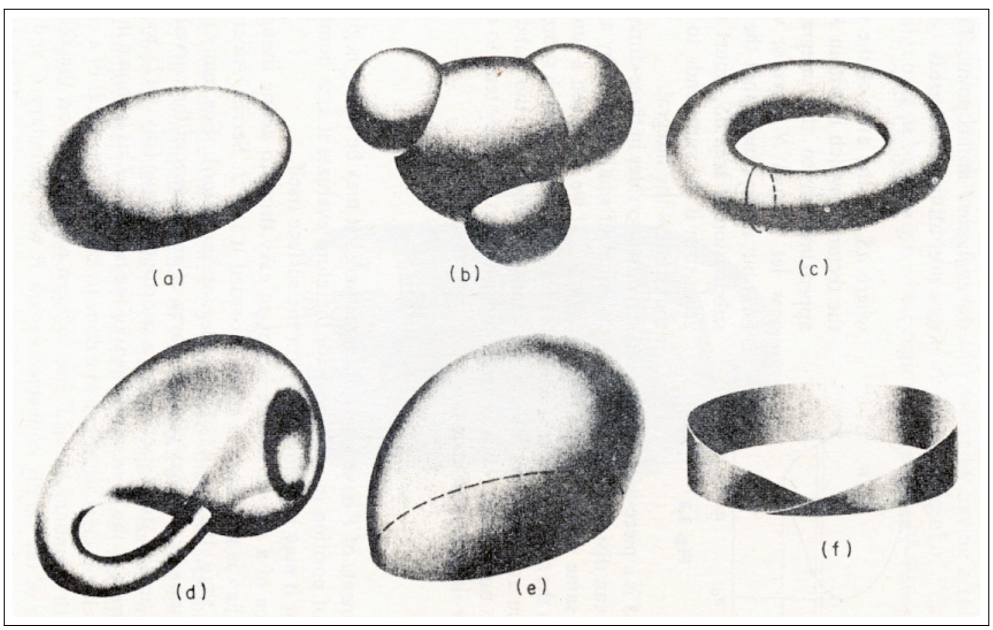

Figure 1.1 Types of surfaces: (a) a smooth closed surface; (b) a piecewise smooth surface; (c) a surface that is not simple connected; (d) a surface that is not closed: Klein bottle; (e) a hemisphere: (f) Mobius strip. From Aris (1962), p. 45.

1.1.2 Fundamental function analysis

A scalar field f is defined in a region D of two- or three-dimensional space with the property that the value of f varies from point to point in D. Some concepts and analysis for scalar functions are listed below.

(1) If $\lim_{x \to c} f(x) = f(c)$, the function $f(x)$ is said to be continuous at the point $x = c$.

(2) The base of natural logarithm is denoted by e, where $e = \lim_{n \to \pm\infty} \left(1 + \frac{1}{n}\right)^n$ $= 2.7182818285 \cdots$. One often writes $\ln(x)$ for $\log_e x$.

(3) By using the Euler formula $e^{i\theta} = \cos\theta + i\,\sin\theta$, the real sine and cosine function can be combined into a single function.

(4) A definite integral of a function $f(x)$ which exists on the interval $a \le x \le b$, can be defined by the limiting process in the sense of Riemann sum: namely,

$$\int_a^b f(x)\,dx = \lim_{N \to \infty} \sum_{i=1}^{N} f\left(a + i\,\frac{b-a}{N}\right) \frac{b-a}{N} \tag{1.1}$$

(5) For function of one variable, the rule for change of variable in a definite integral is

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{u_1}^{u_2} f(x(u))\,\frac{dx}{du}\,du \tag{1.2}$$

where we assume $f(x)$ and $f(x(u))$ are continuous in the range of integration and $x = x(u)$ is continuous and its derivative is continuous for $u_1 \le u \le u_2$.

(6) For functions of two variables, the integral becomes

$$\int_{S_{xy}} f(x,y)\,dx\,dy = \int_{S_{uv}} f(x(u,v),\, y(u,v))\,|J|\,du\,dv, \tag{1.3}$$

where Jacobian $J \equiv \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$.

(7) For $f(x,t)$ and $\dfrac{\partial f}{\partial t}$ in a region S_{xt}, $\quad a(t) \le x \le b(t), t_1 \le t \le t_2$,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f\left[b(t),t\right]\,b'(t) - f\left[a(t),t\right]\,a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx \tag{1.4}$$

This relationship is called Leibnitz's rule. The corresponding expression for the integral over a two or three dimensional region is called Reynolds transport theorem, which will be derived later.

(8) Dirac delta functions

Dirac delta function is defined as the sense of generalized functions:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{1.5}$$

Also, the derivative of the unit-step function:

$$\frac{dU(t)}{dt} = \delta(t) \tag{1.6}$$

The definite integral of Dirac delta function:

$$\int_a^b \delta(t)\,dt = \begin{cases} 1 & \text{if } a < 0 < b \\ 0 & \text{otherwise} \end{cases} \tag{1.7}$$

Dirac delta function is combined with a regular function:

$$\int_a^b g(t)\,\delta(t)\,dt = g(0)\int_a^b \delta(t)\,dt \tag{1.8}$$

(9) Fourier transforms

For $f(x)$ periodic with period $2L$, then $f(x)$ can be expressed in a Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\,\cos\left(\frac{n\pi x}{L}\right) + b_n\,\sin\left(\frac{n\pi x}{L}\right)\right] \tag{1.9}$$

where

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\,\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad b_n = \frac{1}{L}\int_0^{2L} f(x)\,\sin\left(\frac{n\pi x}{L}\right)\,dx \tag{1.10}$$

The Fourier transform of a function and its inverse transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt \tag{1.11}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,e^{i\omega t}\,d\omega \tag{1.12}$$

(10) The Laplace transform:

$$F(s) = \int_0^{\infty} f(t)\,e^{-st}\,dt \tag{1.13}$$

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)\,e^{st}\,ds \tag{1.14}$$

1.2 Vector Calculus

1.2.1 Definition of vector quantity

The simplest physical vector quantity is a *line vector*, that is, a linear displacement.

Now we investigate how a line vector is transformed from one coordinate system to another. Vector quantities are those that transform the same way independent of coordinate systems taken.

Consider two rectangular Cartesian coordinate systems rotated with respect to one another. Let a_{11}, a_{21}, a_{31} denote the direction cosines of the x'_1 axis, with respect to the x_1, x_2, x_3 axes, respectively. Let a_{12}, a_{22}, a_{32} denote those of x'_2, and a_{13}, a_{23}, a_{33} those of x'_3. [2]

[2] a_{ij} represents the cosine of the angle between the x_i and x'_j axes.

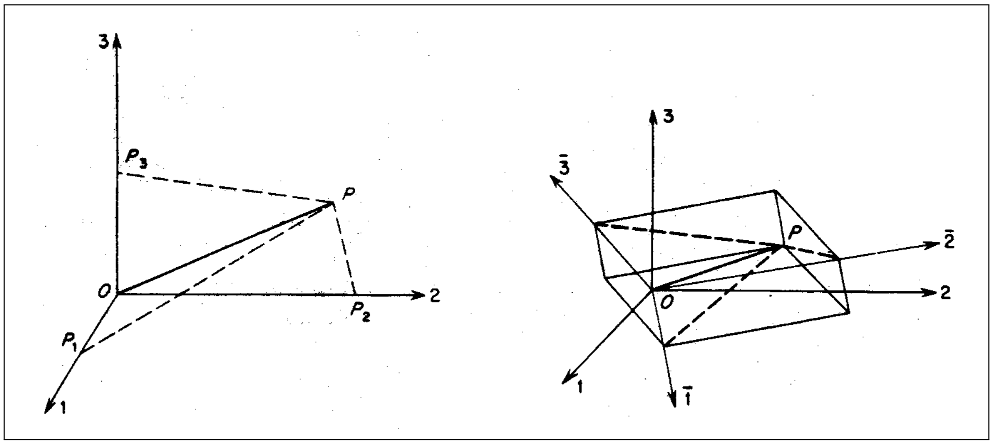

Figure 1.2 Two Cartesian coordinate systems rotated with respect to one another. From Aris (1962), p. 9.

Then, to find the coordinates of a point $P(x, y, z)$ in the primed system, note that in moving a distance x_1 along the x_1 axis you move $a_{11}\, x_1$ along x'_1, $a_{12}\, x_1$ along x'_2, and $a_{13}\, x_1$ along x'_3; etc. Hence, the new coordinates of P are

$$\begin{aligned} x'_1 &= a_{11}\, x_1 + a_{21}\, x_2 + a_{31}\, x_3 \\ x'_2 &= a_{12}\, x_1 + a_{22}\, x_2 + a_{32}\, x_3 \\ x'_3 &= a_{13}\, x_1 + a_{23}\, x_2 + a_{33}\, x_3 \end{aligned} \tag{1.15}$$

Also, if we transform from x'_1, x'_2, x'_3 to x_1, x_2, x_3 by a similar calculation, we find

$$\begin{aligned} x_1 &= a_{11}\, x'_1 + a_{12}\, x'_2 + a_{13}\, x'_3 \\ x_2 &= a_{21}\, x'_1 + a_{22}\, x'_2 + a_{23}\, x'_3 \\ x_3 &= a_{31}\, x'_1 + a_{32}\, x'_2 + a_{33}\, x'_3 \end{aligned} \tag{1.16}$$

Then we express Eqs. (1.15) and (1.16) in a summation notation:

$$x'_i = \sum_{j=1}^{3} a_{ji}\, x_j \quad \text{for} \quad i = 1, 2, 3 \tag{1.17}$$

$$x_i = \sum_{j=1}^{3} a_{ij}\, x'_j \quad \text{for} \quad i = 1, 2, 3 \tag{1.18}$$

A *vector* is defined as a set of three numbers u_1, u_2, u_3, referred to a coordinate system x_1, x_2, x_3, having the property that when transferred to the x'_1, x'_2, x'_3 system the corresponding quantities are given by

$$u'_i = \sum_{j=1}^{3} a_{ji}\, u_j \quad \text{for} \quad i = 1, 2, 3 \tag{1.19}$$

This is really the same as our earlier definition in terms of a line vector, because Eqs. (1.17) and (1.18) are the transformation formulas for a line vector.

It is clear that, to test whether a physical quantity is a vector quantity, one must have a definition that permits examination of its transformation formula. Let us consider two simple examples.

(1) First, consider velocity of a point $P(x_1, x_2, x_3)$. The components of this quantity along the three axes are $\dfrac{dx_1}{dt}$, $\dfrac{dx_2}{dt}$, and $\dfrac{dx_3}{dt}$. Calculating the velocity in the primed system, we find

$$\frac{dx'_i}{dt} = \frac{d}{dt}\sum_{j=1}^{3} a_{ji}\, x_j = \sum_{j=1}^{3} a_{ji}\, \frac{dx_j}{dt}. \tag{1.20}$$

This has exactly the form required by Eq. (1.19). Hence the velocity of a point is a vector quantity.

(2) Next, consider the set of numbers $\dfrac{\partial u}{\partial x_i}$ where u is a scalar function $u(x_1, x_2, x_3)$. We see how $\dfrac{\partial u}{\partial x'_i}$ is expressed in terms of $\dfrac{\partial u}{\partial x_i}$:

$$\frac{\partial u}{\partial x'_i} = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial x'_i} = \sum_{j=1}^{3} \frac{\partial u}{\partial x_j}\, a_{ji} \quad \text{from Eq. (1.18).} \tag{1.21}$$

Hence $\dfrac{\partial u}{\partial x_i}$ is a vector. It is actually a gradient of the scalar function.

1.2.2 Symbol of vectors

In these notes, we denote vectors by underlined symbols and scalar numbers by ordinary symbols. In particular, let us define three *unit vectors*, $\underline{i}, \underline{j}$, and $\underline{k}$, each having unit magnitude and being directed along x, y, and z axes of a rectangular cartesian coordinate system respectively. For general curvilinear coordinate systems, we denote the corresponding unit base vectors as $\underline{e}_1, \underline{e}_2$, and $\underline{e}_3$.

We can write an expression for any vector $\underline{a}$ as the sum of its components; i.e.

$$\underline{a} = a_1\,\underline{i} + a_2\,\underline{j} + a_3\,\underline{k}. \tag{1.22}$$

For instance, the position vector $\underline{x}$, denoting the displacement of any point from the origin, is [3]

$$\underline{x} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}. \tag{1.23}$$

In this case we write the distance of $\underline{x}$ from the origin as $r \equiv |\underline{x}| \equiv \sqrt{x^2 + y^2 + z^2}$. We also denote the distance between two position vectors by r.

Let the three coordinates of a rectangular Cartesian system be called x_1, x_2, and x_3. Then, for a vector whose corresponding components are a_1, a_2, and a_3, we write simply a_n instead of writing down all components and unit vectors, where the subscript is understood to take the values 1, 2, and 3. Thus a vector is recognized by the presence of a subscript; a scalar by the absence of a subscript. This notation of vector (tensor) analysis is the simplest one when we perform vector operations with the least memory work.

1.2.3 Basic unit tensors

In general, in a 3-dimensional space a tensor of order (rank) m has 3^m components,

$$\tau_{ij\cdots k}\,\underline{e}_i\,\underline{e}_j\,\cdots\,\underline{e}_k \quad \text{for} \quad i, j, \cdots k = 1, 2, 3 \tag{1.24}$$

[3] $\underline{r}$ is also used for $\underline{x}$ herein.

1.2.3.1 Kronecker delta tensor

The most useful tensor of order 2 is the unit tensor, denoting by doubly-underlined upper-cased bold face:

$$\underline{\underline{\mathbf{I}}} = \delta_{ij}\ \underline{e}_i\, \underline{e}_j \tag{1.25}$$

with Kronecker delta δ_{ij} being defined by

$$\delta_{ij} = 1 \ \text{ if } \ i = j; \quad \delta_{ij} = 0 \ \text{ if } \ i \neq j \tag{1.26}$$

The contraction (inner product) of 2 unit tensors gives

$$\underline{\underline{\mathbf{I}}} \cdot \underline{\underline{\mathbf{I}}} = \delta_{ij}\, \delta_{jk} = \delta_{ik} = \underline{\underline{\mathbf{I}}} \tag{1.27}$$

The double contraction of 2 unit tensors (denoted by a colon) gives

$$\underline{\underline{\mathbf{I}}} : \underline{\underline{\mathbf{I}}} = \delta_{ij}\, \delta_{ji} = \delta_{ii} = d \tag{1.28}$$

where d is the dimension of the space that we dealt with; e.g., $d = 3$ in 3-dimesions.

1.2.3.2 Permutation tensor

As another example, the important tensor of order 3 is the permutation (alternating) tensor:

$$\underline{\underline{\underline{\mathbf{E}}}} = \epsilon_{ijk}\ \underline{e}_i\, \underline{e}_j\, \underline{e}_k \tag{1.29}$$

where ϵ_{ijk} are the Cartesian components of permutation symbol:

$$\left.\begin{array}{lll} \epsilon_{ijk} & = 0 & \text{if any } \ i, j, k \ \text{ equal} \\ \epsilon_{ijk} & = 1 & \text{if } \ (ijk) = (123), (231), (312) \\ \epsilon_{ijk} & = -1 & \text{if } \ (ijk) = (132), (213), (321). \end{array}\right\} \tag{1.30}$$

1.2.3.3 Multiplication of basic tensors

We can easily see that the following formulas for δ_{ij} and ϵ_{ijk} holds from their definitions:

$$\delta_{ii} = 3, \tag{1.31}$$

$$\delta_{ij}\, u_{klmi} = u_{klmj}, \tag{1.32}$$

$$\delta_{ij}\, \epsilon_{ijk} = 0, \tag{1.33}$$

The permutation tensor is used for cross (vector) product of vectors. If we need more than one cross products, the multiplication of two permutation tensors is involved. Let us start with the rule of vector product: [4]

$$\epsilon_{ijk} = \underline{e}_i \cdot (\underline{e}_j \times \underline{e}_k) = \begin{vmatrix} \underline{e}_i \cdot \underline{e}_1 & \underline{e}_i \cdot \underline{e}_2 & \underline{e}_i \cdot \underline{e}_3 \\ \underline{e}_j \cdot \underline{e}_1 & \underline{e}_j \cdot \underline{e}_2 & \underline{e}_j \cdot \underline{e}_3 \\ \underline{e}_k \cdot \underline{e}_1 & \underline{e}_k \cdot \underline{e}_2 & \underline{e}_k \cdot \underline{e}_3 \end{vmatrix} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \tag{1.34}$$

From Eq. (1.34), the product of two permutation tensors is written as

$$\epsilon_{ijk}\, \epsilon_{mnl} = \left| \begin{bmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{bmatrix} \begin{bmatrix} \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \\ \delta_{l1} & \delta_{l2} & \delta_{l3} \end{bmatrix} \right| = \begin{vmatrix} \delta_{im} & \delta_{in} & \delta_{il} \\ \delta_{jm} & \delta_{jn} & \delta_{jl} \\ \delta_{km} & \delta_{kn} & \delta_{kl} \end{vmatrix} \tag{1.35}$$

Contraction with respect to k, l (i.e., $k = l$) yields

$$\epsilon_{ijk}\, \epsilon_{mnk} = \delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm} \tag{1.36}$$

Making the contraction with respect to j, n and continuing again give

$$\epsilon_{ijk}\, \epsilon_{mjk} = \delta_{im}\, \delta_{jj} - \delta_{ij}\, \delta_{jm} = 3\, \delta_{im} - \delta_{im} = 2\, \delta_{im} \tag{1.37}$$

$$\epsilon_{ijk}\, \epsilon_{ijk} = 2\, \delta_{ii} = 6 \tag{1.38}$$

[4] We will follow the procedure in the text, Wu, J.-Z, Ma, H.-Y. and Zhou, M.-D. (2006), *Vorticity and Vortex Dynamics*, Springer, pp. 697–698.

