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- 보 텍 스 방 법 -

COMPUTATIONAL MARINE HYDRODYNAMICS

-VORTEX METHODS-

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6

VORTICITY BASED METHODS

6.1 Introduction **238**

6.1.1 Various vortical flows 239

6.1.2 Recent developments 241

6.2 Vorticity-Velocity-Pressure Formulation **246**

6.2.1 Navier-Stokes equations in Helmholz decomposition 246

6.2.2 Vorticity transport equation. 250

6.2.3 Pressure Poisson equation 251

6.2.4 Kinematic boundary condition 252

6.2.5 Dynamic boundary condition. 252

6.2.6 Integral approach of formulation 253

6.2.7 Stream function approach: VIC method 256

6.2.8 Particle method in solving the vorticity transport equation 258

6.2.9 Hydrodynamic Forces 259

6.1 Introduction

In this chapter, we will explain the vorticity-based methods as a tool for the numerical simulation of unsteady incompressible viscous flows. We will deal with various numerical methods based on the vorticity-velocity-pressure formulation for solving the Navier-Stokes equations. Specially, the finite volume method and the vortex particle method are comparatively used for temporal evolution of a vorticity field. The velocity, vorticity and pressure field is calculated in the time marching process.

In general, there are three separate types of approach in the solution procedure for the velocity field:

(a) to use the Biot-Savart integral for a presumably given vorticity field,

(b) to solve directly the kinematic relation between the velocity and the vorticity, and

(c) to solve the Poisson equation for the stream function potential.

In this course, the schemes based on the differential approaches (a) and (c) will be employed. The advantage in employing the integral approach (a) is based on its stability. Integral operators are bounded and smoothing, so that discrete approximations would be stable even if the discretized mesh is refined. The approach (c) corresponds to the VIC (Vortex-In-Cell) method.

The present formulation includes the pressure calculation while most of the existing vorticity-based methods have not treated the pressure field. The main feature of the formulation is the use of an integral approach for obtaining the velocity and pressure fields, in conjunction with a finite volume scheme and the vortex particle method for solving the vorticity transport equation. The integral approach may reflect more easily the global coupling among vorticity, velocity and pressure when imposed the boundary condition for vorticity at a solid surface.

The numerical schemes for computing the vorticity evolution and the integral approach for solving the velocity and the pressure are given in Chapter 7 and

Chapter 8. We will take, as test problems, vorticity dominant flows around a simple geometry such as a circular cylinder, driven cavity and hydrofoil, in which certain special features are apparent, notably concerned with the vorticity distribution on the body surface.

Our numerical schemes could be judged by a comparison with the existing analytical solution and experimental/numerical results provided by other researchers. The demonstrated results indicate that the present integral approach can be incorporated into the finite volume scheme and the Lagrangian vortex method from the viewpoint that the evolution of vorticity in the fluid and on the boundary is accurately predicted and are found to be in good agreement with the comparable solutions.

6.1.1 Various vortical flows

Some vortical flows are natural and essential for movement of fluid (Lugt 1983). The vortical flow behavior at a point in space can be related to a vortex definition. Vorticity is related to the angular velocity of matter at a point in continuum space. Such a vortical motion is composed of a basic mode of motion due to deformation along with rigid motion (Batchelor 1967). Figures **6.1** through **6.3** show typical patterns that may be observed in nature and laboratory.

In some aspects, it is convenient to represent the fluid motion in terms of vorticity together with velocity and pressure. The advantages of the vorticity interpretation and computation rely on the fundamental difference between fluid and solid. Shearing process of fluid at solid surface can be precisely represented by vorticity variable as the skew-symmetric part of the velocity gradient. Moreover, a knowledge of vorticity implies knowing not only the fluid motion at a single spatial point, but also the relation of that motion with those of neighbouring points. Thus, the vorticity reflects the dynamic mechanism of the shearing process more directly than velocity variable (Wu & Wu 1993).

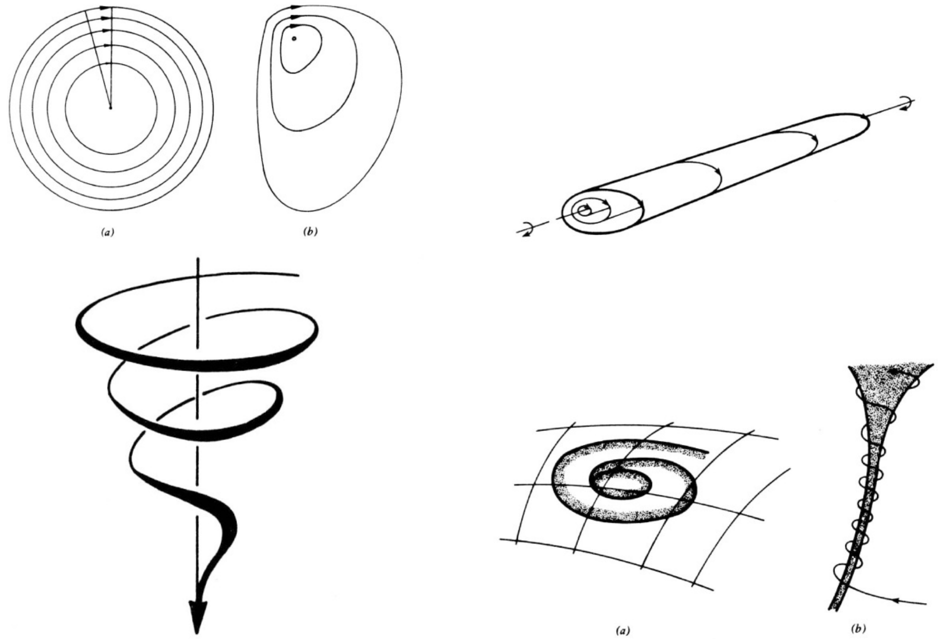

Figure 6.1 Various vortex patterns. Concentric circular vortex and asymmetrical vortex; Cylindrical vortex (perspective view); Spiral vortex; Disk-like and columnar vortices. From Lugt (1983).