The corresponding formulas in a 2-dimensional space are given by

$$\epsilon_{ij3}\,\epsilon_{mn3} = \begin{vmatrix} \delta_{im} & \delta_{in} \\ \delta_{jm} & \delta_{jn} \end{vmatrix} = \delta_{im}\,\delta_{jn} - \delta_{in}\,\delta_{jm} \tag{1.39}$$

$$\epsilon_{ij3}\,\epsilon_{mj3} = \delta_{im}\,\delta_{jj} - \delta_{ij}\,\delta_{jm} = 2\,\delta_{im} - \delta_{im} = \delta_{im} \tag{1.40}$$

$$\epsilon_{ij3}\,\epsilon_{ij3} = 2 \tag{1.41}$$

1.2.3.4 Example of permutation tensor

A special example of the permutation tensor can be observed in definition of vorticity: [5]

$$\underline{\omega} = \omega_i = \nabla \times \underline{q} = \epsilon_{ijk}\,\frac{\partial q_k}{\partial x_j} = \epsilon_{ijk}\,\frac{1}{2}\left(\frac{\partial q_k}{\partial x_j} - \frac{\partial q_j}{\partial x_k}\right) = \frac{1}{2}\,\epsilon_{ijk}\,\Omega_{jk} \tag{1.42}$$

where $\Omega_{jk} \equiv \left(\frac{\partial q_k}{\partial x_j} - \frac{\partial q_j}{\partial x_k}\right)$ is a spin(rotational) tensor. Also it is easily seen that, by multiplying the above equation by ϵ_{lmi} and using Eq. (1.34),

$$\epsilon_{lmi}\,\omega_i = \epsilon_{lmi}\,\frac{1}{2}\epsilon_{ijk}\,\Omega_{jk} = \frac{1}{2}(\delta_{lj}\,\delta_{mk} - \delta_{lk}\,\delta_{jm})\,\Omega_{jk} = \frac{1}{2}(\Omega_{lm} - \Omega_{ml}) = \Omega_{lm} \tag{1.43}$$

from which we have

$$\Omega_{ij} = \epsilon_{ijk}\,\omega_k. \tag{1.44}$$

The inner product of a vector $\underline{a}$ and an antisymmetric tensor $\underline{\underline{\Omega}}$ becomes

$$\underline{a} \cdot \underline{\underline{\Omega}} = a_i\,\epsilon_{ijk}\,\omega_k = \underline{\omega} \times \underline{a}, \quad \underline{\underline{\Omega}} \cdot \underline{a} = \epsilon_{ijk}\,\omega_k\,a_j = \underline{a} \times \underline{\omega}. \tag{1.45}$$

If the relative velocity $\underline{v}$ of any two points is $\underline{\underline{\Omega}} \cdot \underline{x}$ where $\underline{x}$ is the relative position vector of the two points, then the motion is due to a rigid body rotation. Here $\underline{\underline{\Omega}}$ relates to the angular velocity.

[5] See Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 25 and Wu, J.-Z, Ma, H.-Y. and Zhou, M.-D. (2006), *Vorticity and Vortex Dynamics*, Springer, p. 698.

Similarly, we also have

$$\nabla \cdot \underline{\underline{\Omega}} = \frac{\partial}{\partial x_i}\,(\epsilon_{ijk}\,\omega_k) = -\nabla \times \underline{\omega}. \tag{1.46}$$

Such relations between vorticity $\underline{\omega}$ and the spin tensor $\underline{\underline{\Omega}}$ are useful to deduce the physical interpretation in vortex dynamics that will be described in more detail in Chapter 6.

1.2.4 Multiplication of vectors

1.2.4.1 Scalar product

The *scalar product* of two vectors $\underline{a}$ and $\underline{b}$ is defined as the scalar number given by the product of their scalar magnitudes and the cosine of the angle between them: $\underline{a} \cdot \underline{b} = a\,b\,\cos(\underline{a} \cdot \underline{b})$ or $\underline{a} \cdot \underline{b} = a_1\,b_1 + a_2\,b_2 + a_3\,b_3$. According to the notation of Kronecker delta tensors, it becomes $\underline{a} \cdot \underline{b} = \delta_{ij}\,a_i\,b_j = a_i\,b_i$, where summation convention has been used.

1.2.4.2 Vector product

The *vector product* of two vectors $\underline{a}$ and $\underline{b}$ is defined as a vector whose direction is perpendicular to both $\underline{a}$ and $\underline{b}$ and whose magnitude is the product of their magnitudes and the sine of the angle between them; i. e., $\underline{c} = \underline{a} \times \underline{b}$; $\quad w = a\,b\,\sin(\underline{a},\,\underline{b})$.

To determine the expression for $\underline{a} \times \underline{b}$ in terms of Cartesian components, we may write by cyclic substitution of subscripts as an aid to memory; in a form of tensor-notation, $\underline{a} \times \underline{b} = \epsilon_{ijk}\,a_j\,b_k$.

1.2.4.3 Scalar triple product

The *scalar triple product*, $\underline{a}\cdot(\underline{b}\times\underline{c})$, is a scalar number having a value equal to the volume of the parallelepiped erected on $\underline{a}$, $\underline{b}$, and $\underline{c}$. This may be expressed as, using the alternating tensor, $\underline{a} \cdot (\underline{b} \times \underline{c}) = a_i\, \epsilon_{ijk}\, b_j\, c_k$. Obviously, the parentheses used here are unnecessary and we see $\underline{a} \cdot \underline{b} \times \underline{c} = \underline{a} \times \underline{b} \cdot \underline{c} = \underline{b} \cdot \underline{c} \times \underline{a}$ etc.

1.2.4.4 Vector triple product

The *vector triple product*, $\underline{a} \times (\underline{b} \times \underline{c})$, is expressed in terms of components in the plane of $\underline{b}$ and $\underline{c}$:

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c} \tag{1.47}$$

This can be easily verified by using the basic formula for the alternating tensor listed above:

$$\underline{a} \times (\underline{b} \times \underline{c}) = \epsilon_{mli}\, a_l\, \epsilon_{ijk}\, b_j\, c_k = (\delta_{mj}\, \delta_{lk} - \delta_{mk}\, \delta_{lj})\, a_l\, b_j\, c_k = a_k\, b_j\, c_k - a_j\, b_j\, c_k. \tag{1.48}$$

Combining the above results one finds

$$\underline{a} \times (\underline{b} \times \underline{c}) + \underline{b} \times (\underline{c} \times \underline{a}) + \underline{c} \times (\underline{a} \times \underline{b}) = 0 \tag{1.49}$$

1.2.5 Vector derivatives

1.2.5.1 Gradient: ∇u

Consider a scalar function $u = u(x, y, z)$ that is differentiable and has continuous derivatives. Let us define the gradient of u at x, y, z as the limiting value of a certain surface integral over a surface surrounding the point x, y, z, as follows

$$\nabla u \equiv \lim_{\triangle V \to 0} \frac{1}{\triangle V} \oint_S u\, \underline{n}\, dS \tag{1.50}$$

where S is the area enclosing the volume $\triangle V$, dS is the element of area, and $\underline{n}$ is the unit vector normal to the surface at each point of the surface integration. [6]

[6] $\int_S \cdots dS$ and $\oint_S \cdots dS$ are the symbolism to indicate that the integration is over, respectively, an open surface and a closed surface.

Now we can take $\triangle V$ very small, in the form of a cube, say, with sides $\triangle x, \triangle y, \triangle z$. Then, neglecting second-order quantities, $\triangle V = \triangle x \, \triangle y \, \triangle z$, and

$$\begin{aligned}
\int_S u \, \underline{n} \, dS \;&\approx\; -u \, \underline{i} \, \triangle y \, \triangle z - u \, \underline{j} \, \triangle x \, \triangle z - u \, \underline{k} \, \triangle x \, \triangle y \\
&\quad + \left(u + \frac{\partial u}{\partial x} \triangle x\right) \, \underline{i} \, \triangle y \, \triangle z + \left(u + \frac{\partial u}{\partial y} \triangle y\right) \, \underline{j} \, \triangle x \, \triangle z \\
&\quad + \left(u + \frac{\partial u}{\partial z} \triangle z\right) \, \underline{k} \, \triangle x \, \triangle y \qquad (1.51) \\
&\approx\; \left\{ \frac{\partial u}{\partial x} \underline{i} + \frac{\partial u}{\partial u} \underline{j} + \frac{\partial u}{\partial z} \underline{k} \right\} V \qquad (1.52)
\end{aligned}$$

Hence, in limit,

$$\nabla u = \underline{i} \, \frac{\partial u}{\partial x} + \underline{j} \, \frac{\partial u}{\partial y} + \underline{k} \, \frac{\partial u}{\partial z} \qquad (1.53)$$

We recognize this as the vector. Another symbol often used for ∇u is grad u. In a form of tensor notation, it is $\dfrac{\partial u}{\partial x_i}$.

1.2.5.2 Divergence: $\nabla \cdot \underline{v}$

Consider now a *vector function*, $\underline{v} = \underline{v}(x, y, z) \equiv v_1 \; \underline{i} + v_2 \; \underline{j} + v_3 \; \underline{k}$, where v_1, v_2, and v_3 are all scalar functions of x, y, z, having continuous derivatives. We define

$$\nabla \cdot \underline{v} = \lim_{\triangle V \to 0} \frac{1}{\triangle V} \oint_S \underline{n} \cdot \underline{v} \, dS \qquad (1.54)$$

Now, by calculating for a small cubical volume, you can easily confirm the following equality:

$$\nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \qquad (1.55)$$

Another symbol used for $\nabla \cdot \underline{v}$ is div $\underline{v}$. In a form of tensor notation, it is $\dfrac{\partial v_i}{\partial x_i}$.

1.2.5.3 Curl: $\nabla \times \underline{v}$

We define the curl of a vector

$$\nabla \times \underline{v} \equiv \lim_{\triangle V \to 0} \frac{1}{\triangle V} \oint_S \underline{n} \times \underline{v} \, dS \tag{1.56}$$

and find, by considering a small cube, that

$$\nabla \times \underline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \underline{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \underline{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \underline{k} \tag{1.57}$$

or, to assist the memory, purely symbolically we write

$$\nabla \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{1.58}$$

Another symbol used for curl $\underline{v}$ is $\nabla \times \underline{v}$. In a form of tensor notation, it is $\epsilon_{ijk} \dfrac{\partial v_k}{\partial x_j}$.

1.2.5.4 Laplacian: $\nabla^2 u$

The Laplacian of a scalar function $u(x, y, z)$ is defined as

$$\nabla^2 u \equiv \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{1.59}$$

By analogy, the Laplacian of a vector function is the vector whose rectangular Cartesian components are the Laplacian of the vector's corresponding components [7]

$$\nabla^2 \underline{v} = \underline{i} \, \nabla^2 v_1 + \underline{j} \, \nabla^2 v_2 + \underline{k} \, \nabla^2 v_3 \tag{1.60}$$

[7] We must do more work to find its expression in a non-Cartesian system.

1.2.5.5 Differential operator: ∇

From the original definition of grad u, we can deduce that the differential du is given by the formula, in rectangular Cartesian coordinates,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz = d\underline{\ell} \cdot \nabla u \tag{1.61}$$

where $d\underline{\ell}$ is any directed line (vector) element. This means that du is the increment of u corresponding to a position increment $d\underline{\ell}$.

Similarly, for a vector function $\underline{v}(x, y, z)$,

$$\begin{aligned} d\underline{v} &\equiv \underline{i}\,dv_1 + \underline{j}\,dv_2 + \underline{k}\,dv_3 \\ &= \left(dx\,\frac{\partial}{\partial x} + dy\,\frac{\partial}{\partial y} + dz\,\frac{\partial}{\partial z}\right)\left(\underline{i}\,v_1 + \underline{j}\,v_2 + \underline{k}\,v_3\right) \\ &= d\underline{\ell} \cdot \nabla \underline{v} \end{aligned} \tag{1.62}$$

In all of the formulas above, we consider the symbol ∇ as representing a *vector operator* $\underline{i}\,\frac{\partial}{\partial x} + \underline{j}\,\frac{\partial}{\partial y} + \underline{k}\,\frac{\partial}{\partial z}$. If you treat this operator as a vector, with the appropriate vector-multiplication signs, you get the right result. Equations (1.61) and (1.62) are independent of the choice of coordinate system. As will be seen later on, the expressions for *div, grad, curl,* etc. in a more general curvilinear system do not bear much resemblance to one another.

1.2.5.6 Directed derivative

Equations (1.61) and (1.62) lead immediately to the formulas for the *directed derivative* in the direction of a given vector $\underline{s} \equiv s_1\,\underline{i} + s_2\,\underline{j} + s_3\,\underline{k}$ in rectangular Cartesian coordinate:

$$\frac{\partial u}{\partial s} = \underline{e}_s \cdot \nabla u \tag{1.63}$$

$$\frac{\partial \underline{v}}{\partial s} = \underline{e}_s \cdot \nabla \underline{v} \tag{1.64}$$

Again we have defined a new vector operator: $\underline{e}_s \cdot \nabla = \frac{s_1}{s}\frac{\partial}{\partial x} + \frac{s_2}{s}\frac{\partial}{\partial y} + \frac{s_3}{s}\frac{\partial}{\partial z}$, where s is the magnitude of $\underline{s}$.

1.2.6 Expansion formulas

The following formulas are of general utility. Let ϕ denote any differentiable scalar function of x, y, z, and $\underline{u}, \underline{v}$ and $\underline{w}$ any such vector functions.

$$\nabla \cdot (\phi\, \underline{u}) = \underline{u} \cdot \nabla\phi + \phi\, \nabla \cdot \underline{u} \tag{1.65}$$

$$\nabla \times (\phi\, \underline{u}) = (\nabla\phi) \times \underline{u} + \phi\, \nabla \times \underline{u} \tag{1.66}$$

$$\nabla \cdot (\underline{v} \times \underline{w}) = \underline{w} \cdot \nabla \times \underline{v} - \underline{v} \cdot \nabla \times \underline{w} \tag{1.67}$$

$$\nabla \times (\underline{v} \times \underline{w}) = \underline{w} \cdot \nabla\underline{v} + \underline{v}\nabla \cdot \underline{w} - \underline{w}\nabla \cdot \underline{v} - \underline{v} \cdot \nabla\underline{w} \tag{1.68}$$

$$\nabla(\underline{v} \cdot \underline{w}) = \underline{v} \cdot \nabla\underline{w} + \underline{w} \cdot \nabla\underline{v} + \underline{v} \times (\nabla \times \underline{w}) + \underline{w} \times (\nabla \times \underline{v}) \tag{1.69}$$

$$\nabla \cdot (\nabla \times \underline{v}) = 0 \tag{1.70}$$

$$\nabla \times (\nabla\, \phi) = 0 \tag{1.71}$$

$$\nabla \times (\nabla \times \underline{v}) = \nabla(\nabla \cdot \underline{v}) - \nabla^2\underline{v} \tag{1.72}$$

Operation on the position vector $\underline{x} = x_1\, \underline{i} + x_2\, \underline{j} + x_3\, \underline{k}$ whose magnitude is denoted by $r = |\underline{x}| = \sqrt{\underline{x} \cdot \underline{x}}$, with a constant vector $\underline{a}$, is illustrated as follows:

$$\nabla r = \underline{x}/r \tag{1.73}$$

$$\nabla \cdot \underline{x} = 3 \tag{1.74}$$

$$\nabla \times \underline{x} = 0 \tag{1.75}$$

$$\nabla r^n = n\, r^{n-2}\, \underline{x} \tag{1.76}$$

$$\nabla \cdot (r^n\, \underline{x}) = (n+3)\, r^n \tag{1.77}$$

$$\nabla \times (r^n\, \underline{x}) = 0 \tag{1.78}$$

$$\nabla^2(r^n) = n(n+1)\, r^{n-2} \tag{1.79}$$

$$\nabla \cdot (\underline{a} \times \underline{x}) = 0 \tag{1.80}$$

$$\nabla(\underline{a} \cdot \underline{x}) = \underline{a} \tag{1.81}$$

$$\nabla \times (\underline{a} \times \underline{x}) = 2\,\underline{a} \tag{1.82}$$

$$\nabla \cdot (\underline{a} \times \nabla r) = 0 \tag{1.83}$$

$$\nabla \cdot (r\,\underline{a}) = (\underline{x} \cdot \underline{a})/r \tag{1.84}$$

$$\nabla \times (r\,\underline{a}) = (\underline{x} \times \underline{a})/r \tag{1.85}$$

1.3 Integral Theorems

1.3.1 Divergence theorem

Let u and $\underline{v}$ denote arbitrary scalar and vector functions of x, y, z as before. These are assumed to be defined, continuous, and single-valued in a certain region of space, and, moreover, that their first derivatives with respect to x, y, and z satisfy the same requirements.

Now consider the surface integral $\oint_S u\,\underline{n}\,dS$, carried over any closed surface S within the region, enclosing a volume V, where $\underline{n}$ is the unit normal vector directed outward. It is clear that, if the volume V is subdivided into small volume V_i, this integral equals the sum of all the integrals $\oint_{S_i} u\,\underline{n}\,dS$ taken over the small surfaces S_i. Since integration over neighboring elements will cancel one another, and only the integration over the outside will remain:

$$\oint_S u\,\underline{n}\,dS = \sum \oint_{S_i} u\,\underline{n}\,dS \tag{1.86}$$

But, in the limit, the surface integral over the small surface become $\nabla u\,dV$, according to our definition of the gradient, Eq. (1.50), and the summation becomes a volume integration:

$$\oint_S u\,\underline{n}\,dS = \int_V \nabla u\,dV \tag{1.87}$$

In particular, if $u = const.$, Eq. (1.87) becomes

$$\oint_S \underline{n}\, dS = 0. \tag{1.88}$$

It means that the integral of vectorial surface element over a closed surface must vanish.

If u is taken as a negative of static pressure acting on a body submerged fully into a fluid (i.e., $u = -p = \rho g z$, where z is vertically upward coordinate), the force acting on the body is

$$\underline{F} = \oint_S (-p)\, \underline{n}\, dS = \int_V \nabla(\rho\, g\, z)\, dV = \int_V (\rho\, g\, \underline{k})\, dV = \rho\, g\, V\, \underline{k} \tag{1.89}$$

This relation is well known as the Archimedes principle for buoyancy force of a submerged body.

By entirely analogous reasoning, using the definitions of the divergence and curl, we have

$$\oint_S \underline{n} \cdot \underline{v}\, dS = \int_V \nabla \cdot \underline{v}\, dV \tag{1.90}$$

and

$$\oint_S \underline{n} \times \underline{v}\, dS = \int_V \nabla \times \underline{v}\, dV \tag{1.91}$$

Equation (1.90) is known as the divergence theorem, or Gauss theorem. If we take $\underline{v}$ as fluid velocity, Eqs. (1.90) and (1.91) become, respectively,

$$\oint_S \underline{n} \cdot \underline{v}\, dS = \int_V \theta\, dV \tag{1.92}$$

and

$$\oint_S \underline{n} \times \underline{v}\, dS = \int_V \underline{\omega}\, dV \tag{1.93}$$

These equations show that the velocity components over boundary are directly related with the field distribution of expansion (or compressing process) and vorticity in fluid region.

The three types of the theorem above can be unified by a general form:

$$\oint_S (\underline{n} * f)\, dS = \int_V (\nabla * f)\, dV \tag{1.94}$$

where $*$ denotes one of differential operator, scalar product and vector product, and f is a scalar or vector function depending on the choice.