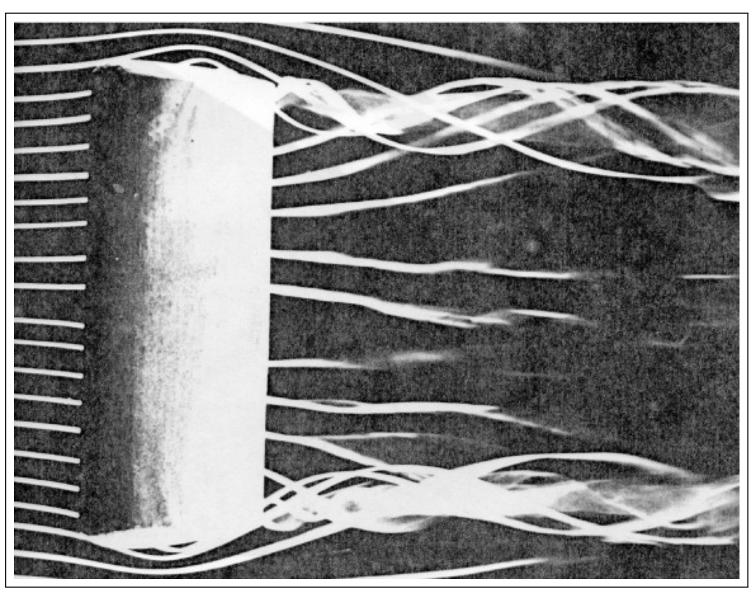

Figure 6.2 Trailing vortices from a rectangular wing. From Van Dyke (1982).

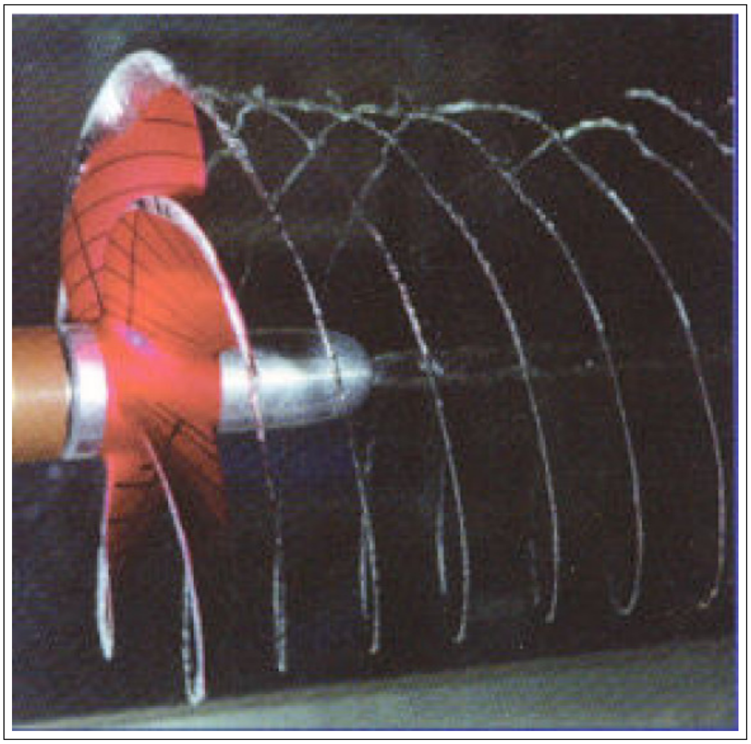

Figure 6.3 Tip vortex cavitation of a marine propeller.

6.1.2 Recent developments

6.1.2.1 CFD modeling

Application of Computational Fluid Dynamics (CFD) might cover the range from the automation of well-established engineering design methods to the use of detailed solutions of the Navier-Stokes (referred to as 'N-S' below) equations as substitutes for experimental research into the nature of complex flows (Ferziger & Perić (1996)). In recent, the advancement in computer hardware technology has made it possible to perform numerical treatment of complex flow fields.

There is yet to be found the most appropriate mathematical formulation of the Navier-Stokes equations to simulate these flows is still open, considering the fact that the choice is strictly dependent on the problem domain and the boundary conditions. For suitable dynamical, spatial and steadiness approximations of Navier-Stokes equations for incompressible viscous flow, there exist so many mathematical models or discretization techniques. The computational procedures are shown in Figure 6.4 .