As an example, take $\underline{f} = \nabla u$ to yield

$$\int_V \nabla^2 u\, dV = \int_V \nabla \cdot (\nabla u)\, dV = \oint_S \underline{n} \cdot \nabla u\, dS = \oint_S \frac{\partial u}{\partial n}\, dS \tag{1.95}$$

where $\partial u/\partial n$ is the directed derivative in the outward direction as defined in Eq (1.63).

1.3.2 Stokes theorem

Let us apply Eq. (1.50) for definition of ∇u to a very small volume element of a thin disk with uniform height $\triangle h$ and base area $\triangle S$. Its volume then becomes $\triangle S\ \triangle h$. Consider the product of ∇u with the outward unit normal vector to the upper surface $\underline{n}_u$. Then it is not difficult to prove that,

$$\underline{n}_u \times \nabla u \approx \underline{n}_u \times \frac{1}{\triangle V} \oint_S u\, \underline{n}\, dS \approx \frac{1}{\triangle S} \oint_C u\, d\underline{\ell} \tag{1.96}$$

where C is the small contour that forms the boundary of $\triangle S$. The line integral in Eq. (1.96) is taken in the direction that would advance a right-hand screw in the $\underline{n}$ direction.

Now consider a volume element with the uniform thin height and an arbitrary base surface S. If this volume is subdivided into very small volume V_i with the same height, the above product in an integral sense can be expressed as the sum of all the integrals taken over the small line integrals:

$$\int_S \underline{n} \times \nabla u\, dS = \lim_{V_i \to 0} \sum \oint_{C_i} u\, d\underline{\ell} \tag{1.97}$$

Since the line integration over neighboring contour elements will cancel one another, and only the integration over the outside contour will remain:

$$\int_S \underline{n} \times \nabla u \, dS = \oint_C u \, d\underline{\ell} \tag{1.98}$$

With this knowledge, two more important transformation theorems follow:

$$\int_S \underline{n} \cdot \nabla \times \underline{v} \, dS = \oint_C \underline{v} \cdot d\underline{\ell} \tag{1.99}$$

$$\int_S (\underline{n} \times \nabla) \times \underline{v} \, dS = \oint_C d\underline{\ell} \times \underline{v} \tag{1.100}$$

The first of these is known as *Stokes theorem.* [8]

If u is constant, Eq. (1.98) becomes

$$0 = \oint_C u \, d\underline{\ell} \tag{1.101}$$

and if $\underline{v} = \underline{x}$, Eq. (1.100) becomes, since $(\underline{n} \times \nabla) \times \underline{x} = -2\,\underline{n}$,

$$\int_S \underline{n} \, dS = \frac{1}{2} \oint_C \underline{x} \times d\underline{\ell}. \tag{1.102}$$

If we consider $\underline{v}$ as fluid velocity, we have the well-known relation between vorticity flux through an open surface and circulation along the boundary of the surface:

$$\int_S \underline{n} \cdot \underline{\omega} \, dS = \oint_C \underline{v} \cdot d\underline{\ell} \tag{1.103}$$

By analogous reasoning, we have used the relationship,

$$\underline{n} \cdot \nabla \times \underline{v} \approx \frac{1}{S} \oint_C \underline{v} \cdot d\underline{\ell} \tag{1.104}$$

The conditions on u and $\underline{v}$ are analogous to those imposed above; that is, the functions and their first derivations must be finite, continuous, and single-valued

[8] For rigorous proof, see Arfken, G. (1970), *Mathematical Methods for Physicists*, 2nd ed., Academic Press, pp. 51–53.

in the region. The surface S enclosed by the contour C need not be flat; $\underline{n}$ is normal to S at every point, and the direction of C is chosen as described above.

The unified form of Stokes theorem may be written by,

$$\int_S (\underline{n} \times \nabla) * f \, dS = \oint_C d\underline{\ell} * f \tag{1.105}$$

1.3.3 Volume integrals of a vector

Using integration by parts, we can express the integration of $f(x)$ by the moment of $f'(x)$:

$$\int_a^b f(x)\, dx = b\, f(b) - a\, f(a) - \int_a^b x\, f'(x)\, dx \tag{1.106}$$

In a similar fashion to this one-dimensional formula, a surface or volume integral can be cast to the integrals of the first moment of the derivative of f plus boundary integrals.

With $d = 2, 3$ being the space dimesion and $\underline{x}$ the position vector, we find the vector expansion formulas:

$$\begin{aligned} \nabla \cdot (\underline{f}\, \underline{x}) &= \underline{f} + \underline{x}\, (\nabla \cdot \underline{f}) && (1.107) \\ \nabla \cdot (\underline{x}\, \underline{f}) &= d\, \underline{f} + \underline{x} \cdot \nabla \underline{f} && (1.108) \\ \nabla (\underline{x} \cdot \underline{f}) &= \underline{f} + \underline{x} \cdot \nabla \underline{f} + \underline{x} \times (\nabla \times \underline{f}) && (1.109) \\ \underline{x} \times (\underline{n} \times \underline{f}) &= \underline{n}\, (\underline{f} \cdot \underline{x}) - (\underline{n} \cdot \underline{x})\, \underline{f}, && (1.110) \end{aligned}$$

From the volume integral for Eq. (1.107), we apply the divergence theorem to find an identity:

$$\int_V \underline{f}\, dV = -\int_V \underline{x}\, (\nabla \cdot \underline{f})\, dV + \oint_S \underline{x}\, (\underline{n} \cdot \underline{f})\, dS \tag{1.111}$$

Another form of Eq. (1.111) can be provided as follows.
First, subtracting Eq. (1.109) from Eq. (1.108) yields

$$\nabla \cdot (\underline{x}\, \underline{f}) - \nabla (\underline{x} \cdot \underline{f}) = (d - 1)\, \underline{f} - \underline{x} \times (\nabla \times \underline{f}) \tag{1.112}$$

Now we take volume integrals of this equation and apply the divergence theorem to find another identity, using Eq. (1.110):

$$\begin{aligned}\int_V \underline{f}\, dV &= \frac{1}{d-1}\int_V \underline{x}\times(\nabla\times\underline{f})\, dV + \frac{1}{d-1}\int_V \left\{\nabla\cdot(\underline{x}\,\underline{f}) - \nabla(\underline{f}\cdot\underline{x})\right\}\, dV \\ &= \frac{1}{d-1}\int_V \underline{x}\times(\nabla\times\underline{f})\, dV + \frac{1}{d-1}\oint_S \left\{(\underline{n}\cdot\underline{x})\,\underline{f} - \underline{n}\,(\underline{f}\cdot\underline{x})\right\}\, dS \\ &= \frac{1}{d-1}\int_V \underline{x}\times(\nabla\times\underline{f})\, dV - \frac{1}{d-1}\oint_S \underline{x}\times(\underline{n}\times\underline{f})\, dS \qquad (1.113)\end{aligned}$$

We note that the left-hand side of Eq. (1.112) and Eq. (1.113) is independent of the choice of the origin of $\underline{x}$, so must be the right-hand side.

1.3.3.1 Volume integral of first moment

We can also cast the first vector moment $\underline{x}\times\underline{f}$ to the second moments of $\nabla\times\underline{f}$:

$$2\int_V \underline{x}\times\underline{f}\, dV = -\int_V x^2\,(\nabla\times\underline{f})\, dV + \oint_S x^2\,\underline{n}\times\underline{f}\, dS \qquad (1.114)$$

$$\int_V \underline{x}\times\underline{f}\, dV = \int_V \underline{x}\left\{\underline{x}\cdot(\nabla\times\underline{f})\right\}\, dV - \oint_S \left\{(\underline{n}\times\underline{f})\cdot\underline{x}\right\}\underline{x}\, dS \qquad (1.115)$$

$$3\int_V \underline{x}\times\underline{f}\, dV = \int_V \underline{x}\times\left\{\underline{x}\times(\nabla\times\underline{f})\right\}\, dV - \oint_S \underline{x}\times\left\{\underline{x}\times(\underline{n}\times\underline{f})\right\}\, dS \qquad (1.116)$$

To derive Eq. (1.114) and Eq. (1.115), we have used the following relations and then applied the divergence theorem:

$$\begin{aligned}\nabla\times(x^2\underline{f}) &= \nabla(x^2)\times\underline{f} + x^2\nabla\times\underline{f} \\ &= 2\,\underline{x}\times\underline{f} + x^2\nabla\times\underline{f} \qquad (1.117)\\ \nabla\cdot(\underline{f}\times\underline{x}\,\underline{x}) &= \nabla\cdot(\underline{f}\times\underline{x})\,\underline{x} + \underline{f}\times\underline{x} \\ &= \underline{x}\left\{\underline{x}\cdot(\nabla\times\underline{f})\right\} + \underline{f}\times\underline{x} \qquad (1.118)\\ (\underline{n}\times\underline{f})\cdot\underline{x} &= \underline{n}\cdot(\underline{f}\times\underline{x}) \qquad (1.119)\end{aligned}$$

Equation (1.116) is the sum of Eq. (1.114) and Eq. (1.115) by aid of the relation $\underline{x}\times(\underline{x}\times\underline{a}) = \underline{x}\,(\underline{x}\cdot\underline{a}) - x^2\,\underline{a}$.

1.3.4 Surface integrals of a vector

The corresponding transformation rule on surface integral is, since $(\underline{n} \times \nabla) \times (\phi\, \underline{x}) = (\underline{n} \times \nabla\phi) \times \underline{x} - (d-1)\phi\, \underline{n}$,

$$\int_S \phi\, \underline{n}\, dS = -\frac{1}{d-1}\int_S \underline{x} \times (\underline{n} \times \nabla\phi)\, dS + \frac{1}{d-1}\oint_C \phi\, \underline{x} \times d\underline{x} \qquad (1.120)$$

and for $d = 3$, the integral of tangent vector becomes

$$\int_S \underline{n} \times \underline{f}\, dS = -\int_S \underline{x} \times \left\{(\underline{n} \times \nabla) \times \underline{f}\right\} dS + \oint_C \underline{x} \times (d\underline{x} \times \underline{f}) \qquad (1.121)$$

Equation (1.120) is a special case of Eq. (1.113) with $\underline{f} = \nabla\phi$. Here we apply the divergence theorem for a volume integral, in which the closed boundary surface can be regarded as being composed of two open surfaces (S_1 and S_2):

$$\int\limits_{S_1+S_2} \phi\, \underline{n}\, dS = -\frac{1}{d-1}\int\limits_{S_1+S_2} \underline{x} \times (\underline{n} \times \nabla\phi)\, dS \qquad (1.122)$$

Then the surface integral over the first surface S_1 can be expressed in terms of integrals over the second part of the closed surface:

$$\begin{aligned}\int_{S_1} \phi\, \underline{n}\, dS &= -\frac{1}{d-1}\int_{S_1} \underline{x} \times (\underline{n} \times \nabla\phi)\, dS \\ &\quad -\frac{1}{d-1}\int_{S_2} \underline{x} \times (\underline{n} \times \nabla\phi)\, dS - \int_{S_2} \phi\, \underline{n}\, dS \qquad (1.123)\end{aligned}$$

Use of the relation $(\underline{n} \times \nabla) \times (\phi\, \underline{x}) = (\underline{n} \times \nabla\phi) \times \underline{x}) - (d-1)\ \phi\ \underline{n}$ for the integral over the second surface makes us to have

$$\begin{aligned}\int_{S_1} \phi\, \underline{n}\, dS &= -\frac{1}{d-1}\int_{S_1} \underline{x} \times (\underline{n} \times \nabla\phi)\, dS + \frac{1}{d-1}\int_{S_2} (\underline{n} \times \nabla) \times (\phi\, \underline{x})\, dS \\ &= -\frac{1}{d-1}\int_{S_1} \underline{x} \times (\underline{n} \times \nabla\phi)\, dS + \frac{1}{d-1}\oint_{-C_2} d\underline{x} \times (\phi\, \underline{x}) \\ &= -\frac{1}{d-1}\int_{S_1} \underline{x} \times (\underline{n} \times \nabla\phi)\, dS + \frac{1}{d-1}\oint_{C_2} (\phi\, \underline{x}) \times d\underline{x} \qquad (1.124)\end{aligned}$$

by applying the Stokes theorem for the second integral.

Proof of Eq. (1.121)

According to the associated vector expansion formulas, we have

$$\begin{aligned}
(\underline{n}\times\nabla)\times(\underline{f}\times\underline{x}) &= \epsilon_{pil}\,\epsilon_{ijk}\,n_j\,\frac{\partial}{\partial x_k}(\epsilon_{lmn}\,f_m\,x_n)\\
&= \epsilon_{lpi}\,\epsilon_{lmn}\,\epsilon_{ijk}\,n_j\left(\frac{\partial f_m}{\partial x_k}\,x_n + f_m\,\delta_{kn}\right)\\
&= (\delta_{pm}\,\delta_{in}-\delta_{pn}\,\delta_{im})\left(\epsilon_{ijk}\,n_j\,\frac{\partial f_m}{\partial x_k}\,x_n + \epsilon_{ijk}\,n_j\,f_m\,\delta_{kn}\right)\\
&= \epsilon_{ijk}\,n_j\,\frac{\partial f_m}{\partial x_k}\,x_i - \epsilon_{ijk}\,n_j\,\frac{\partial f_i}{\partial x_k}\,x_n + \epsilon_{ijk}\,n_j\,f_m\,\delta_{ki} - \epsilon_{ijk}\,n_j\,f_i\\
&= \underline{x}\cdot(\underline{n}\times\nabla\underline{f}) - \left\{(\underline{n}\times\nabla)\cdot\underline{f}\right\}\underline{x} + \underline{n}\times\underline{f} \qquad (1.125)
\end{aligned}$$

and, with a similar manipulation,

$$\begin{aligned}
(\underline{n}\times\nabla)\times\underline{f}\times\underline{x} &= \underline{x}\cdot(\underline{n}\times\nabla\underline{f}) - (\underline{n}\times\nabla\underline{f})\cdot\underline{x} \qquad (1.126)\\
(\underline{n}\times\nabla)(\underline{f}\cdot\underline{x}) &= (\underline{n}\times\nabla\underline{f})\cdot\underline{x} + \underline{n}\times\underline{f} \qquad (1.127)\\
(\underline{n}\times\nabla)\cdot(\underline{f}\,\underline{x}) &= \left\{(\underline{n}\times\nabla)\cdot\underline{f}\right\}\underline{x} - \underline{n}\times\underline{f} \qquad (1.128)
\end{aligned}$$

Now, adding Eqs. (1.125) and (1.128) and subtracting Eqs. (1.126) and (1.127) from its result, we have

$$\begin{aligned}
&(\underline{n}\times\nabla)\times(\underline{f}\times\underline{x}) + (\underline{n}\times\nabla)\cdot(\underline{f}\,\underline{x}) - (\underline{n}\times\nabla)\times\underline{f}\times\underline{x} - (\underline{n}\times\nabla)(\underline{f}\cdot\underline{x})\\
&= -\underline{n}\times\underline{f} \qquad (1.129)
\end{aligned}$$

We then take a surface integral for Eq. (1.129) and use Stokes theorem to yield

$$\begin{aligned}
&\oint_C d\underline{x}\times(\underline{f}\times\underline{x}) + \int_S \underline{x}\times\left\{(\underline{n}\times\nabla)\times\underline{f}\right\}\,dS - \oint_C d\underline{x}\,(\underline{f}\cdot\underline{x}) + \oint_C d\underline{x}\cdot(\underline{f}\,\underline{x})\\
&= -\int_S \underline{n}\times\underline{f}\,dS \qquad (1.130)
\end{aligned}$$

Rearranging this equation, we provide Eq. (1.121):

$$\begin{aligned}\int_S \underline{n} \times \underline{f}\, dS &= -\oint_C \left\{\underline{f}\,(\underline{x}\cdot d\underline{x}) - \underline{x}\,(\underline{f}\cdot d\underline{x})\right\} - \int_S \underline{x} \times \left\{(\underline{n}\times\nabla)\times\underline{f}\right\}\, dS \\ &\quad + \oint_C d\underline{x}\,(\underline{f}\cdot\underline{x}) - \oint_C d\underline{x}\cdot(\underline{f}\,\underline{x}) \\ &= \oint_C \underline{x}\times(d\underline{x}\times\underline{f}) - \int_S \underline{x}\times\left\{(\underline{n}\times\nabla)\times\underline{f}\right\}\, dS \qquad (1.131)\end{aligned}$$

■■■

1.3.4.1 Surface integrals of first moment

The surface integrals of the first moment $\underline{x} \times \underline{n}\phi$ and $\underline{x} \times (\underline{x} \times \underline{f})$ can also be transformed to the following alternative forms: [9]

$$\int_S \underline{x}\times\underline{n}\,\phi\, dS = \frac{1}{2}\int_S x^2\,\underline{n}\times\nabla\phi\, dS - \frac{1}{2}\oint_C x^2\,\phi\, dl \qquad (1.132)$$

$$\begin{aligned}\int_S \underline{x}\times\underline{n}\,\phi\, dS &= -\int_S \underline{x}\left\{\underline{x}\cdot(\underline{n}\times\nabla\phi)\right\}\, dS \\ &\quad + \oint_C \phi\,\underline{x}\,(\underline{x}\cdot d\underline{x}) \qquad (1.133)\end{aligned}$$

$$\begin{aligned}\int_S \underline{x}\times\underline{n}\,\phi\, dS &= -\frac{1}{3}\int_S \underline{x}\times\left\{\underline{x}\times(\underline{n}\times\nabla\phi)\right\}\, dS \\ &\quad + \frac{1}{3}\oint_C \phi\,\underline{x}\times(\underline{x}\times d\underline{x}) \qquad (1.134)\end{aligned}$$

$$\begin{aligned}\int_S \underline{x}\times(\underline{n}\times\underline{f})\, dS &= \int_S \underline{\underline{S}}\cdot\left\{(\underline{n}\times\nabla)\times\underline{f}\right\}\, dS \\ &\quad - \oint_C \underline{\underline{S}}\cdot(d\underline{x}\times\underline{f}) \qquad (1.135)\end{aligned}$$

where $\underline{\underline{S}}$ is the second order tensor depending only on $\underline{x}$:

$$\underline{\underline{S}} = \frac{1}{2}x^2\,\underline{\underline{I}} - \underline{x}\,\underline{x} \quad \text{or} \quad S_{ij} = \frac{1}{2}x^2\,\delta_{ij} - x_i\,x_j \qquad (1.136)$$

Equation (1.133) is obtained from applying the Stokes' theorem to the sur-

[9] See Wu, J.-Z, Ma, H.-Y. and Zhou, M.-D. (2006), *Vorticity and Vortex Dynamics*, Springer, p. 702.

face integrals of the following identity:

$$
\begin{aligned}
(\underline{n} \times \nabla) \cdot (\phi \, \underline{x} \, \underline{x}) &= \underline{n} \cdot \{\nabla \times (\phi \, \underline{x} \, \underline{x})\} = n_i \left\{ \epsilon_{ijk} \frac{\partial}{\partial x_j} (\phi \, x_k \, x_l) \right\} \\
&= n_i \left\{ \epsilon_{ijk} \frac{\partial \phi}{\partial x_j} x_k \, x_l + \phi \, \epsilon_{ijk} \, x_k \, \delta_{jl} \right\} \\
&= n_i \left\{ \epsilon_{ijk} \frac{\partial \phi}{\partial x_j} x_k \, x_l + \phi \, \epsilon_{ilk} \, x_k \right\} \\
&= \underline{n} \cdot (\nabla \phi \times \underline{x}) \, \underline{x} - \phi \, \underline{n} \times \underline{x} \\
&= \{\underline{n} \times \nabla \phi) \cdot \underline{x}\} \, \underline{x} + \underline{x} \times (\phi \, \underline{n}) \qquad (1.137)
\end{aligned}
$$

Note that Eq. (1.134) is obtained from a linear combination of Eq. (1.132) and Eq. (1.133).