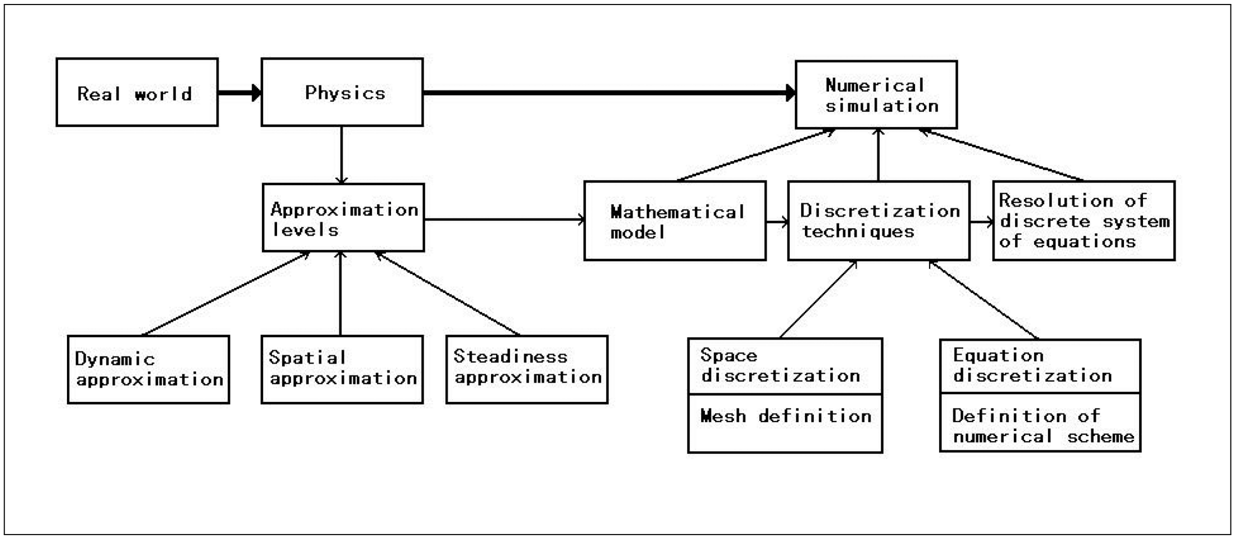

Figure 6.4 Computational procedure for solving Navier - Stokes equations. From Hirsch (1988).

As one candidate for solving Navier-Stokes equations , many researchers have introduced various numerical methods based on the vorticity-velocity formulation. The vorticity-velocity formulation has a few advantages over the primitive variable formulation. A particular numerical algorithm developed for the solution of the vorticity transport equation in an inertial reference frame may be applied to that in a moving frame with correspondingly redefined boundary and initial conditions without any extra consideration of stability problems caused by the additional source terms (Speziale 1987).

6.1.2.2 Physical interpretation

Since the physical interpretation by Lighthill (1963) and Batchelor (1967) of the vorticity dynamics, many researchers have introduced various numerical methods based on the vorticity-velocity formulation for solving the Navier-Stokes equations as an alternative to the primitive variable formulation.

The vorticity-velocity formulation has a few advantages over the primitive variable formulation. The vorticity-velocity formulation is mathematically natural since the inertia force (including the external body force) term in the Navier-Stokes equations can be expressed as a Helmholtz decomposition form. Then, the pressure and the vorticity become a pair of potentials of the inertia force term (Wu & Wu 1993).

In externally attached flow problems where the viscous region occupies only the boundary layer and wake, a computational region for vorticity evolution can be confined to this region of the entire flow field (Wu 976).

Furthermore, the use of the vorticity field may be desirable to understand certain features of established vortical flows. A particular numerical algorithm developed for the solution of the vorticity transport equation in an inertial reference frame may be applied to that in a non-inertial frame with correspondingly redefined boundary and initial conditions without any extra consideration of stability problems caused by the additional source terms (Speziale 1987).

The fluid around a solid body adheres to the body surface at any instant in time. This no-slip characteristics for fluid velocity must produce a proper quantity of vorticity at the surface. This vorticity then enters and is distributed throughout the fluid by convection and diffusion. The production and redistribution of the vorticity is governed by the vorticity transport equation. One of the most difficult problems encountered in the vorticity-velocity formulation is the introduction of the proper value of vorticity or vorticity flux at the solid surface (Gresho 1991).

Mathematical identity for a vector or scalar field is used to define field values of a quantity of interest, which involves an integral of singularities distributed over a surface and over a field. This concept that was well established for the potential flow analysis have been extensively introduced to solve viscous flow problems (see Morino (1990) for general description). This approach has been recognized to accompany a large amount of computational time, not to ensure a reasonable accuracy in numerical implementation.

Anderson (1989) was the first to present dynamic boundary conditions appropriate for the vorticity formulation of the two-dimensional ‘N-S’ equations. The boundary conditions do not reveal the inherent vorticity-pressure coupling due to an additional compatibility condition, implying a special force balance on a solid wall through the N-S equations.

The dynamic mechanism of the viscous shearing process at the solid body surface must be interpreted in terms of the vorticity and the pressure variables together (Wu & Wu 1993). From a different approach, Wu *et al.* (1994) presented a systematic theoretical analysis for these dynamic boundary conditions. They proposed a fully decoupled scheme based on fractional step methods (in which the vorticity transport equation is separated into convection and diffusion equations) applicable for high Reynolds numbers.

6.1.2.3 Vortex particle method

In recent times, great efforts have been made towards solving this problem especially in two-dimensional flow cases by Koumoutsakos & Leonard (1995). In their work, a fractional two-step algorithm is employed in a similar way to the work of Wu *et al.* (1994).

In the first step, discrete point-vortices updated at previous time steps in the time-marching procedure are convected during a time interval ($\triangle t$) via the Biot-Savart integral with smoothed integral kernels (see Figure **6.5**) and their strength is modified based on the scheme of the particle strength exchange scheme. In the second step, a spurious vortex sheet (γ) which is observed on the surface of a body at the end of the first step is computed and related to a vorticity flux (σ) generating from the solid wall in the fluid: $\sigma = \gamma / \triangle t$. In order to reveal dynamical interaction between the vorticity and the pressure, a tangential gradient of the pressure on the right-hand side of this equation should be added.

Essentials of the vortex methods are

(1) one of numerical techniques to solve the N.-S. equations,

(2) suitable simulation for vortical flows,

(3) use of vorticity as a variable,

(4) Lagrangian concept computation,

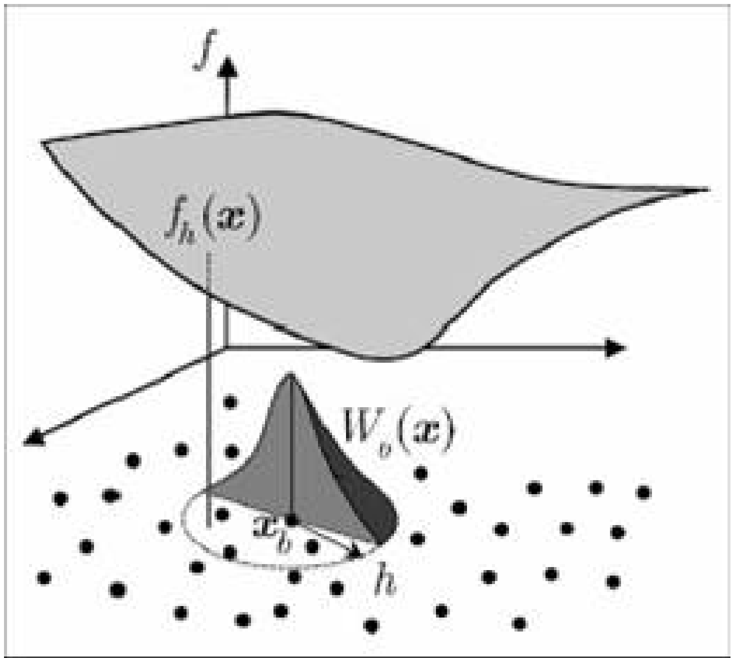

Figure 6.5 Smoothed integral kernel function in particle methods.

(5) confined computational region of non-zero vorticity,

(6) gridless or regular grid system in flow field, and

(7) automatically satisfied far-field boundary condition.

6.1.2.4 Vortex-In-Cell method

The major category of vortex method is distinguished by the scheme of calculation of the velocity field. Generally the vortex method can be divided into grid free method based on the Biot-Savart law (Ploumhans et al (2002)) and vortex-in-cell method where a grid is used for the velocity calculation but particles are used to track the vorticity (Cottet & Poncet (2003)). Vortex-in-cell method has been considered computationally efficient for the evaluation of velocity.

Table 6.1 reproduces the comparison, introduced in Cottet (1999), of the run parameters used for a VIC(Vortex-In-Cell) code and a second order compact finite-difference scheme for 2-D driven cavity flow. The table shows that the VIC method can have economic cost due to the less restrictive time step.

As extentive work, Cottet & Poncet (2003) designed an immersed boundary vortex-in-cell method for the investigation of a cylinder wake. They computed the velocity and the vorticity strain based on grid Poisson solver.

Table 6.1 Comparison of CPU times between vortex-in-cell method and finite difference method for 2-D driven cavity flow for various Reynolds numbers.

Reynolds number	100	2000	10000
N_{FDM}	64	128	256
N_{VIC}	64	128	256
Δt_{FDM}	0.01	0.008	0.004
Δt_{VIC}	0.01	0.02	0.04
$\mathrm{CPUtime}_{\mathrm{FDM}}$	3	24	225
$\mathrm{CPUtime}_{\mathrm{VIC}}$	5	16	32

6.2 Vorticity-Velocity-Pressure Formulation

6.2.1 Navier-Stokes equations in Helmholz decomposition

In Chapter 2, we have described the equations of motion, being a relation between the rate of change of momentum of a material volume of a fluid and all forces acting on that portion of fluid,

$$\frac{d}{dt}\int_V \rho\, \underline{q}\, dV \;=\; \int_V \rho\, \underline{f}\, dV \;+\; \oint_S \underline{\tau}\, dS \tag{6.1}$$

where $\underline{f}$ is the external body force per unit mass of fluid and $\underline{\tau}$ is the stress vector (the surface force per unit area).