1.4 Curvilinear Coordinates on Lines and Surfaces

We are used to encounter Cartesian components in vector and tensor analysis. In some situations, local curvilinear coordinates along a line or surface are more convenient, especially when they orthogonal. Therefore they are as intrinsic as possible, with an arbitrarily moving origin thereon. [10]

1.4.1 Intrinsic line frame

If we are interested in the flow behavior along a smooth line C with arc length element ds, say a streamline or a vortex line, the intrinsic coordinate system with origin $O(\underline{x})$ on C has three orthonormal basis vectors: the tangent vector $\underline{t} = \dfrac{\partial \underline{x}}{\partial s}$, the binormal $\underline{b} = \underline{t} \times \underline{n}$, and the principal normal $\underline{n}$ (toward the center of curvature), see Figure **1.3** .

[10]Most of material covered in this section has been taken from
(1) Kreyszig, E. (2006), *Advanced Engineering Mathematics*, 9th ed., Wiley, pp. 397-398,
(2) Wu, J.-Z, Ma, H.-Y. and Zhou, M.-D. (2006), *Vorticity and Vortex Dynamics*, Springer, pp. 705-712,
(3) Farin G. & Hansford D. (2000), *The Essentials of CAGD*, A K Peters, Natick, MA, pp. 119-121, and
(4) Farin G. (2002), *Curves and Surfaces for CAGD-A Practrical Guide*, 5th ed., Morgan Kaufmann Publishers, pp. 181-187, 419-421.

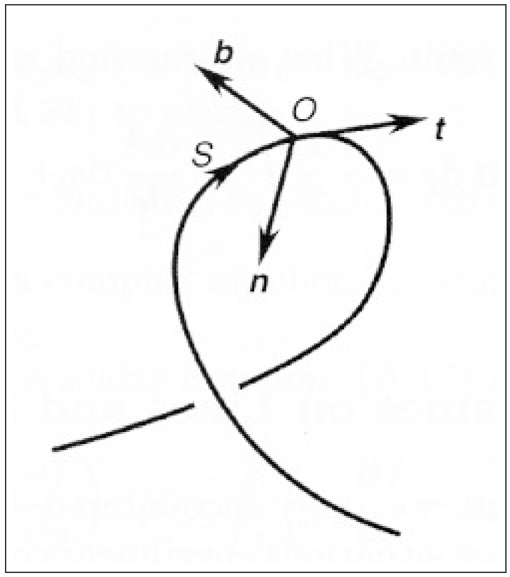

Figure 1.3 Intrinsic 3 orthonomal basis vectors along a curve in a local curvilinear coordinate system. From Wu, Ma and Zhou (2006), p. 706.

Three axes $\underline{t}$, $\underline{b}$ and $\underline{n}$ are defined as

$$\underline{t}(s) = \frac{\partial \underline{x}}{\partial s} \Big/ \left| \frac{\partial \underline{x}}{\partial s} \right| \tag{1.138}$$

$$\underline{b}(s) = \left(\frac{\partial \underline{x}}{\partial s} \times \frac{\partial^2 \underline{x}}{\partial s^2} \right) \Big/ \left| \frac{\partial \underline{x}}{\partial s} \times \frac{\partial^2 \underline{x}}{\partial s^2} \right| \tag{1.139}$$

$$\underline{n}(s) = \underline{b} \times \underline{t} \tag{1.140}$$

The plane spanned by the point $\underline{x}$, $\underline{t}$, and $\underline{n}$ is called the **osculating plane**. The planes spanned by ($\underline{x}$, $\underline{t}$, $\underline{b}$) and ($\underline{x}$, $\underline{b}$, $\underline{n}$) are called, respectively, the **rectifying plane** and the **normal plane**. [11]

The key of using this frame is to know how the basis vectors change their directions as s varies. This is given by the 'Frenet-Serret formulas', form the entire basis of spatial curve theory in classical differential geometry:

$$\frac{\partial \underline{t}}{\partial s} = \kappa\, \underline{n}, \quad \frac{\partial \underline{n}}{\partial s} = -\kappa\, \underline{t} + \tau\, \underline{b}, \quad \frac{\partial \underline{b}}{\partial s} = -\tau\, \underline{n} \tag{1.141}$$

[11](1) The equation of the osculating plane is $det \left[\underline{y} - \underline{x}, \frac{\partial \underline{x}}{\partial s}, \frac{\partial^2 \underline{x}}{\partial s^2} \right] = 0$, where $\underline{y}$ denotes any point on the plane.

(2) A curve C in a surface is a *geodesic* of the surface if the second derivative $\left(\frac{\partial^2 \underline{x}}{\partial s^2} \right)$ of the position vector of C is always normal to the surface. (See O'Neill (1966), p. 228 or Aris (1962), p. 201).

where κ and τ are the curvature and torsion of C, respectively. The curvature radius is $r_c = 1/\kappa$ with $d\,r = -d\,n$. A formula for the curvature is given by

$$\kappa(s) \;=\; \frac{\left|\dfrac{\partial \underline{x}}{\partial s} \times \dfrac{\partial^2 \underline{x}}{\partial s^2}\right|}{\left|\dfrac{\partial \underline{x}}{\partial s}\right|^3} \tag{1.142}$$

A point where the curvature changes sign is called *inflection points*. The torsion measures how much a curve deviates from a plane curve, i.e., it is the curvature of the projection of C onto the $(\underline{n}, \underline{b})$ plane (i.e., the normal plane). Similarly, a formula for the torsion is given by

$$\tau(s) \;=\; \frac{det\left[\dfrac{\partial \underline{x}}{\partial s}, \dfrac{\partial^2 \underline{x}}{\partial s^2}, \dfrac{\partial^3 \underline{x}}{\partial s^3}\right]}{\left|\dfrac{\partial \underline{x}}{\partial s} \times \dfrac{\partial^2 \underline{x}}{\partial s^2}\right|^2} \tag{1.143}$$

The Taylor expansion of $\underline{x}(s + \triangle s)$ can be written as

$$\begin{aligned} \underline{x}(s + \triangle s) \;&=\; \underline{x}(s) + \triangle s\, \underline{t} + \frac{1}{2}\kappa\, \triangle s^2\, \underline{n} \\ &\quad -\frac{1}{6}\kappa^2\, \triangle s^3\, \underline{t} + \frac{1}{6}\kappa'\, \triangle s^3\, \underline{n} + \frac{1}{6}\kappa\, \tau\, \triangle s^3\, \underline{b} \\ &\quad + \cdots \end{aligned} \tag{1.144}$$

For 2-D curves only, the slope κ' is given by, [12]

$$\frac{d\kappa}{ds} = \frac{det\left[\dfrac{\partial \underline{x}}{\partial s}, \dfrac{\partial^3 \underline{x}}{\partial s^3}\right]}{\left|\dfrac{\partial \underline{x}}{\partial s}\right|^4} - 3\left(\frac{\partial \underline{x}}{\partial s}\right)\left(\frac{\partial^2 \underline{x}}{\partial s^2}\right)\frac{det\left[\dfrac{\partial \underline{x}}{\partial s}, \dfrac{\partial^2 \underline{x}}{\partial s^2}\right]}{\left|\dfrac{\partial \underline{x}}{\partial s}\right|^6} \tag{1.145}$$

Now, let the differential distance form O along the directions of $\underline{n}$ and $\underline{b}$ be

[12] See Farin G. (2002), *Curves and Surfaces for CAGD-A Practrical Guide*, 5th ed., Morgan Kaufmann Publishers, p. 421.

dn and db, respectively. Then

$$\nabla = \underline{t}\,\frac{\partial}{\partial s} + \underline{n}\,\frac{\partial}{\partial n} + \underline{b}\,\frac{\partial}{\partial b} \tag{1.146}$$

It involves curves along $\underline{n}$ and $\underline{b}$ directions, for which the Frenet-Serret formulas can be applied to complete the gradient operation.

1.4.1.1 Example: Propeller pitch helix

Let us consider the constant-pitch helix of a propeller blade. The position vector is expressed as, by converting a cylindrical coordinates (r, θ, x) into the Cartersian coordinates,

$$\underline{x}(\theta) = r\,\theta\,\tan\phi\,\underline{i} + r\cos\theta\,\underline{j} + r\sin\theta\,\underline{k} \tag{1.147}$$

where ϕ is the constant pitch angle. Simple calculations yield

$$\frac{\partial \underline{x}}{\partial \theta} = r\tan\phi\,\underline{i} - r\sin\theta\,\underline{j} + r\cos\theta\,\underline{k} \tag{1.148}$$

$$\frac{\partial^2 \underline{x}}{\partial \theta^2} = -r\cos\theta\,\underline{j} - r\sin\theta\,\underline{k} \tag{1.149}$$

$$\frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial^2 \underline{x}}{\partial \theta^2} = -r^2\cos\theta\tan\phi\,\underline{k} + r^2\tan\phi\sin\theta\,\underline{j} + r^2\,\underline{i} \tag{1.150}$$

from which the curvature becomes

$$\kappa = \frac{r^2(1+\tan^2\phi)^{1/2}}{r^3(1+\tan^2\phi)^{3/2}} = \frac{\cos^2\phi}{r} \tag{1.151}$$

The radius of curvature is given by $r_c = \dfrac{1}{\kappa} = \dfrac{r}{\cos^2\phi}$. The center of the *osculating circle* is, in the direction of the normal vector $\underline{n}$,

$$\underline{x}_c = \underline{x} + r_c\,\underline{n} \tag{1.152}$$

By developing of such an osculating circle onto a (2-D) plane, we can take two end points of the circular arc with the radius r_c that correspond to, respectively, the leading edge and the trailing edge of the blade section.

By assembling the corresponding two end points of the osculating circular arcs for pitch helices at other radial positions, we can produce a developed outline of the propeller blade. In general, the corresponding radii of curvature might be different each other.

1.4.1.2 Example: Streamline intrinsic frame

For example, if C is a streamline such that $\underline{u} = q\,\underline{t}$, then the divergence of the velocity becomes

$$\nabla \cdot \underline{u} = \frac{\partial q}{\partial s} + q\,\nabla \cdot \underline{t} = \frac{\partial q}{\partial s} + q\left(\underline{n} \cdot \frac{\partial \underline{t}}{\partial n} + \underline{b} \cdot \frac{\partial \underline{t}}{\partial b}\right) \tag{1.153}$$

Similarly, the curl operation for $\underline{t}$ is

$$\nabla \times \underline{t} = \left(\underline{n} \times \frac{\partial \underline{t}}{\partial n} + \underline{b} \times \frac{\partial \underline{t}}{\partial b}\right) + \kappa\,\underline{b} \tag{1.154}$$

Here, since $|\underline{t}| = 1$, it follows that

$$\underline{n} \cdot \left(\underline{n} \times \frac{\partial \underline{t}}{\partial n} + \underline{b} \times \frac{\partial \underline{t}}{\partial b}\right) = \underline{t} \cdot \frac{\partial \underline{t}}{\partial b} = \frac{1}{2}\frac{\partial |\underline{t}|^2}{\partial b} = 0 \tag{1.155}$$

$$\underline{b} \cdot \left(\underline{n} \times \frac{\partial \underline{t}}{\partial n} + \underline{b} \times \frac{\partial \underline{t}}{\partial b}\right) = -\underline{t} \cdot \frac{\partial \underline{t}}{\partial n} = -\frac{1}{2}\frac{\partial |\underline{t}|^2}{\partial b} = 0 \tag{1.156}$$

The first term of $\nabla \times \underline{t}$ in (1.154) must be along the $\underline{t}$ direction, with the magnitude known as the 'torsion of neighboring vector lines'.

$$\xi \equiv \underline{t} \cdot (\nabla \times \underline{t}) = \underline{b} \cdot \frac{\partial \underline{t}}{\partial n} - \underline{n} \cdot \frac{\partial \underline{t}}{\partial b} \tag{1.157}$$

Using this notation we obtain

$$\nabla \times \underline{t} = \xi\,\underline{t} + \kappa\,\underline{b} \tag{1.158}$$

This result enables us to derive the vorticity expression in the streamline intrinsic frame

$$\underline{\omega} = \nabla \times (q\,\underline{t}) = \nabla q \times \underline{t} + q\,\nabla \times \underline{t} = \nabla q \times \underline{t} + q\,\xi\,\underline{t} + q\,\kappa\,\underline{b} \tag{1.159}$$

The first term of this equation is

$$(\nabla q) \times \underline{t} = \frac{\partial q}{\partial b}\,\underline{n} - \frac{\partial q}{\partial n}\,\underline{b} \tag{1.160}$$

so we obtain

$$\underline{\omega} = q\,\xi\,\underline{t} + \frac{\partial q}{\partial b}\,\underline{n} + \left(q\,\kappa - \frac{\partial q}{\partial n}\right)\underline{b} \tag{1.161}$$

Thus, $\xi = 0$ if $\underline{\omega} \cdot \underline{u}\,(= q^2\,\xi) = 0$.

1.4.2 Curvilinear orthogonal coordinates

We will have need for the expressions of several vector differential operators in terms of curvilinear orthogonal coordinates. [13] Suppose x_1, x_2, x_3 are mutually orthogonal curvilinear coordinates.

1.4.2.1 Line element

When the line-element vector in the orthogonal system is expressed in terms of a scalar multiple, the scalar multiple is usually written h_i and is called a scale factor:

$$d\underline{s} = (h_1\,dx_1, h_2\,dx_2, h_3\,dx_3) \tag{1.162}$$

where $h_1 = h_1(x_1, x_2, x_3) = \left|\frac{\partial \underline{s}}{\partial x_1}\right| = \left\{\left(\frac{\partial s_1}{\partial x_1}\right)^2 + \left(\frac{\partial s_2}{\partial x_1}\right)^2 + \left(\frac{\partial s_3}{\partial x_1}\right)^2\right\}^{1/2}$ etc.

The base vectors, $\frac{\partial \underline{s}}{\partial x_i}$, is then expressed in terms of the scale factor and a

[13] For example, expressions for the related common differentials in spherical, cylindrical and polar coordinate systems are found in Batchelor, G. K. (1967), *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, pp. 598–603.

unit vector, e.g.

$$\frac{\partial \underline{s}}{\partial x_1} = h_1(x_1, x_2, x_3)\ \underline{e}_1(x_1, x_2, x_3) \tag{1.163}$$

For example, if we take spherical coordinates $x_1 = r, x_2 = \theta$, and $x_3 = \phi$ where ϕ is the azimuthal angle about the axis $\theta = 0$, the line element is $d\underline{s} = (dr,\ r\,d\theta,\ r\sin\theta\,d\phi)$; hence the scale factors $h_1 = 1, h_2 = r, h_3 = r\sin\theta$.

If we take cylindrical coordinates $x_1 = \rho, x_2 = \phi$, and $x_3 = z$ where ϕ is the azimuthal angle about the axis $\rho = 0$, the line element is $d\underline{s} = (d\rho, \rho\,d\phi, dz)$; hence the scale factors $h_1 = 1, h_2 = \rho, h_3 = 1$.

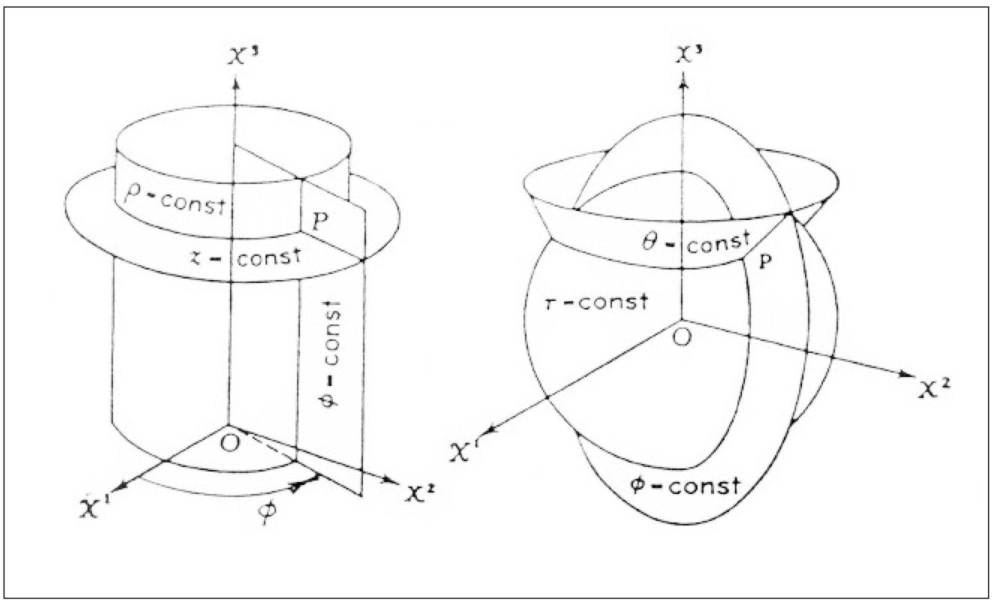

Figure 1.4 Cylindrical and spherical coordinate systems. From Brockett(Lecture note NA520, 1988), p.1-30a.