For incompressible Newtonian fluid, the stress tensor is related to the pressure and the strain rate linearly.

$$\tau_{ij} = -p\, \delta_{ij} + 2\mu\, D_{ij} \tag{6.2}$$

where

$$D_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right) \tag{6.3}$$

Then, substitution of Eq. (6.2) in Eq. (6.1) gives

$$\rho\frac{Dq_i}{Dt} = \rho\, f_i - \frac{\partial p}{\partial x_i} + \mu\,\frac{\partial^2 q_i}{\partial x_j \partial x_j} \tag{6.4}$$

Eq. (6.4) becomes, in vector notation,

$$\rho\frac{D\underline{q}}{Dt} = \rho\,\underline{f} - \nabla p + \mu\,\nabla^2 \underline{q} \tag{6.5}$$

Alternatively, for an incompressible flow, Eq. (6.2) can be reduced to

$$\tau_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right) + 2\mu\,\frac{\partial q_j}{\partial x_i} \tag{6.6}$$

As represented by the surface integral of Eq. (6.1), the stress vector is derived as

$$\begin{aligned} \underline{\tau} = \tau_{ij}\,n_j &= \left\{-p\,\delta_{ij} + \mu\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right) + 2\mu\,\frac{\partial q_j}{\partial x_i}\right\} n_j \\ &= -p\,\underline{n} + \mu\,\underline{\omega}\times\underline{n} + 2\mu\,(\nabla \underline{q})\cdot\underline{n} \end{aligned} \tag{6.7}$$

The equation of motion Eq. (6.1) then gives

$$\begin{aligned} \int_V \rho\,\frac{D\underline{q}}{Dt}\,dV &= \int_V \rho\,\underline{f}\,dV + \oint_S \left\{-p\,\underline{n} + \mu\,\underline{\omega}\times\underline{n} + 2\mu\,(\nabla\underline{q})\cdot\underline{n}\right\} dS \\ &= \int_V \rho\,\underline{f}\,dV + \int_V \left\{-\nabla p - \nabla\times(\mu\,\underline{\omega})\right\} dV \end{aligned} \tag{6.8}$$

Here we have ignored the contribution of the surface integral $J = \oint_S \nabla\underline{q}\cdot\underline{n}\,dS$ because it becomes zero as outlined below.

Contribution of $J = \oint_S \nabla\underline{q}\cdot\underline{n}\,dS$ in Eq. (6.8)

From the vector expansion,

$$(\underline{n}\times\nabla)\times\underline{q} = \nabla\underline{q}\cdot\underline{n} - (\nabla\cdot\underline{q})\,\underline{n} \tag{6.9}$$

$$J = \oint_S \left\{ (\underline{n} \times \nabla) \times \underline{q} + (\nabla \cdot \underline{q})\, \underline{n} \right\} dS \tag{6.10}$$

For incompressible flow, $\nabla \cdot \underline{q} = 0$. Then, dividing the surface region S into two parts S_u and S_l by introducing a line C,

$$J = \int_{S_u} (\underline{n} \times \nabla) \times \underline{q}\, dS + \int_{S_l} (\underline{n} \times \nabla) \times \underline{q}\, dS \tag{6.11}$$

Use the Stokes theorem for each term,

$$J = \oint_C d\ell \times \underline{q} + \oint_{-C} d\ell \times \underline{q} = 0 \tag{6.12}$$

Thus the contribution of $2\mu\nabla\underline{q} \cdot \underline{n}$ to the surface force becomes zero. ■■■

Consequently, we can introduce the reduced stress vector that has the Helmholtz decomposition form:

$$\underline{\tau}^* = -p\,\underline{n} + \mu\,\underline{\omega} \times \underline{n} \tag{6.13}$$

Figure 6.6 shows the directions of the stresses related to the surface vorticity. Viscous stress makes 45^o with principal axes of strain rate tensor.

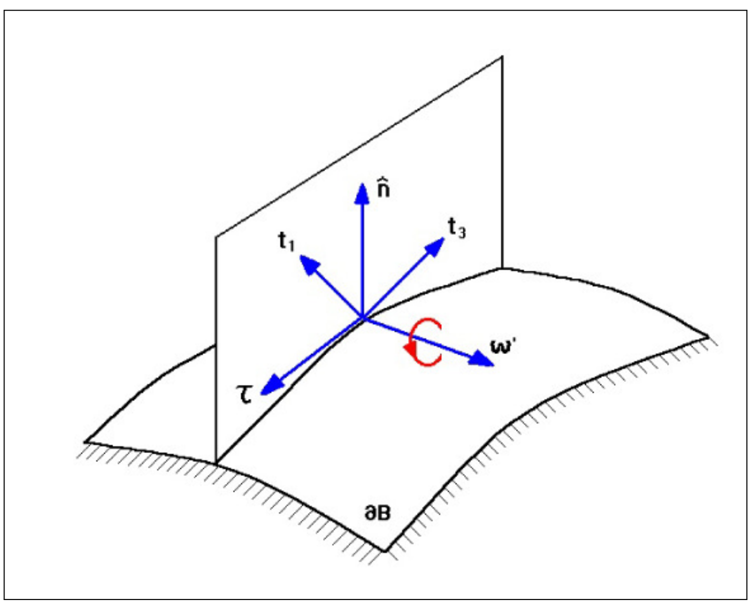

Figure 6.6 Interaction between shearing process and surface vorticity. The principal axes t_1 and t_3 rotate around $\underline{\omega}'$. From Wu *et al.* (1993).

Figure 6.7 shows the erection of the hairpin vortex structures in boundary layers. Vortex stretching (distortion) interacts on vorticity field. Such interaction provides the physiccal background on generation of turbulent flows. Primary hairpin vortex may induce a pressure gradient near the wall surface by which strong secondary or tertiary haipin vortex is ejected and then the multiple breakup of a single turbulent streak occurs.