The scalar differential arc length, denoted by ds is determined from

$$\begin{aligned} ds^2 &= d\underline{s}\cdot d\underline{s} = \left(\frac{\partial \underline{s}}{\partial x_i}\,dx_i\right)\cdot\left(\frac{\partial \underline{s}}{\partial x_j}\,dx_j\right) \\ &= h_i\,h_j\,dx_i\,dx_j\,\underline{e}_i\cdot\underline{e}_j \end{aligned} \tag{1.164}$$

When the unit base vectors are orthogonal, this expression reduces to the simple form

$$ds^2 = h_1^2\,dx_1^2 + h_2^2\,dx_2^2 + h_3^2\,dx_3^2 \tag{1.165}$$

By the triple scalar product, the volume element can be obtained from the elemental arc length vectors:

$$dV = \pm \left(\frac{\partial \underline{s}}{\partial x_1} dx_1 \right) \cdot \left(\frac{\partial \underline{s}}{\partial x_2} \times \frac{\partial \underline{s}}{\partial x_3} \, dx_2 \, dx_3 \right) \tag{1.166}$$

where the $\pm$ sign is necessary to provide a positive element of volume. For an orthogonal coordinate system, with $\dfrac{\partial \underline{s}}{\partial x_1} = h_1 \, \underline{e}_1$, etc, the volume element is

$$dV = h_1 \, h_2 \, h_3 \; dx_1 \, dx_2 \, dx_3 \tag{1.167}$$

since $\underline{e}_1 \cdot (\underline{e}_2 \times \underline{e}_3) = \pm 1$. Multiplication of the scale factors corresponds to the Jacobian $J = h_1 \, h_2 \, h_3$.

1.4.2.2 Gradient (∇u)

We have the formula $du = d\underline{s} \cdot \nabla u$, which is completely general. Also in any coordinate system, we have

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 \tag{1.168}$$

Equating these two relations gives

$$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 = h_1 \, dx_1 \, (\nabla u)_1 + h_2 \, dx_2 \, (\nabla u)_2 + h_3 \, dx_3 \, (\nabla u)_3 \tag{1.169}$$

Now dx_1, dx_2, dx_3 are completely arbitrary; hence this equation can be true only if their coefficients are equal. Thus

$$\nabla u = \left(\frac{1}{h_1} \frac{\partial u}{\partial x_1}, \; \frac{1}{h_2} \frac{\partial u}{\partial x_2}, \; \frac{1}{h_3} \frac{\partial u}{\partial x_3} \right) \tag{1.170}$$

1.4.2.3 Divergence $(\nabla \cdot \underline{v})$

For this operator we return to the original definition; thus, denoting by v_1, v_2, v_3 the components of $\underline{v}$ in the 1, 2, 3 directions at any point,

$$\begin{aligned}
\nabla \cdot \underline{v} \;\approx\; & (h_1\, h_2\, h_3 \triangle x_1 \triangle x_2 \triangle x_3)^{-1} \\
& \Big\{ - v_1\, h_2\, h_3 \triangle x_2 \triangle x_3 - v_2\, h_3\, h_1 \triangle x_3 \triangle x_1 - v_3 h_1 h_2 \triangle x_1 \triangle x_2 \\
& \quad + \left[v_1\, h_2\, h_3 + \frac{\partial}{\partial x_1}(v_1\, h_2\, h_3) \triangle x_1 \right] \triangle x_2 \triangle x_3 \\
& \quad + \left[v_2\, h_3\, h_1 + \frac{\partial}{\partial x_2}(v_2\, h_3\, h_1) \triangle x_2 \right] \triangle x_3 \triangle x_1 \\
& \quad + \left[v_3\, h_1\, h_2 + \frac{\partial}{\partial x_3}(v_3\, h_1\, h_2) \triangle x_3 \right] \triangle x_1 \triangle x_2 \Big\} \\
= \; & \frac{1}{h_1\, h_2\, h_3} \left\{ \frac{\partial}{\partial x_1}(h_2\, h_3\, v_1) + \frac{\partial}{\partial x_2}(h_3\, h_1\, v_2) + \frac{\partial}{\partial x_3}(h_1\, h_2\, v_3) \right\}
\end{aligned} \tag{1.171}$$

1.4.2.4 Curl $(\nabla \times \underline{v})$

Apply Stoke's Theorem to one face of the element of a cube, say the 1-3 face:

$$\begin{aligned}
& \int_S \underline{n} \cdot \nabla \times \underline{v}\, dS = \oint_C \underline{v} \cdot d\underline{s} \\
= \; & v_1\, h_1 \triangle x_1 - v_3\, h_3 \triangle x_3 \\
& + \left[v_3\, h_3 + \frac{\partial}{\partial x_1}(v_3\, h_3) \triangle x_1 \right] \triangle x_3 - \left[v_1\, h_1 + \frac{\partial}{\partial x_3}(v_1\, h_1) \triangle x_3 \right] \triangle x_1 \\
= \; & \left[\frac{\partial}{\partial x_1}(h_3\, v_3) - \frac{\partial}{\partial x_3}(h_1\, v_1) \right] \triangle x_1 \triangle x_3
\end{aligned} \tag{1.172}$$

But also

$$\int_S \underline{n} \cdot \nabla \times \underline{v}\, dS \approx -h_1\, h_3 \triangle x_1 \triangle x_3\, (\nabla \times \underline{v})_2 \tag{1.173}$$

Thus, by cyclic substitution,

$$
\begin{aligned}
(\nabla \times \underline{v})_2 &= \frac{1}{h_3\, h_1}\left[\frac{\partial}{\partial x_3}(h_1\, v_1) - \frac{\partial}{\partial x_1}(h_3\, v_3)\right] \\
(\nabla \times \underline{v})_3 &= \frac{1}{h_1\, h_2}\left[\frac{\partial}{\partial x_1}(h_2\, v_2) - \frac{\partial}{\partial x_2}(h_1\, v_1)\right] \\
(\nabla \times \underline{v})_1 &= \frac{1}{h_2\, h_3}\left[\frac{\partial}{\partial x_2}(h_3\, v_3) - \frac{\partial}{\partial x_3}(h_2\, v_2)\right]
\end{aligned}
\tag{1.174}
$$

or, symbolically

$$
\nabla \times \underline{v} = \frac{1}{h_1\, h_2\, h_3}\begin{vmatrix} h_1\, \underline{i}_1 & h_2\, \underline{i}_2 & h_3\, \underline{i}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1\, v_1 & h_2\, v_2 & h_3\, v_3 \end{vmatrix}
\tag{1.175}
$$

For example, if we take spherical coordinates $x_1 = r$, $x_2 = \theta$, and $x_3 = \alpha$ where α is the azimuthal angle about the axis $\theta = 0$,

$$
\begin{aligned}
\nabla \times \underline{v} &= \frac{\underline{e}_r}{r \sin\theta}\left\{\frac{\partial(v_\alpha\, \sin\theta)}{\partial\theta} - \frac{\partial v_\theta}{\partial\alpha}\right\} \\
&\quad + \frac{\underline{e}_\theta}{r}\left\{\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\alpha} - \frac{\partial(r\, v_\alpha)}{\partial r}\right\} \\
&\quad + \frac{\underline{e}_\alpha}{r}\left\{\frac{\partial(r\, v_\theta)}{\partial r} - \frac{\partial(r\, v_r)}{\partial\theta}\right\}
\end{aligned}
\tag{1.176}
$$

1.4.2.5 Laplacian ($\nabla^2 u$)

For $\nabla^2 u$, we simply employ Eqs. (1.170) and (1.171) above:

$$
\nabla^2 u = \nabla \cdot (\nabla u) = \\
\frac{1}{h_1\, h_2\, h_3}\left\{\frac{\partial}{\partial x_1}\left(\frac{h_2\, h_3}{h_1}\frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_3\, h_1}{h_2}\frac{\partial u}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_1\, h_2}{h_3}\frac{\partial u}{\partial x_3}\right)\right\}
\tag{1.177}
$$

The most convenient way to write out $\nabla^2 \underline{v}$ is by use of expansion formula

Eq. (1.72):

$$\nabla^2 \underline{v} = \nabla(\nabla \cdot \underline{v}) - \nabla \times (\nabla \times \underline{v}) \tag{1.178}$$

which can be expanded by use of formulas, Eqs. (1.170), (1.171), and (1.175).

1.4.2.6 Convection term $(\underline{u} \cdot \nabla \underline{v})$

This useful vector appears in the Navier-Stokes equation when we write the time rate of flow momentum in Eulerian description sense. Performing very complicated procedure but straightforward manipulation, we arrive at the following result:

$$\begin{aligned}
(\underline{u} \cdot \nabla \underline{v})_1 \;=\; & \frac{1}{h_1}\Big[u_1 \frac{\partial v_1}{\partial x_1} + u_2 \frac{\partial v_2}{\partial x_1} + u_3 \frac{\partial v_3}{\partial x_1} \\
& + \frac{1}{h_2}(u_1 v_2 - u_2 v_1) \frac{\partial h_1}{\partial x_2} + \frac{1}{h_3}(u_1 v_3 - u_3 v_1) \frac{\partial h_1}{\partial x_3}\Big] \\
& - \frac{u_2}{h_1 h_2}\left[\frac{\partial(h_2 v_2)}{\partial x_1} - \frac{\partial(h_1 v_1)}{\partial x_2}\right] + \frac{u_3}{h_3 h_1}\left[\frac{\partial(h_1 v_1)}{\partial x_3} - \frac{\partial(h_3 v_3)}{\partial x_1}\right]
\end{aligned} \tag{1.179}$$

$$\begin{aligned}
(\underline{u} \cdot \nabla \underline{v})_2 \;=\; & \frac{1}{h_2}\Big[u_1 \frac{\partial v_1}{\partial x_2} + u_2 \frac{\partial v_2}{\partial x_2} + u_3 \frac{\partial v_3}{\partial x_2} \\
& + \frac{1}{h_3}(u_2 v_3 - u_3 v_2) \frac{\partial h_2}{\partial x_3} + \frac{1}{h_1}(u_2 v_1 - u_1 v_2) \frac{\partial h_2}{\partial x_1}\Big] \\
& - \frac{u_3}{h_2 h_3}\left[\frac{\partial(h_3 v_3)}{\partial x_2} - \frac{\partial(h_2 v_2)}{\partial x_3}\right] + \frac{u_1}{h_1 h_2}\left[\frac{\partial(h_2 v_2)}{\partial x_1} - \frac{\partial(h_1 v_1)}{\partial x_2}\right]
\end{aligned} \tag{1.180}$$

$$\begin{aligned}
(\underline{u} \cdot \nabla \underline{v})_3 \;=\; & \frac{1}{h_3}\Big[u_1 \frac{\partial v_1}{\partial x_3} + u_2 \frac{\partial v_2}{\partial x_3} + u_3 \frac{\partial v_3}{\partial x_3} \\
& + \frac{1}{h_1}(u_3 v_1 - u_1 v_3) \frac{\partial h_3}{\partial x_1} + \frac{1}{h_2}(u_3 v_2 - u_2 v_3) \frac{\partial h_3}{\partial x_2}\Big] \\
& - \frac{u_1}{h_3 h_1}\left[\frac{\partial(h_1 v_1)}{\partial x_3} - \frac{\partial(h_3 v_3)}{\partial x_1}\right] + \frac{u_2}{h_2 h_3}\left[\frac{\partial(h_3 v_3)}{\partial x_2} - \frac{\partial(h_2 v_2)}{\partial x_3}\right]
\end{aligned} \tag{1.181}$$

1.5 Tensors of Second Order

For example, let us consider a stress tensor that is a key quantity in continuum mechanics. [14] A stress is a force per unit area, in which force and an element of area are vectors. The area element have to specify both its magnitude and the direction of its normal. If $\underline{F}$ denotes the force and $\underline{S}$ is the area element, the stress tensor $\underline{\underline{T}}$ might be thought of as $\underline{F}/\underline{S}$. This quotient of two vectors cannot be defined, but rather we can define $\underline{F}$ as $\underline{S} \cdot \underline{\underline{T}}$. The stress tensor at a point $\underline{\underline{T}}$ becomes a newly physical quantity associated with two directions. In fact, it needs 9 numbers to specify the stress tensor in a reference system corresponding to the 9 possible combinations of 2 base vectors.

A <u>second-order tensor</u> is a set of nine numbers τ_{ij}, having the property that when transferred from the x_1, x_2, x_3 system to the x'_1, x'_2, x'_3 system the corresponding quantities are given by

$$\tau'_{ij} = \sum_{k=1}^{3} \sum_{\ell=1}^{3} a_{ki}\, a_{\ell j}\, \tau_{k\ell}, \quad \text{for} \quad i, j = 1, 2, 3 \tag{1.182}$$

1.5.1 Dyadic products

Much of our work can be simplified if we extend our definitions of vector multiplication to include the *dyadic product* $\underline{u}\ \underline{v}$. For our purpose, this need only be defined by the relations

$$\begin{aligned} (\underline{u}\ \underline{v}) \cdot \underline{w} &\equiv \underline{u}\,(\underline{v} \cdot \underline{w}) \\ \underline{w} \cdot (\underline{u}\ \underline{v}) &\equiv (\underline{w} \cdot \underline{u})\,\underline{v} \end{aligned} \tag{1.183}$$

Actually the dyadic product $\underline{u}\ \underline{v}$ is a special form of *second-order tensor*; it can easily be seen to satisfy the definition of such a tensor. This definition may be stated as follows, with reference to the x_i and x'_i coordinate systems.

In the case of $\underline{u}\ \underline{v}$, of course, the nine numbers involved are the products $u_i\, v_j\, (i, j = 1, 2, 3)$.

[14] See Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 5.

Let us consider some examples:

(1) For $\nabla(\underline{u}\cdot\underline{v})$, using dyadic notation, $\nabla(\underline{u}\cdot\underline{v}) = (\nabla\underline{u})\cdot\underline{v} + (\nabla\underline{v})\cdot\underline{u}$.

(2) Laplacian $\nabla^2\underline{v} = \nabla\cdot(\nabla\underline{v})$.

(3) When we define $(\underline{u}\;\underline{v})\times\underline{w} \equiv \underline{u}(\underline{v}\times\underline{w})$ and $\underline{w}\times(\underline{u}\;\underline{v}) \equiv (\underline{w}\times\underline{u})\underline{v}$, these are obviously dyadics.

(4) If ϕ is any dyadic product, $\phi\cdot(\underline{a}\times\underline{b}) = (\phi\times\underline{a})\cdot\underline{b}$.

(5) Let us look at the more important example. Let u_i be a vector, and consider the set of nine numbers $\partial u_i/\partial x_j$. This is easily shown to be a second-order tensor. It might be represented by the symbol *grad* $\underline{u}$ or $\nabla\underline{u}$.

1.5.2 Gradient of a vector

Now, consider the gradient of a vector, $\nabla\underline{u}$, which is involved into the convection and the diffusion terms of the Navier-Stokes equations.
The velocity change at a point $d\underline{u}$ is

$$d\underline{u} = (d\underline{x}\cdot\nabla)\,\underline{u} \tag{1.184}$$

The gradient of a vector is defined by, in a similar fashion to the gradient of a scalar,

$$\nabla\underline{u} = \lim_{V\to 0}\frac{1}{V}\oint_S \underline{n}\;\underline{u}\,dS = \frac{\partial u_j}{\partial x_i} \tag{1.185}$$

In a rectangular Cartesian coordinate system, the gradient of a vector $\underline{u} = u_1\,\underline{i} + u_2\,\underline{j} + u_3\,\underline{k}$ is

$$\nabla\underline{u} = \frac{\partial u_1}{\partial x_1}\underline{i}\;\underline{i} + \frac{\partial u_2}{\partial x_1}\underline{i}\;\underline{j} + \frac{\partial u_3}{\partial x_1}\underline{i}\;\underline{k} + \cdots \text{ similar 6 terms} \tag{1.186}$$

In general orthogonal curvilinear coordinates, the gradient of a vector $\underline{u} = u_1\,\underline{e}_1 + u_2\,\underline{e}_2 + u_3\,\underline{e}_3$, is [15]

$$\begin{aligned}\nabla \underline{u} &= \frac{1}{h_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial h_1}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial h_1}{\partial x_3}\right)\underline{e}_1\,\underline{e}_1 + \frac{1}{h_1}\left(\frac{\partial v_2}{\partial x_1} - \frac{v_1}{h_2}\frac{\partial h_1}{\partial x_2}\right)\underline{e}_1\,\underline{e}_2 \\ &\quad + \frac{1}{h_1}\left(\frac{\partial v_3}{\partial x_1} - \frac{v_1}{h_3}\frac{\partial h_1}{\partial x_3}\right)\underline{e}_1\,\underline{e}_3 + \cdots \text{ similar 6 terms}\end{aligned} \tag{1.187}$$

If the vector $\underline{v}$ is a velocity vector in the field of fluid mechanics, this is often resolved into a symmetric and antisymmetric form:

$$\begin{aligned}\nabla \underline{v} &= \frac{1}{2}\left[(\nabla \underline{v} + \nabla \underline{v}^T) + (\nabla \underline{v} - \nabla \underline{v}^T)\right] \\ &= \frac{1}{2}\,def(\underline{v}) + \frac{1}{2}\,rot(\underline{v})\end{aligned} \tag{1.188}$$

where, if we consider a second-order tensor to be a 3×3 matrix, the superscript T stand for transpose of the matrix which is the operation described by interchanging the rows and columns of the matrix. The first term is called the *strain rate tensor*, having 6 independent components. It represents (i) normal strain rate and (ii) shear strain rate which cause stress in fluid.
The second term is called the *spin tensor* or *vorticity tensor* $\underline{\underline{\Omega}}$, having only off-diagonal components. It represents rigid body rotation of a fluid element.

1.6 Transport Theorem

We will have need for the rate of change of an integral taken over a volume moving through a field

$$\frac{d}{dt}\int_{V(t)} F(\underline{x}, t)\, dV \tag{1.189}$$

where $F(\underline{x}, t)$ may be a scalar, vector or tensor variable. We assume the path of points in $V(t)$ are known:

$$\underline{x} = \underline{x}(\underline{\xi}, t) \tag{1.190}$$

[15] For details, see Milne-Thomson, L. M. (1968), *Theoretical Hydrodynamics*, Fifth edition, Macmillan, London, pp. 62–66 and Batchelor, G. K. (1967), *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, pp. 598–603.

where $\underline{\xi}$ is the initial point of $\underline{x}$. Hence we can invert the integral to the $\underline{\xi}$ variable:

$$\int_{V(t)} F(\underline{x}, t)\, dV \;=\; \int_{V(0)} F^*(\underline{\xi}, t)\, J\, d\xi_1\, d\xi_2\, d\xi_3 \tag{1.191}$$

where Jacobian J is written as

$$J = \frac{\partial(x_1,\, x_2,\, x_3)}{\partial(\xi_1,\, \xi_2,\, \xi_3)} = \epsilon_{ijk}\, \frac{\partial x_1}{\partial \xi_i}\, \frac{\partial x_2}{\partial \xi_j}\, \frac{\partial x_3}{\partial \xi_k} \tag{1.192}$$

and the integrand

$$F^*(\underline{\xi}, t) = F\left\{\underline{x}(\underline{\xi}, t), t\right\} \tag{1.193}$$

Hence

$$\frac{d}{dt}\int_{V(0)} F^*(\underline{\xi}, t)\, J\, d\xi_1\, d\xi_2\, d\xi_3 \;=\; \int_{V(0)} \left(\frac{\partial F^*}{\partial t} J + F^* \frac{\partial J}{\partial t}\right)\, d\xi_1\, d\xi_2\, d\xi_3 \tag{1.194}$$

Now

$$\frac{\partial J}{\partial t} = \epsilon_{ijk}\, \frac{\partial}{\partial t}\left(\frac{\partial x_1}{\partial \xi_i}\, \frac{\partial x_2}{\partial \xi_j}\, \frac{\partial x_3}{\partial \xi_k}\right) \tag{1.195}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial x_1}{\partial \xi_i}\right) = \frac{\partial}{\partial \xi_i}\left(\frac{\partial x_1}{\partial t}\right) = \frac{\partial v_1}{\partial \xi_i} \tag{1.196}$$

If $v_1 = v_1(x_1,\, x_2,\, x_3)$,

$$\frac{\partial v_1}{\partial \xi_i} = \frac{\partial v_1}{\partial x_j}\, \frac{\partial x_j}{\partial \xi_i} \tag{1.197}$$

Since $\epsilon_{ijk}\, \dfrac{\partial v_1}{\partial x_2}\, \dfrac{\partial x_2}{\partial \xi_i}\, \dfrac{\partial x_2}{\partial \xi_j} \dfrac{\partial x_3}{\partial \xi_k}$ and similar terms are zero, the non-zero terms

$$\epsilon_{ijk}\left(\frac{\partial v_1}{\partial x_1}\, \frac{\partial x_1}{\partial \xi_i}\, \frac{\partial x_2}{\partial \xi_j}\, \frac{\partial x_3}{\partial \xi_k} + \frac{\partial v_2}{\partial x_2}\, \frac{\partial x_1}{\partial \xi_i}\, \frac{\partial x_2}{\partial \xi_j}\, \frac{\partial x_3}{\partial \xi_k} + \frac{\partial v_3}{\partial x_3}\, \frac{\partial x_1}{\partial \xi_i}\, \frac{\partial x_2}{\partial \xi_j}\, \frac{\partial x_3}{\partial \xi_k}\right) \tag{1.198}$$

remain. So

$$\frac{\partial J}{\partial t} = (\nabla \cdot \underline{v})\, J \tag{1.199}$$

where $\underline{v}$ is the velocity of the point $\underline{x}$. Hence

$$\int_{V(0)} \left(\frac{\partial F^*}{\partial t} + F^* \, \nabla \cdot \underline{v} \right) J \, d\xi_1 \, d\xi_2 \, d\xi_3 = \int_{V(t)} \left[\left(\frac{\partial F^*}{\partial t} \right)_{\underline{\xi}=const} + F^* \, \nabla \cdot \underline{v} \right] dV \tag{1.200}$$

Now

$$\left. \frac{\partial F^*}{\partial t} \right|_{(\underline{\xi}=const)} = \left. \frac{F \left\{ \underline{x}(\underline{\xi}, t), t \right\}}{\partial t} \right|_{\underline{\xi}} = \frac{\partial F}{\partial t} + \frac{\partial \underline{x}}{\partial t} \cdot \nabla F = \frac{\partial F}{\partial t} + \underline{v} \cdot \nabla F \tag{1.201}$$

Hence

$$\frac{d}{dt} \int_{V(t)} F \, dV = \int_V \left[\frac{\partial F}{\partial t} + \nabla \cdot (\underline{v} \, F) \right] dV \tag{1.202}$$

or

$$\boxed{\frac{d}{dt} \int_{V(t)} F \, dV = \int_{V(t)} \frac{\partial F}{\partial t} \, dV + \oint_{S(t)} \underline{n} \cdot (\underline{v} \, F) \, dS} \tag{1.203}$$

We can apply this relation at any instant in time.

The first integral implies rate of change in volume and the second one rate of change associated with motion of surface bounding volume. [16] It is noted that this is similar to Leibnitz's rule for an integral over one dimensional region:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t) \, dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} \, dx + f[b(t), t] \, b'(t) - f[a(t), t] \, a'(t) \tag{1.204}$$

We can apply extensively the transport theorem to the case that there is a discontinuity interface Σ within a volume V. [17] The volume V is considered to be composed of two volumes V_1 and V_2 divided by an internal surface Σ. V is a material volume but as Σ moves with arbitrary velocity $\underline{u}$ and across it F suffers a discontinuity, F_1 and F_2 being its values on either side. If $\underline{\nu}$ is the normal to Σ in the direction form V_1 to V_2, Eq. (1.203) may be generalized to

$$\frac{d}{dt} \int_{V(t)} F \, dV = \int_{V(t)} \frac{\partial F}{\partial t} \, dV + \oint_{S(t)} \underline{n} \cdot (\underline{v} \, F) \, dS + \oint_{\Sigma(t)} \underline{\nu} \cdot (\underline{u} \, F) \, dS \tag{1.205}$$

[16] See Newman, J. N. (1977), *Marine Hydrodynamics*, MIT Press, for depicted interpretation.

[17] Refer to Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 86.

1.7 Moving Coordinate Systems

1.7.1 Velocity due to rigid body rotation

Suppose a rigid body rotates about an axis through the origin of a coordinate system with an angular velocity $\underline{\omega} = \omega\,\underline{n}$, where the direction of the axis is given by a unit vector $\underline{n}$ and ω is the magnitude of the angular velocity (see Figure **1.5**). [18]

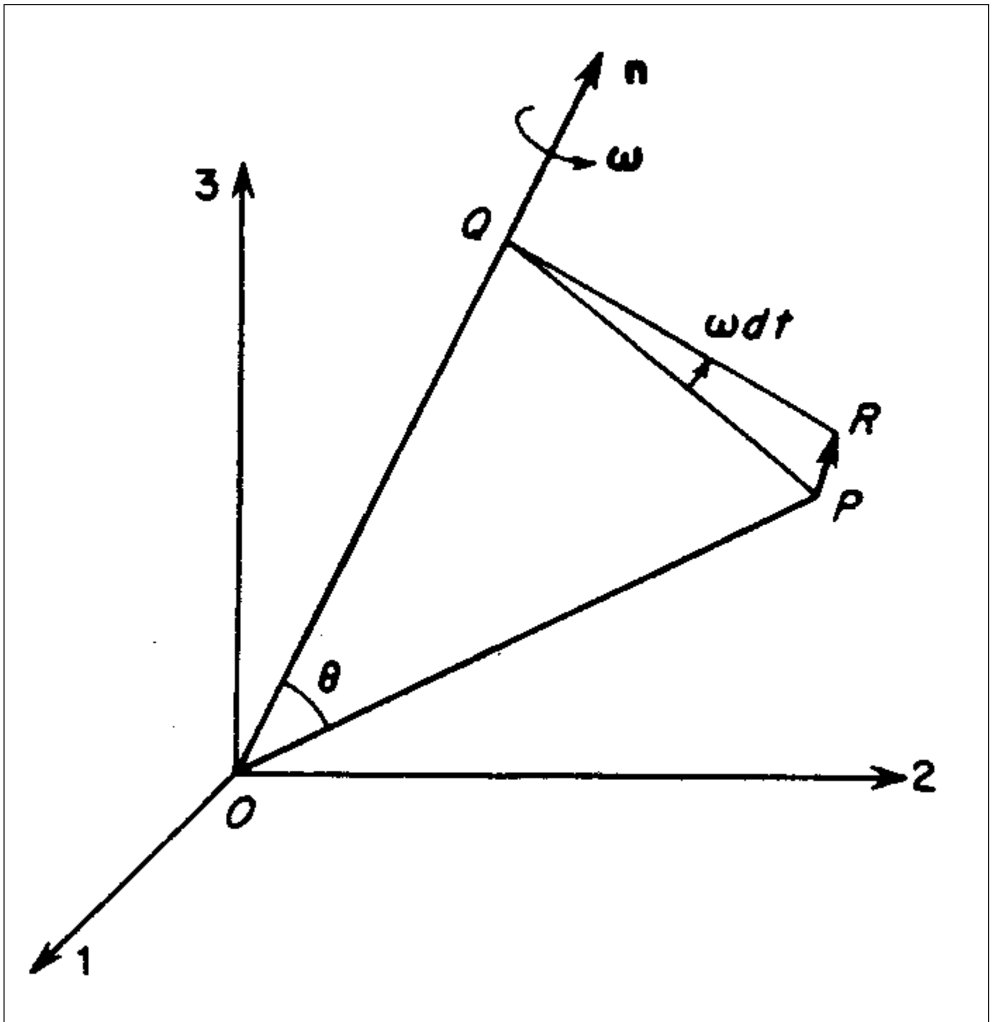

Figure 1.5 Rotation of a rigid body. From Aris (1962), p. 17.

Let P be any point in the body at position $\underline{x}$. Then $\underline{n} \times \underline{x}$ is a vector in the direction of PR of which magnitude is $|\underline{x}| \sin\theta$. However, $|\underline{x}| \sin\theta = PQ$ is the

[18]The description herein is based on Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 17.

perpendicular distance from P to the axis of rotation. In a small interval of time δt, the radius PQ moves through an angle $\omega\ \delta t$ and hence P moves through a distance $(PQ)\,\omega\,\delta t$.

It follows that the small short distance PR is a vector $\delta \underline{x}$ perpendicular to the plane of OP and the axis of rotation:

$$\delta \underline{x} = (\underline{n} \times \underline{x})\,\omega\,\delta t = (\underline{\omega} \times \underline{x})\,\delta t \tag{1.206}$$

Dividing both sides by δt and taking the limit $\delta t \to 0$ provide the velocity of the point P. Thus the linear velocity $\underline{v}$ of the point $\underline{x}$ due to a rotation $\underline{\omega}$ is

$$\underline{v} = \underline{\omega} \times \underline{x} \tag{1.207}$$

This result can be directly applied to moving coordinate systems. Details are given in the following subsection.

1.7.2 Transformations of moving coordinates

Let us introduce two coordinate systems: one system fixed to space and the other moving relative to the space-fixed system. The moving (the unprimed) coordinate system is supposed to be in motion of both translation and rotation relative to the space-fixed (the unprimed) system. Then the position vector $\underline{x}'$ defined in the space-fixed system is related to the position vector $\underline{x}$ defined in the moving system as follows:

$$\underline{x}' = \underline{x} + \underline{R} \tag{1.208}$$

where $\underline{R}$ is the distance vector between the origins of two coordinate systems. (See figure **1.6**)

Because of the relative motion, time-derivative will appear different to observers in the two coordinate systems. For example, a vector that is constant in either system would seem to vary with time to an observer fixed in the other system. We can write the relationship between the derivative(d'/dt) observed in the space-fixed system and the derivative(d/dt) observed in the moving system,

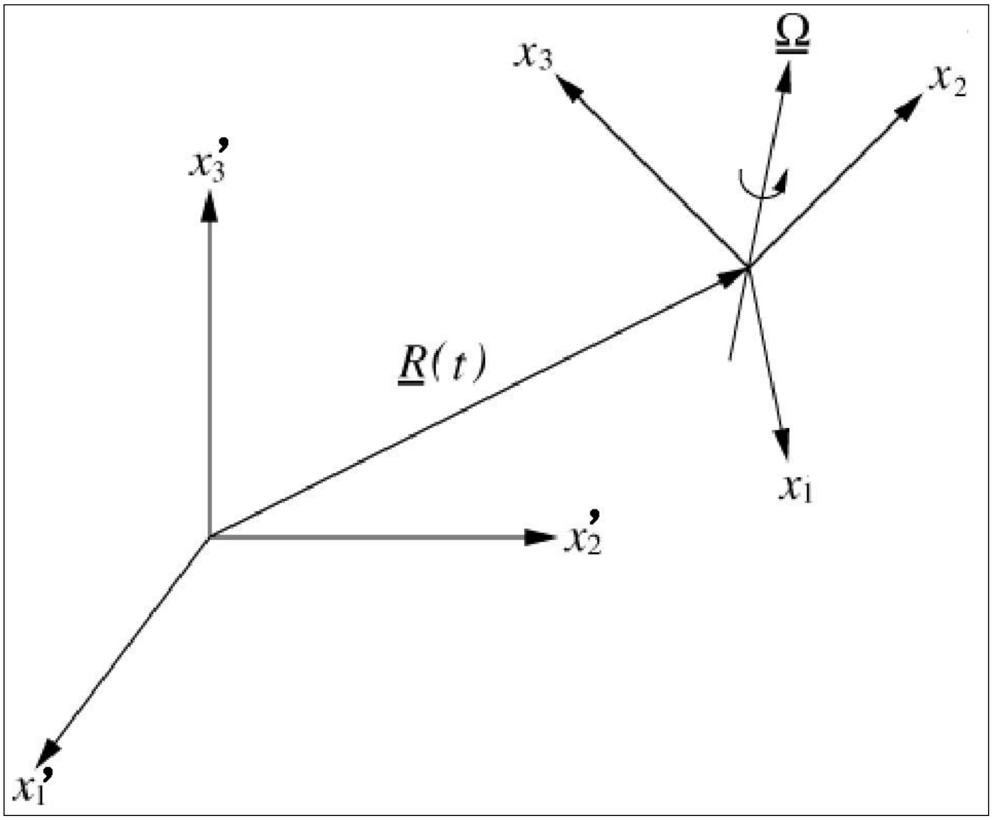

Figure 1.6 Moving coordinate system.

for an arbitrary vector:

$$\frac{d\underline{A}'}{dt} = \frac{d\underline{A}}{dt} + \underline{\Omega} \times \underline{A} \tag{1.209}$$

where $\underline{\Omega}$ is the vector angular velocity of the moving system. The last term in Eq. (1.209) implies a rotation of a rigid body. [19]

If this formula is applied to the special case of the position vector $\underline{x}$ given in Eq. (1.208), we have the velocity:

$$\underline{q}' = \underline{q} + \underline{\Omega} \times \underline{x} + \underline{\dot{R}} \tag{1.210}$$

where $\underline{\dot{R}}$ represents the translation velocity of the moving frame. Therefore this equation implies that the absolute velocity is the sum of the velocity($\underline{q}$) measured by an observer in the moving system and the frame velocity of the moving system ($\underline{\Omega} \times \underline{x} + \underline{\dot{R}}$).

[19]See 김 형 종 (1999), 미적분학, 총 2권, 서울대학교 출판부, pp. 317–318, and Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 17.

In a similar manner, we can obtain the relation between acceleration vectors by making use of the general rule Eq. (1.209):

$$\underline{a}' \equiv \frac{d'^2\underline{x}'}{dt^2} = \underline{a} + 2\,\underline{\Omega} \times \underline{q} + \frac{d\underline{\Omega}}{dt} \times \underline{x} + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) + \ddot{\underline{R}} \tag{1.211}$$

Here we have written $d\underline{\Omega}/dt$ instead of $d'\underline{\Omega}/dt$ because $\underline{\Omega}$ is a vector that is always the same in both systems.

The first term of Eq. (1.211) ($\underline{a}$) is the acceleration viewed in the moving system. The second is the *Coriolis acceleration*, which depends on the velocity in the moving system. The meaning of the third term is not clear. The fourth term is the generalized centripetal acceleration, since

$$|\underline{\Omega} \times (\underline{\Omega} \times \underline{x})| = \Omega^2\, \underline{x}\, \sin(\underline{\Omega}, \underline{x}) \tag{1.212}$$

It is noted that, if we consider the self-rotation of earth with constant angular speed, this term becomes a form of gradient of a scalar function and its effect was already included in gravitational acceleration for treatment as a body force term of the momentum equations.

1.8 Mathematical Identities

1.8.1 Green's scalar identity

If $\underline{u} = \psi\nabla\phi$ in Eq. (1.90), we obtain Green's first identity:

$$\int_V \left[\psi\nabla^2\phi + \nabla\psi \cdot \nabla\phi\right]\, dV \;=\; \oint_S \psi\, \underline{n} \cdot \nabla\phi\, dS \tag{1.213}$$

And if $\underline{u} = \phi\nabla\psi$, use Eq. (1.90) and add the result to Green's first identity, we obtain Green's second(scalar) identity:

$$\int_V \left[\psi\nabla^2\phi - \phi\nabla^2\psi\right]\, dV \;=\; \oint_S \left[\psi\frac{\partial\phi}{\partial n} - \phi\frac{\partial\psi}{\partial n}\right]\, dS \tag{1.214}$$

where $\underline{n} \cdot \nabla\phi = \dfrac{\partial \phi}{\partial n}$. For these relations to be valid, ϕ and ψ must be continuous in the volume and on the surface and the second derivatives must be continuous within the volume while on the surface only the first derivatives need be continuous. [20]

As an practical application, an arbitrary scalar field defined in a volume V can be represented in terms of integrals over the enclosing surfaces plus an integral of $\nabla^2\phi$ over the volume.
From the expansion formulas, we see that $\dfrac{1}{|\underline{x}|} = \dfrac{1}{|\underline{r}|} = \dfrac{1}{r}$ satisfies Laplace's equation: $\nabla^2\left(\dfrac{1}{r}\right) = 0$ if $\underline{r} \neq 0$. Similarly $\nabla^2\left(\dfrac{1}{|\underline{y}-\underline{x}|}\right) = 0$ for $\underline{y}$ a constant vector. Since $\nabla^2\left(\dfrac{1}{|\underline{y}-\underline{x}|}\right)$ does not exist at $\underline{x} = \underline{y}$, we exclude this point from the volume by surrounding it with a sphere.