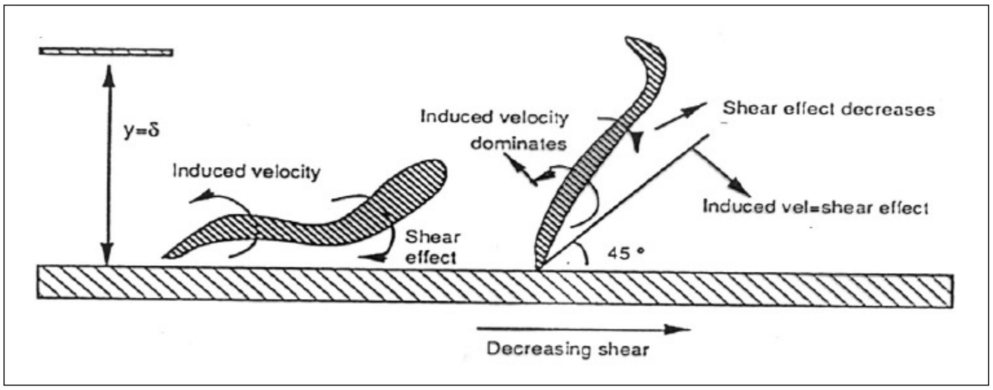

Figure 6.7 Possible effect on the hairpin vortex structures. Adapted from Taylor & Smith (1990).

We also have a natural form of Helmholtz decomposition for the Navier-Stokes equations:

$$\rho \frac{D\underline{q}}{Dt} - \rho \underline{f} = -\nabla p - \nabla \times (\mu\, \underline{\omega}) \tag{6.14}$$

Moreover, the first term in Eq. (6.14) can be rewritten as, by using vector identities:

$$\begin{aligned} \frac{D\underline{q}}{Dt} &= \frac{\partial \underline{q}}{\partial t} + \underline{q} \cdot \nabla \underline{q} \\ &= \frac{\partial \underline{q}}{\partial t} + \nabla \left(\frac{1}{2} \underline{q} \cdot \underline{q} \right) - \underline{q} \times \underline{\omega} \end{aligned} \tag{6.15}$$

According to Eqs. (6.14) and (6.15), the Navier-Stokes equations for an incompressible flow of a Newtonian fluid are written as:

$$\frac{\partial \underline{q}}{\partial t} + \nabla \left(\frac{p}{\rho} + \frac{1}{2} \underline{q} \cdot \underline{q} \right) = \underline{f} + \underline{q} \times \underline{\omega} - \nabla \times (\nu\, \underline{\omega}) \tag{6.16}$$

6.2.2 Vorticity transport equation

The vorticity transport equation is obtained by taking the curl of Eq. (6.16):

$$\frac{\partial \underline{\omega}}{\partial t} = \nabla \times \underline{f} + \nabla \times \left(\underline{q} \times \underline{\omega}\right) - \nu \nabla \times (\nabla \times \underline{\omega}) \tag{6.17}$$

Using the vector expansion formulas:

$$\begin{aligned} \nabla \times \left(\underline{q} \times \underline{\omega}\right) &= \underline{\omega} \cdot \nabla \underline{q} + \underline{q} (\nabla \cdot \underline{\omega}) - \underline{\omega} \left(\nabla \cdot \underline{q}\right) - \underline{q} \cdot \nabla \underline{\omega} && (6.18) \\ \nabla \times (\nabla \times \underline{\omega}) &= \nabla (\nabla \cdot \underline{\omega}) - \nabla^2 \underline{\omega} && (6.19) \end{aligned}$$

Since $\nabla \cdot \left(\nabla \times \underline{q}\right) = \nabla \cdot \underline{\omega} = 0$, the vorticity transport equation is equaivalently represented as

$$\frac{\partial \underline{\omega}}{\partial t} + \left(\underline{q} \cdot \nabla\right) \underline{\omega} = (\underline{\omega} \cdot \nabla) \underline{q} + \nu \nabla^2 \underline{\omega} + \nabla \times \underline{f} \tag{6.20}$$

The corresponding vorticity transport equation for a compressible fluid with variable viscosity and density is, [1]

$$\begin{aligned} \frac{\partial \underline{\omega}}{\partial t} = & - \left(\underline{q} \cdot \nabla\right) \underline{\omega} + (\underline{\omega} \cdot \nabla) \underline{q} + \nu \nabla^2 \underline{\omega} + \nabla \times \underline{f} - \underline{\omega} \left(\nabla \cdot \underline{q}\right) \\ & + \frac{1}{\rho^2} (\nabla \rho \times \nabla p) + \frac{\mu}{\rho} \{\nabla \rho \times (\nabla \times \underline{\omega})\} - \frac{4\mu}{3\rho^2} \left\{\nabla \rho \times \nabla(\nabla \cdot \underline{q})\right\} \\ & + \left[\nabla \times \left\{\frac{1}{\rho}\left(-\frac{2}{3}(\nabla \cdot \underline{q})(\nabla \mu) + 2(\nabla \underline{q}) \cdot (\nabla \mu) + (\nabla \mu) \times \underline{\omega}\right)\right\}\right] \end{aligned} \tag{6.21}$$

The vorticity transport equation in Eulerian and Lagrangian description can be expressed as, respectively, ignoring the body force term,

[1] For details, see Zabusky, N. J. (1999), "Vortex paradigm for accelerated inhomogeneous flows: Visiometrics for the Rayleigh-Taylor and Richtmyer-Meshkov environments," *Annual Review of Fluid Mechanics*, vol. 31, pp. 495–536.