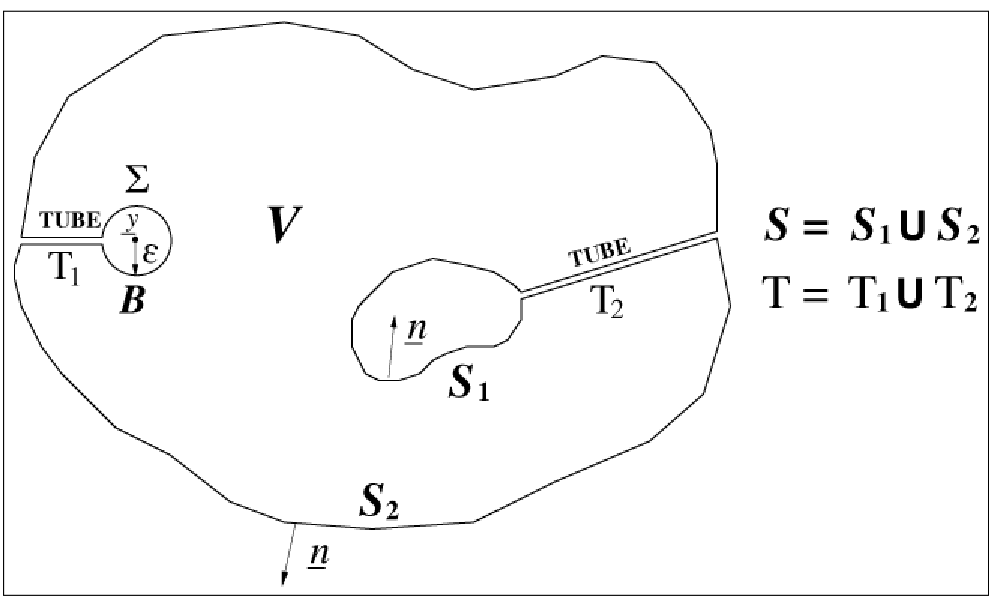

Figure 1.7 Two-dimensional drawing of a simply connected region for deriving the scalar identity.

[20] More detailed explanation can be found in mathematical texts, e.g., Kreyszig, E. (1993), *Advanced Engineering Mathematics*, Seventh ed.,Wiley, pp. 553–554.

Hence if we take $\psi = \dfrac{1}{|\underline{y}-\underline{x}|}$, Green's second identity becomes:

$$\oint_{S+T+\sum(\underline{y},\epsilon)} \left[\frac{\underline{n}\cdot\nabla\phi}{|\underline{y}-\underline{x}|} - \phi\,\underline{n}\cdot\nabla\frac{1}{|\underline{y}-\underline{x}|}\right] dS = \int_{V-B(\underline{y},\epsilon)} \left[\frac{1}{|\underline{y}-\underline{x}|}\nabla^2\phi\right] dV \tag{1.215}$$

where $B(\underline{y},\epsilon)$ is a sphere of radius ϵ centered at $\underline{y}$ and bounded by Σ.

In this application, the surface is in three-dimensional space and the integration variable is $\underline{x}$. We illustrate the situation with a two-dimensional drawing as shown in Figure 1.7 . Integrations over the small tubes joining Σ and S_2, and S_1 and S_2 vanish by continuity of ϕ.

On the surface Σ surrounding the point $\underline{y}$, as shown in Figure 1.8 for an enlarged view, we have

$$\begin{aligned}
\underline{y}-\underline{x} &= -\epsilon\,\underline{e}_r && (1.216)\\
\underline{n} &= -\underline{e}_r && (1.217)\\
dS &= (\epsilon\,d\theta)\,(\epsilon\,\sin\theta\,d\alpha) && (1.218)\\
\phi(\underline{x}) &= \phi(\underline{y}) + \epsilon\left.\frac{\partial\phi}{\partial r}\right|_{\underline{y}} + \cdots && (1.219)\\
\nabla\frac{1}{|\underline{y}-\underline{x}|} &= \frac{(\underline{y}-\underline{x})}{|\underline{x}-\underline{y}|^3} = -\frac{\epsilon\,\underline{e}_r}{\epsilon^3} && (1.220)
\end{aligned}$$

where $\underline{e}_r$ is the unit vector in the radial direction. Hence the integration for the surface Σ and the small ball B becomes, respectively,

$$\begin{aligned}
&\oint_{\Sigma}\left[\frac{\underline{n}\cdot\nabla\phi}{|\underline{y}-\underline{x}|} - \phi\,\underline{n}\cdot\nabla\frac{1}{|\underline{y}-\underline{x}|}\right] dS\\
&= -\phi(\underline{y})\int_0^{2\pi} d\alpha\int_0^{\pi}\left[\epsilon^2\,\frac{\underline{e}_r\cdot(\epsilon\,\underline{e}_r)}{\epsilon^3}\sin\theta\right] d\theta + O(\epsilon)\\
&= -4\pi\,\phi(\underline{y}) + O(\epsilon)
\end{aligned} \tag{1.221}$$

and

$$\int_B\left[\nabla^2\phi\,\frac{1}{|\underline{y}-\underline{x}|}\right] dV = \left.\nabla^2\phi\right|_{\underline{y}}\,\left(O(\epsilon^2)\right) \tag{1.222}$$

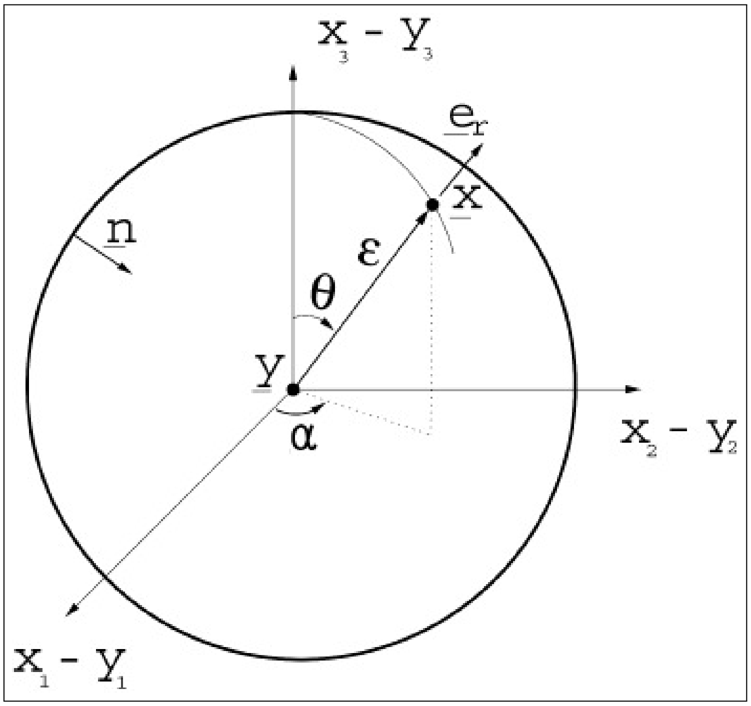

Figure 1.8 Small sphere region containing a singular point.

Hence, taking the limit as $\epsilon \to 0$, we find

$$\boxed{\phi(\underline{y}) = \frac{1}{4\pi}\oint_S \left[\frac{\underline{n}\cdot\nabla\phi}{|\underline{y}-\underline{x}|} - \phi\,\frac{\underline{n}\cdot(\underline{y}-\underline{x})}{|\underline{y}-\underline{x}|^3}\right] dS - \frac{1}{4\pi}\int_V \frac{\nabla^2\phi}{|\underline{y}-\underline{x}|}\,dV} \quad (1.223)$$

If the point $\underline{y}$ had been outside V, the left-hand side would have been zero.

For a two-dimensional field, $\psi = \ln\dfrac{1}{\sqrt{x_1^2+x_2^2}}$ in Green's second identity and a similar expression is obtained.

1.8.2 Uniqueness of scalar identity

Let us consider the uniqueness of this integral representation. If another scalar field, say $\phi'(\underline{x})$ had the same value of $\nabla^2\phi$ in V and the same value of ϕ or $\underline{n}\cdot\nabla\phi$ on S, then we could construct a third solution which had $\nabla^2\phi'' = 0$ in V, and either $\phi'' = 0$ or $\underline{n}\cdot\nabla\phi''$ on S. If $\phi = \psi = \phi''$ in Green's first identity, then

$$\int_V \left[\phi'' \nabla^2 \phi'' + \nabla \phi'' \cdot \nabla \phi'' \right] dV = \oint_S \left[\phi'' \underline{n} \cdot \nabla \phi'' \right] dS \tag{1.224}$$

and this reduces to only

$$\int_V \nabla \phi'' \cdot \nabla \phi'' \, dV = 0 \tag{1.225}$$

Since $(\nabla \phi)^2$ is always greater than or equal to zero, the only solution is

$$\nabla \phi'' \cdot \nabla \phi'' = 0 \tag{1.226}$$

This requires that ϕ'' be at most a constant. If ϕ were specified on the boundary, the constant is zero. If $\underline{n} \cdot \nabla \phi$ is specified on the boundary, ϕ is uniquely determined by the integral to within a constant. It is important to recognize that our expression for ϕ is in terms of ϕ and $\underline{n} \cdot \nabla \phi$ and the above consideration shows we need specify only one of these on the boundary. Hence to find the unknown on the boundary, one must first solve an integral equation.

Also we have assumed that the field boundaries are fixed. If they were to depend on the field, then special conditions must be specified to insure the solution is unique. In addition to this uniqueness, we should also consider the far-field behavior of ϕ as the distance r goes to infinity. [21]

1.8.3 Type of boundary conditions

(1) Dirichlet boundary condition (1st type)
The Dirichlet (or first type) boundary condition is perhaps the easiest one to understand. When we solve a differential equation, we put specified values on the boundary of the domain where a solution needs to take. For example, when Poisson equation such as $\nabla^2 \psi = -\omega$ for stream function ψ and vorticity ω is satisfied in a domain Ω, the Dirichlet boundary condition

[21] Detailed consideration may be found in Batchelor, G. K. (1967), *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge.

takes the form $\psi(\underline{x}) = f(\underline{x})$ on the boundary $\partial\Omega$, where $f(\underline{x})$ is a known function defined on the boundary.

(2) Neumann boundary condition (2nd type)
The Neumann (or second type) boundary condition specifies the values that the derivative of a solution is to take on the boundary of the domain, when imposed on an ordinary or a partial differential equation. For example, for Laplace equation $\nabla^2\phi = 0$ which we will present later on, the Neumann boundary condition takes the form $\dfrac{\partial\phi(\underline{x})}{\partial n} = g(\underline{x})$. Here, n denotes the (typically exterior) normal to the boundary and g is a given scalar function.

(3) Robin boundary condition (3rd type)
The Robin (or third type) boundary condition is a type of hybrid boundary condition; it is a linear combination of Dirichlet and Neumann boundary conditions, namely, it is a specification of a linear combination of the values of a function and the values of its derivative on the boundary of the domain. Robin boundary conditions are a weighted combination of Dirichlet boundary conditions and Neumann boundary conditions, such as $a\,\phi + b\,\dfrac{\partial\phi}{\partial n} = h(\underline{x})$ where a and b are non-zero constants or functions more generally. Robin boundary conditions are commonly used in solving Sturm-Liouville problems (Stakgold, 1986). These boundary conditions should not be confused with mixed boundary conditions, which are boundary conditions of different types specified on different subsets of the boundary.

(4) Mixed boundary condition
The mixed boundary condition for a partial differential equation implies that different types of boundary condition are used on different parts of the boundary. For example, in partial sheet cavity problems for a hydrofoil, if ϕ is a solution to Laplace equation on a fluid domain and the boundary is divided into two portions of cavity and non-cavity, one would impose a Dirichlet boundary condition on the cavity portion and a Neumann boundary condition on the non-cavity portion.

(5) Cauchy boundary condition

A Cauchy boundary condition imposed on an ordinary differential equation or a partial differential equation specifies both the values a solution of a differential equation is to take on the boundary of the domain and the normal derivative at the boundary. It corresponds to imposing both a Dirichlet and a Neumann boundary condition.

Cauchy boundary conditions can be understood from the theory of second order, ordinary differential equations, where to have a particular solution one has to specify the value of the function and the value of the derivative at a given initial or boundary point.

For a second order partial differential equation, we now need to know the value of the function at the boundary, and its normal derivative in order to solve the partial differential equation.

When the variable is specially time, Cauchy conditions can also be called initial value conditions.

1.8.4 Vector identity

Another identity involving vectors can be constructed from divergence theorems for a vector and a dyadic. In the third divergence theorem given by Eq. (1.91), let the vector be $\underline{u} \times \underline{v}$, then

$$\int_V \left[\nabla \times (\underline{u} \times \underline{v})\right] dV = \oint_S \left[\underline{n} \times (\underline{u} \times \underline{v})\right] dS \tag{1.227}$$

According to the expansion formula on vector triple products, we know

$$\begin{aligned} \underline{n} \times (\underline{u} \times \underline{v}) &= (\underline{n} \times \underline{u}) \times \underline{v} + (\underline{v} \times \underline{n}) \times \underline{u} \\ &= (\underline{n} \times \underline{u}) \times \underline{v} - \underline{v}\,(\underline{n} \cdot \underline{u}) + \underline{n}\,(\underline{u} \cdot \underline{v}) \end{aligned} \tag{1.228}$$

Hence

$$\oint_S \left[\underline{n} \times (\underline{u} \times \underline{v})\right] dS = \oint_S \left[(\underline{n} \times \underline{u}) \times \underline{v} - (\underline{n} \cdot \underline{u})\,\underline{v} + \underline{n}\,(\underline{u} \cdot \underline{v})\right] dS \tag{1.229}$$

These integrals can be rearranged by the divergence theorem:

$$\oint_S [(\underline{n} \times \underline{u}) \times \underline{v}]\, dS = \int_V [\nabla \times (\underline{u} \times \underline{v}) + \nabla \cdot (\underline{u}\, \underline{v}) - \nabla(\underline{u} \cdot \underline{v})]\, dV \tag{1.230}$$

Now adding the results of the divergence theorem for a dyadic $\underline{u}\,\underline{v}$ to both sides:

$$\begin{aligned} &\oint_S (\underline{n} \times \underline{u}) \times \underline{v} + (\underline{n} \cdot \underline{u})\, \underline{v}]\, dS \\ &= \int_V [\nabla \times (\underline{u} \times \underline{v}) + 2\, \underline{v}\, (\nabla \cdot \underline{u}) + 2\, \underline{u}\, \cdot \nabla \underline{v} - \nabla(\underline{u} \cdot \underline{v})]\, dV \end{aligned} \tag{1.231}$$

Using the expansion formulas

$$\nabla \times (\underline{u} \times \underline{v}) = \underline{v} \cdot \nabla \underline{u} + \underline{u}\, (\nabla \cdot \underline{v}) - \underline{v}\, (\nabla \cdot \underline{u}) - \underline{u} \cdot \nabla \underline{v} \tag{1.232}$$

$$\nabla(\underline{u} \cdot \underline{v}) = \underline{v} \cdot \nabla \underline{u} + \underline{v} \times (\nabla \times \underline{u}) + \underline{u} \times (\nabla \times \underline{v}) + \underline{u} \cdot \nabla \underline{v} \tag{1.233}$$

and subtracting one from the other, we obtain

$$\begin{aligned} \nabla \times (\underline{u} \times \underline{v}) - \nabla(\underline{u} \cdot \underline{v}) \;&=\; \underline{u}\, (\nabla \cdot \underline{v}) - \underline{v}\, (\nabla \cdot \underline{u}) - 2\, \underline{u} \cdot \nabla \underline{v} \\ &\quad - \underline{v} \times (\nabla \times \underline{u}) - \underline{u} \times (\nabla \times \underline{v}) \end{aligned} \tag{1.234}$$

Hence

$$\begin{aligned} &\oint_S [(\underline{n} \times \underline{u}) \times \underline{v} + (\underline{n} \cdot \underline{u})\, \underline{v}]\, dS \\ = &\int_V [\underline{v}\, (\nabla \cdot \underline{u}) + \underline{u}\, (\nabla \cdot \underline{v}) - \underline{u} \times (\nabla \times \underline{v}) - \underline{v} \times (\nabla \times \underline{u})]\, dV \end{aligned} \tag{1.235}$$

This is called <u>vector identity</u>.

An arbitrary vector field can be represented by this vector identity by choosing

$$\underline{v} = \nabla \frac{1}{|\underline{y} - \underline{x}|} = \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \quad \text{for } \underline{y} \text{ fixed} \tag{1.236}$$

For which, we have

$$\left.\begin{aligned} \nabla \times \underline{v} &= 0 \\ \nabla \cdot \underline{v} &= 0 \end{aligned}\right\} \quad \text{for } \underline{x} \neq \underline{y} \tag{1.237}$$

Hence for $\underline{y}$ not in V, Eq. (1.235) becomes, without any restriction,

$$\begin{aligned} &\oint_S \left[(\underline{n} \times \underline{u}) \times \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} + (\underline{n} \cdot \underline{u}) \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \right] dS \\ &= \int_V \left[\frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} (\nabla \cdot \underline{u}) - \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \times (\nabla \times \underline{u}) \right] dV \end{aligned} \tag{1.238}$$

For the case when $\underline{y}$ is in V, $\left(\frac{1}{|\underline{y} - \underline{x}|} \right)$ becomes singular as $\underline{y}$ tends to $\underline{x}$. The point $\underline{y}$ can be excluded from V by surrounding it with a sphere of radius ϵ centered at $\underline{y}$, as shown in Fig. **1.7** . This sphere plus any other surfaces inside V can be connected to the exterior surface by small tubes to make all the surfaces continuous and the region remains simply connected, in the same manner as for the scalar identity.

The vector identity applies to the region V as defined with the exclusions:

$$\begin{aligned} &\oint\limits_{S+T+\sum(\underline{y},\epsilon)} \left[(\underline{n} \times \underline{u}) \times \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} + (\underline{n} \cdot \underline{u}) \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \right] dS \\ &= \int\limits_{V-B} \left[\frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} (\nabla \cdot \underline{u}) - \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \times (\nabla \times \underline{u}) \right] dV \end{aligned} \tag{1.239}$$

Integrations over the small tubes T_1 and T_2 vanish by continuity as they become increasingly small.

On the surface Σ surrounding the point $\underline{y}$ (see Figure **1.8**):

$$
\begin{aligned}
\underline{y} - \underline{x} &= -\epsilon\, \underline{e}_r && (1.240)\\
\underline{n} &= -\underline{e}_r && (1.241)\\
dS &= (\epsilon\, d\theta)\,(\epsilon\, \sin\theta\, d\phi) && (1.242)\\
\frac{(\underline{y} - \underline{x})}{|\underline{x} - \underline{y}|^3} &= \frac{-\epsilon\, \underline{e}_r}{\epsilon^3} && (1.243)
\end{aligned}
$$

where $\underline{e}_r$ is the unit vector in the radial direction. Furthermore,

$$\underline{u}|_{\sum} = \underline{u}(\underline{y}) + (\underline{x} - \underline{y}) \cdot \nabla \underline{u} + \cdots = \underline{u}(\underline{y}) + O(\epsilon) \qquad (1.244)$$

$$(\underline{n} \times \underline{u}) \times (\underline{y} - \underline{x}) = \epsilon\,(-\underline{e}_r \times \underline{u}) \times (-\underline{e}_r) = \epsilon\,\{\underline{u} - \underline{e}_r(\underline{u} \cdot \underline{e}_r)\} \qquad (1.245)$$

$$(\underline{n} \cdot \underline{u})\,(\underline{y} - \underline{x}) = \epsilon\,(\underline{e}_r \cdot \underline{u})\,\underline{e}_r \qquad (1.246)$$

Hence,

$$
\begin{aligned}
&\oint_{\sum} \left[(\underline{n} \times \underline{u}) \times \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} + (\underline{n} \cdot \underline{u})\, \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \right] dS \\
=\;& \underline{u}(\underline{y}) \int_0^{2\pi} d\alpha \int_0^{\pi} \frac{\epsilon^3 \sin\theta\, d\theta}{\epsilon^3} + O(\epsilon) \\
=\;& 4\pi\, \underline{u}(\underline{y}) + O(\epsilon) \qquad (1.247)
\end{aligned}
$$

And for $\epsilon \to 0$,

$$
\begin{aligned}
4\pi\, \underline{u}(\underline{y}) =& -\oint_S \left[(\underline{n} \times \underline{u}) \times \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} + (\underline{n} \cdot \underline{u})\, \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \right] dS_x \\
+& \lim_{\epsilon \to 0} \int\limits_{V - B(\underline{y}, \epsilon)} \left[\frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} (\nabla \cdot \underline{u}) - \frac{(\underline{y} - \underline{x})}{|\underline{y} - \underline{x}|^3} \times (\nabla \times \underline{u}) \right] dV_x \qquad (1.248)
\end{aligned}
$$

This is a representation of $\underline{u}$ in terms of both components on the boundary, the normal component $\underline{n} \cdot \underline{u}$, and the tangential component, $\underline{n} \times \underline{u}$, plus the divergence and the curl integrated over the field.

If $\underline{u}$ is divided into two components after interchanging the variables $\underline{x}$ and

$\underline{y}$, Eq. (1.248) is rewritten as

$$
\boxed{
\begin{aligned}
4\pi\,\underline{u} &= \underline{u}_1 + \underline{u}_2 && (1.249)\\
\underline{u}_1(\underline{x}) &= +\fint_V \frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^3}(\nabla\cdot\underline{u})\,dV_y - \oint_S (\underline{n}\cdot\underline{u})\frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^3}\,dS_y && (1.250)\\
\underline{u}_2(\underline{x}) &= -\fint_V \frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^3}\times(\nabla\times\underline{u})\,dV_y - \oint_S (\underline{n}\times\underline{u})\times\frac{(\underline{x}-\underline{y})}{|\underline{x}-\underline{y}|^3}\,dS_y && (1.251)
\end{aligned}
}
$$

where the bar through the integral sign indicates the limit integration.

1.8.5 Integral expression of Helmholtz decomposition

For a vector field $\underline{u}$ given in a domain V, we define a vector $\underline{F}$ by

$$\underline{F}(\underline{x}) = -\int_V G(\underline{x}-\underline{y})\,\underline{u}(\underline{y})\,dV_y \tag{1.252}$$

where $G(\underline{r})$ is the fundamental solution (Green function) of Poisson equation

$$\nabla^2 G(\underline{r}) = \delta(\underline{r}). \tag{1.253}$$

For example, $G(\underline{r}) = -\dfrac{1}{4\pi|\underline{r}|}$ in 3-D dimesional free space.

By Eqs. (1.252) and (1.253) and the definition of the Dirac delta function, we have

$$-\nabla^2\underline{F} = -\int_V \delta(\underline{x}-\underline{y})\,\underline{u}(\underline{y})\,dV_y = \underline{u}(\underline{x}) \tag{1.254}$$

According to Eq. (1.72),

$$\underline{u}(\underline{x}) = -\nabla^2\underline{F} = -\nabla(\nabla\cdot\underline{F}) + \nabla\times(\nabla\times\underline{F}) \tag{1.255}$$

By comparing this expression with the Helmholtz decomposition form $\underline{u} = \nabla\phi + \nabla\times\underline{A}$, the scalar and the vector potentials are simply given by

$$\phi = -\nabla\cdot\underline{F}, \quad \underline{A} = \nabla\times\underline{F} \tag{1.256}$$

We can then perform the integration of Eq. (1.252) to yield

$$
\begin{aligned}
\phi = -\nabla \cdot \underline{F} &= \int_V \nabla \cdot \left\{ G(\underline{x} - \underline{y})\, \underline{u}(\underline{y}) \right\} \, dV_y \\
&= \int_V \nabla G(\underline{x} - \underline{y}) \cdot \underline{u}(\underline{y}) \, dV_y \\
&= - \int_V \nabla_y G(\underline{x} - \underline{y}) \cdot \underline{u}(\underline{y}) \, dV_y \\
&= - \int_V \{ \nabla_y \cdot (G\, \underline{u}) - G\, \nabla_y \cdot \underline{u} \} \, dV_y \\
&= - \oint_S G\, \underline{n} \cdot \underline{u} \, dS_y + \int_V G\, \theta \, dV_y
\end{aligned} \tag{1.257}
$$

$$
\begin{aligned}
\underline{A} = \nabla \times \underline{F} &= - \int_V \nabla \times \left\{ G(\underline{x} - \underline{y})\, \underline{u}(\underline{y}) \right\} \, dV_y \\
&= - \int_V \nabla G(\underline{x} - \underline{y}) \times \underline{u}(\underline{y}) \, dV_y \\
&= \int_V \nabla_y G(\underline{x} - \underline{y}) \times \underline{u}(\underline{y}) \, dV_y \\
&= \int_V \{ \nabla_y \times (G\, \underline{u}) - G\, \nabla_y \times \underline{u} \} \, dV_y \\
&= \oint_S G\, \underline{n} \times \underline{u} \, dS_y - \int_V G\, \underline{\omega} \, dV_y
\end{aligned} \tag{1.258}
$$

Here we denote the gradient operator with respect to the integration variables $\underline{y}$ by ∇_y so that $\nabla G = -\nabla_y G$. Equations (1.257) and (1.258) provide the mathematical background of the Helmholtz decomposition for any vector field. Therefore the irrotational vector $\nabla\phi$ and the solenoidal vector $\nabla \times \underline{A}$ can be expressed in terms of dilatation and vorticity, respectively:

$$
\nabla \phi = - \oint_S (\underline{n} \cdot \underline{u}) \, \nabla G \, dS_y + \int_V \theta \, \nabla G \, dV_y \tag{1.259}
$$

$$
\nabla \times \underline{A} = - \oint_S (\underline{n} \times \underline{u}) \times \nabla G \, dS_y + \int_V \underline{\omega} \times \nabla G \, dV_y \tag{1.260}
$$

Note that we have dropped the subscript y in ∇G for brevity, and hence it denotes the operator with respect to the integration variables $\underline{y}$. This result is the same as the expression of the vector identity Eqs. (1.250) and (1.251) derived in the previous subsection.

1.8.6 Green functions

When other surfaces can be included in the problem of the Laplace equation (more generally other partial differential equations, not necessarily the Laplace equation) that governs flow fields, additional boundary conditions are imposed. Then the Green function is often taken instead of the elementary function for computational advantage.

(1) Green fuction is defined as an elementary singularity plus another non-singular component that satisfies Laplace equation as well as boundary contions on the other surfaces.

(2) What is left unsatisfied is boundary conditions on a body.

(3) Scalar (velocity potential) at $\underline{x}$ in terms of a distribution of elementary singularities $\psi = \dfrac{1}{|\underline{x}-\underline{y}|}$. When we add a function (say $H(\underline{x},\underline{y})$) that also satisfies the Laplace equation and is not singular within the field to ψ, identity is unchanged except that we have a modified singularity element. It is necessary but not easy to find a function H with the following properties.

(4) If there were surfaces near a body, construct new singularity element $G(\underline{x},\underline{y})$ with $\nabla^2 G = 0$ and such that

   (a) G satifies given boundary conditions on non-body surfaces

   (b) G contains elementary singularity element (say $\dfrac{1}{|\underline{x}-\underline{y}|}$) to give the field point value $\phi(\underline{x})$

   (c) G results in integral equation over only the body surface.

(5) The formulation is as follows

$$G(\underline{x},\underline{y}) = \frac{1}{|\underline{x}-\underline{y}|} + H(\underline{x},\underline{y}) \tag{1.261}$$

where $H(\underline{x}, \underline{y})$ is non-singular for all $\underline{x} \in V, \quad \nabla^2 H = 0$ and

$$\phi\, \underline{n} \cdot \nabla G - G\, \underline{n} \cdot \nabla \phi = 0 \quad \text{on} \quad \mathrm{S} \neq \mathrm{S_B} \tag{1.262}$$

(6) For a simple example, if a wall is aligned with onset flow, H is image of elementary singularity.

1.8.7 Uniqueness of vector identity

To examine uniqueness of the solution as before, suppose that vectors $\underline{u}_1$ and $\underline{u}_2$ satisfy $\nabla \cdot \underline{u}_1 = \nabla \cdot \underline{u}_2$ and $\nabla \times \underline{u}_1 = \nabla \times \underline{u}_2$ in V. Then the difference vector $\underline{u}_3 = \underline{u}_1 - \underline{u}_2$ satisfies $\nabla \cdot \underline{u}_3 = 0$ and $\nabla \times \underline{u}_3 = 0$ in V. The condition that the curl and divergence of $\underline{u}_3$ are both zero is necessary and sufficient to establish that $\underline{u}$ is the gradient of a scalar function P which satisfies Laplace's equation:

$$\underline{u}_3 = \nabla P \tag{1.263}$$

$$\nabla^2 P = 0 \tag{1.264}$$

Green's first identity, Eq. (1.213),

$$\int_V \left[\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi\right] \, dV = \oint_S \psi\, \underline{n} \cdot \nabla \phi \, dS \tag{1.265}$$

with $\psi = \phi = P$ reduces to

$$\int_V \underline{u}_3 \cdot \underline{u}_3 \, dV = \oint_S P\, \underline{n} \cdot \underline{u}_3 \, dS \tag{1.266}$$

If the normal component of the two solution vector is specified equal on the boundary, then $\underline{n} \cdot \underline{u}_3 = 0$ on S and hence

$$\int_V \underline{u}_3 \cdot \underline{u}_3 \, dV = 0 \tag{1.267}$$

Since $\underline{u}_3 \cdot \underline{u}_3$ is always greater than or equal to zero, the only possible solution is

$$\underline{u}_3 = 0 \tag{1.268}$$

When the boundary condition uniquely defines the normal component of the vector, Eq. (1.249) represents a unique representation of an arbitrary vector and no information need be given about the tangential component of the vector.

1.8.8 Classification of vector fields

We have noted two distinct types of vector field; 'solenoidal' and 'irrotational'. Apart from these types, several other types of fields have been named. [22]

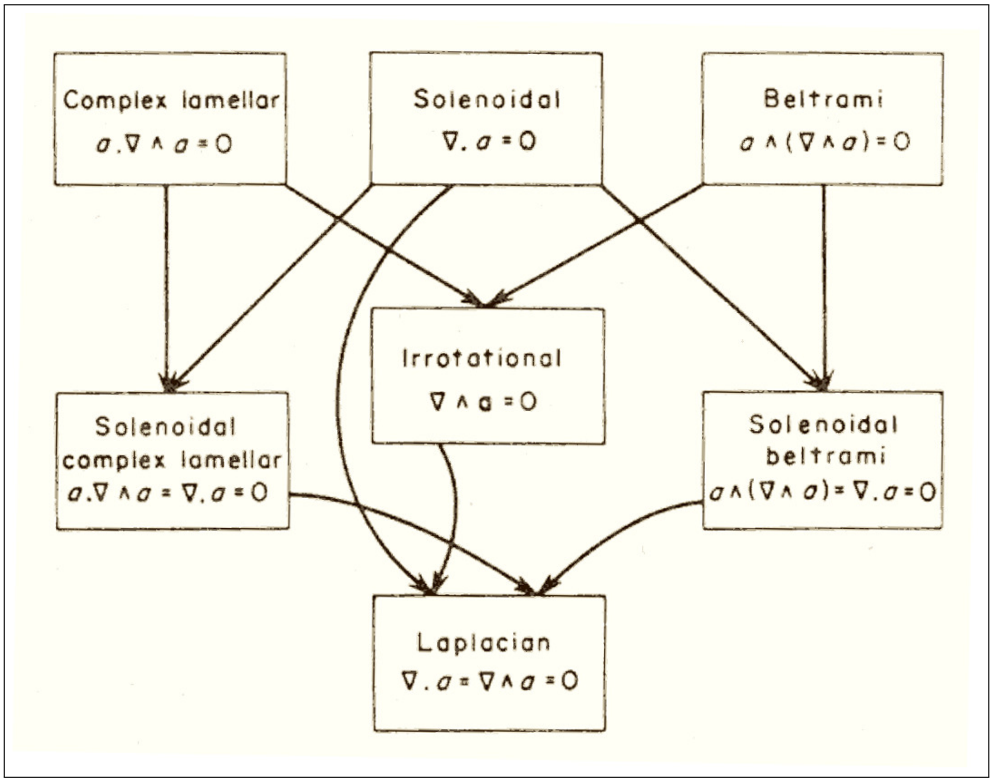

Figure 1.9 Classification of vector fields. From Aris (1962), p. 64.

(1) Laplacian: A field which is both solenoidal and irrotational is called Laplacian. It is the gradient of a potential function. The potential function is taken to be either scalar or vectorial.

(2) Complex lamellar: The condition for a field to be 'complex lamellar' is $\underline{a} \cdot (\nabla \times \underline{a}) = 0$. This field is orthogonal to its curl if $\nabla \times \underline{a} \neq 0$. The

[22] See Aris, R. (1962), *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice Hall, p. 64.

name ‘lamellar’ is also applied to an irrotational vector field $\nabla \times \underline{a} = 0$. The ‘lamellar’ is therefore a special case of the ‘complex lamellar’.

(3) Beltrami: The field is parallel to its curl, i.e, $\underline{a} \times (\nabla \times \underline{a}) = 0$. As a special case, if its curl is proportional to the original vector $\underline{a}$ with a contant (i.e., $\nabla \times \underline{a} = k\,\underline{a}, \quad \nabla k = 0$), it is called ‘Trkalian’.

The relations between these types are shown in a schematic diagram (Figure 1.9). If a field is both a complex lamellar and Beltrami field, it is irrotational if $\underline{a} \neq 0$.

1.9 Improper Integrals

1.9.1 Examples

Most of integrals involved in physics are well defined as a limit of a Rieman sum for which integrand and range of integration are well behaved. [23] Several types of integrals occur in hydrodynamic problems that involve quantities that tend to infinity. Some of these integrals have meaning in the classical mathematical sense that the integral is to be interpreted as a limit process, but a some additional insight is also required. Two general types of integrals are of concern:

(1) those with a range of integration that tends to infinity and

(2) those that have integrands that are singular at points within the range of integration.

As an example of the second type of improper integral, suppose $f(x)$ has singularities at the start of the range and at an intermediate point x_0 within the range of integration, then the definition of the improper integral of $f(x)$ is

$$\int_a^b f(x)\,dx = \lim_{a_1,b_1,c_1 \to 0} \left[\int_{a+a_1}^{x_0-b_1} f(x)\,dx + \int_{x_0+c_1}^{b} f(x)\,dx \right] \qquad (1.269)$$

[23] See, e.g., Kaplan, W. (1952), *Advanced Calculus*, Addison-Wesley.

if it exist.

Such an interpretation of the integral does not always exist. The improper integral $\int_1^\infty \frac{dx}{x}$ has an infinite range of integration but no singularities in the integrand over the range of integration. It is to be interpreted as

$$\lim_{R\to\infty}\left[\int_1^R \frac{dx}{x}\right] = \lim_{R\to\infty}[\ln(R)] \to \infty \tag{1.270}$$

and thus does not produce a finite value. Hence the integral is both improper and unbounded, even the integrand is well behaved over the range of integration.

Meanwhile, the integrand of $\int_0^1 \frac{dx}{\sqrt{x}}$ is singular at $x = 0$. Hence we interpret it as

$$\lim_{\epsilon\to 0}\int_\epsilon^1 \frac{dx}{\sqrt{x}} = \lim_{\epsilon\to 0}\left[2-\sqrt{\epsilon}\right] = 2, \tag{1.271}$$

and hence it exists by construction. We say the integral is convergent improper.

1.9.2 Principal value integrals

There are a class of improper integrals that are fundamental to investigations of the flow about bodies. These are Principal Value Integrals and are defined with some aspect of symmetry relative to the infinities involved. For integrals with infinite limits this may be

$$(P.V.)\int_{-\infty}^{\infty} f(x)\,dx \equiv ⨍_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \tag{1.272}$$

and for integrals with integrands that are singular at points within the range of integration, say at the point x_0 such that $\lim_{x\to x_0} f(x)\to\infty$,

$$(P.V.)\int_a^b f(x)\,dx \equiv ⨍_a^b f(x)\,dx = \lim_{\epsilon\to 0}\left[\int_a^{x_0-\epsilon} f(x)\,dx + \int_{x_0+\epsilon}^b f(x)\,dx\right] \tag{1.273}$$

Some integrals have both an infinite range of integration and singularities in the integrand at some points. They may exist in the principal-value sense by cancelling the positive and negative values around the singularities.

A specific form of an improper integral, with a well-behaved numerator and specific singular denominator, called a Cauchy Principal Value Integral is defined in the same manner. In application, we will arrive at such integrals when a form of the general solution for the flow about a body(obtained with sources, sinks, dipoles or vortices distributed over the body surface) is derived for the case that a field point approaches the body surface. However we will treat the limiting process in such a way that not only is the Cauchy Principal Value Integral obtained but a local contribution from the excluded region is defined.

In the previous section, we have already treated two cases for which an improper integral was evaluated for the representation of scalar and vector field values in terms of surface and volume integrals. For the case with a singular point in the field, a small sphere around the point excluded it from the field. The sphere had constant radius so the integral is a principal-value one with the symmetry appropriate for such integrals. Such an approach is similar to the classical mathematical one in the sense that we saw the integrand had a singular point after we selected the specific form of the function(i.e., Green function satisfying $\nabla^2 G(\underline{x}, \underline{y}) = \delta(\underline{x} - \underline{y})$) [24] to put into an identity and we found a way to define a finite value for the expression obtained. It is, however, worthy to note that exclusion of the singular point for the scalar function $\phi(\underline{y})$ as a principal value in the previous section is not required in potential flow theory. The exclusion need be only as defined for an improper integral. In our later treatment of the values as a field point tends to a surface point, we will find some integrals are principal-value ones and some are simply improper.

[24] The function is of a form $1/r$, which can be also obtained by taking Fourier transform of this equation.