(1) *Eulerian Description*

$$\underbrace{\frac{\partial \underline{\omega}}{\partial t}}_{\text{Local time rate of vorticity}} + \underbrace{\underline{q} \cdot \nabla \underline{\omega}}_{\text{Convective rate of vorticity}} = \underbrace{\underline{\omega} \cdot \nabla \underline{q}}_{\text{Stretching effect of vorticity}} + \underbrace{\nu \nabla^2 \underline{\omega}}_{\text{Viscous diffusion}} + \underbrace{\nabla \times \underline{f}}_{\text{Body force effect}} \tag{6.22}$$

(2) *Lagrangian Description*

$$\underbrace{\frac{D\underline{\omega}}{Dt}}_{\text{Rate of change of vorticity}} = \underbrace{\underline{\omega} \cdot \nabla \underline{q}}_{\text{Stretching effect of vorticity}} + \underbrace{\nu \nabla^2 \underline{\omega}}_{\text{Viscous diffusion}} + \underbrace{\nabla \times \underline{f}}_{\text{Body force effect}} \tag{6.23}$$

The vorticity transport equation for 2-D incompressible flow of a viscous fluid, ignoring the external body force, is represented as

$$\frac{\partial \underline{\omega}}{\partial t} + \left(\underline{q} \cdot \nabla\right) \underline{\omega} = \nu \nabla^2 \underline{\omega} + \nabla \times \underline{f} \tag{6.24}$$

6.2.3 Pressure Poisson equation

The Poisson pressure equation, by taking the divergence of Eq. (6.16) is also derived as

$$\nabla^2 \left(\frac{p}{\rho} + \frac{1}{2} \underline{q} \cdot \underline{q}\right) = \nabla \cdot \left(\underline{q} \times \underline{\omega}\right) + \nabla \cdot \underline{f} \tag{6.25}$$

or equivalently

$$\nabla^2 H = \nabla \cdot \left(\underline{q} \times \underline{\omega}\right) + \nabla \cdot \underline{f} \tag{6.26}$$

Here, the external force is ignored and the pressure p is related to the total pressure H (the static and the dynamic pressure) defined by

$$H = \frac{p - p_\infty}{\rho} + \frac{1}{2}\left(q^2 - q_\infty^2\right) \tag{6.27}$$

where the constants p_∞ and q_∞ are the reference pressure and velocity at infinity (or at a reference point), respectively. With this definition, the boundary condition at infinity for H is expressed by $H \to 0$ as $|\underline{x}| = r \to \infty$. Thus the contribution due to H at infinity is not considered.

6.2.4 Kinematic boundary condition

Equations (6.24) and (6.26) should be solved in the fluid domain with the boundary, being subject to the boundary conditions for velocity, vorticity and pressure on the surface of a solid body.

At a solid boundary, kinematics dictates that the tangential component of the flow velocity on the wall must be equal to the tangential velocity of the body.

$$\underline{q}(\underline{x}_s, t) \cdot \underline{t} = \underline{U}_B \cdot \underline{t} \tag{6.28}$$

where, if a body translates with a speed $\underline{U}_\infty$ and rotates with angular velocity $\underline{\Omega}_b$ around its center of mass located at $\underline{x}_b$, $\underline{U}_B = \underline{U}_\infty + \underline{\Omega}_b \times (\underline{x}_s - \underline{x}_b)$. This boundary condition results from experimental fact and is valid that the fluid is, to a good approximation, a continuum. This is usually called the *no-slip boundary condition*. Also, the normal component of the velocity of the fluid and the velocity of the body should be the same:

$$\underline{q}(\underline{x}_s, t) \cdot \underline{n} = \underline{U}_B \cdot \underline{n} \tag{6.29}$$

This is usually called the *no-through-flow boundary condition*. Equations (6.28) and (6.29) are the constituents of the kinematic boundary condition:

$$\boxed{\underline{q}(\underline{x}_s, t) = \underline{U}_B} \tag{6.30}$$

Fluid element in contact with the wall is subject to the flow velocity and the motion of the wall. This may result in a net torque onto the fluid element that may in turn impart a rotational motion to the fluid.

6.2.5 Dynamic boundary condition

The boundary condition for the vorticity at the solid surface can be derived by taking the cross product of Navier-Stokes equations Eq. (6.16) with a normal vector $\underline{n}$:

$$\underline{n} \times (\rho\, \underline{a}) \;+\; \underline{n} \times \nabla p \;=\; -\,\underline{n} \times (\nabla \times (\mu\, \underline{\omega})) \tag{6.31}$$

where the acceleration is expressed as $\underline{a} = d\underline{q}/dt$ and the external body force $\underline{f}$ is ignored. This condition corresponds to the force equilibrium in the directiontangent to the solid surface. The second term on the right-hand side of Eq. (6.31) also becomes by using vector expansion formulas,

$$\underline{n} \times (\nabla \times (\mu\, \underline{\omega})) = \nabla\, (\mu\, \underline{\omega}) \cdot \underline{n} - \frac{\partial(\mu\, \underline{\omega})}{\partial n} \tag{6.32}$$

Substitution of this relation in Eq. (6.31) then gives

$$\boxed{\frac{\partial(\mu\, \underline{\omega})}{\partial n} = \underline{n} \times (\rho\, \underline{a}) + \underline{n} \times \nabla p + \nabla\, (\mu\, \underline{\omega}) \cdot \underline{n}} \tag{6.33}$$

The boundary condition for the pressure at the solid surface can be derived by taking the scalar product of N.-S. equations (6.16) with a normal vector $\underline{n}$:

$$\underline{n} \cdot \frac{\partial \underline{q}}{\partial t} + \frac{\partial}{\partial n}\left(\frac{p}{\rho} + \frac{1}{2}\underline{q} \cdot \underline{q}\right) = \underline{n} \cdot \left(\underline{q} \times \underline{\omega}\right) - \underline{n} \cdot (\nabla \times (\nu\, \underline{\omega})) \tag{6.34}$$

This condition corresponds to the force equilibrium in the direction normal to the solid surface. Equation (6.34) is also expressed by using the total pressure in Eq. (6.27) and ignoring the external force as

$$\boxed{\frac{\partial H}{\partial n} = -\underline{n} \cdot \frac{\partial \underline{q}}{\partial t} + \underline{n} \cdot \left(\underline{q} \times \underline{\omega}\right) - \underline{n} \cdot (\nabla \times (\nu\, \underline{\omega}))} \tag{6.35}$$

6.2.6 Integral approach of formulation

The governing equations for the unsteady flow of a Newtonian incompressible fluid can be written as,

$$\begin{aligned}
\nabla \cdot \underline{q} &= 0, && (6.36)\\
\underline{\omega} &= \nabla \times \underline{q}, && (6.37)\\
\frac{\partial \underline{\omega}}{\partial t} + \underline{q} \cdot \nabla \underline{\omega} &= \underline{\omega} \cdot \nabla \underline{q} + \nu\, \nabla^2 \underline{\omega} + \nabla \times \underline{f}, && (6.38)\\
\nabla^2 \left(\frac{p}{\rho} + \frac{1}{2} q^2\right) &= \nabla \cdot \left(\underline{q} \times \underline{\omega}\right) + \nabla \cdot \underline{f}, && (6.39)
\end{aligned}$$

where $\underline{q}$, $\underline{\omega}$ and p are the velocity, the vorticity and the pressure, respectively, ν is the kinematic viscosity, and ρ is the density of the fluid. The set of Eqs. (6.37), (6.38) and (6.39) is one of the basic differential vorticity-velocity-pressure formulations. In VIC (Vortex-In-Cell) method, a Poisson equation for the stream function $\nabla^2 \underline{\psi} = -\underline{\omega}$, is used instead Eq. (6.37). In the next section, we will describe in detail about the VIC method to be employed.

According to the mathematical vector identity, an equivalent integral formulation of Eq. (6.37) is written as, with use of Eq. (6.36),

$$\boxed{\underline{q} = \oint_S \left[(\underline{n} \cdot \underline{q}) \, \nabla G + (\underline{n} \times \underline{q}) \times \nabla G \right] \, dS + \int_V \underline{\omega} \times \nabla G \, dV,} \tag{6.40}$$

where $\underline{n}$ is the unit normal pointing into the fluid at the boundary S (C in 2-dimensions) of a fluid domain V (S in 2-dimensions) and ∇ denotes the differential operator with respect to the variable of integration $\underline{\xi}$. Here, G is the fundamental solution of the Laplace equation for an unbounded fluid domain, defined by $G = -\dfrac{1}{4\pi r}$ in 3-dimensions and $G = +\dfrac{1}{2\pi} \ln r$ in 2-dimensions, where r is the distance between a field point $\underline{x}$ and an integration point $\underline{\xi}$.

The velocity field $\underline{q}$ is considered as the sum of two components: the velocity of undisturbed onset flows and the disturbance velocity due to the existence of a solid body. The first integral of Eq. (6.40) represents the contribution from the irrational component of the flows (i.e., $\underline{q}_o + \nabla \phi$ plus the motion of a moving reference frame if introduced). The second one known as the Biot-Savart law represents the disturbance velocity field ($\underline{u}_\omega$) induced by a vorticity field. The use of the Biot-Savart law in computing the velocity field guarantees the enforcement of the boundary condition for the velocity at infinity.

Correspondingly, an integral formulation of Eq. (6.39) can be written as:

$$\boxed{H = \oint_S \left[H \frac{\partial G}{\partial n} - \frac{\partial H}{\partial n} G \right] \, dS + \int_V \left\{ \nabla \cdot (\underline{q} \times \underline{\omega}) \; + \; \nabla \cdot \underline{f} \right\} G \, dV.} \tag{6.41}$$

6.2.6.1 Two-dimensional formulation

Ignoring the external body force $\underline{f}$, the two-dimensional version of the system of Eqs. (6.38), (6.40) and (6.41) can be written as, in non-dimensional form,

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\underline{q}\, \omega) = \frac{1}{Re} \nabla^2 \omega, \tag{6.42}$$

$$\underline{q} = \underline{q}_\infty + \nabla \phi - \frac{1}{2\pi} \int_S \underline{\omega} \times \nabla (\ln r)\, dS, \tag{6.43}$$

$$H = -\frac{1}{2\pi} \oint_C \left[H \frac{\partial (\ln r)}{\partial n} - \frac{\partial H}{\partial n} (\ln r) \right] dl + \frac{1}{2\pi} \int_S \nabla \cdot (\underline{q} \times \underline{\omega}) (\ln r)\, dS, \tag{6.44}$$

where Re is the Reynolds number and ω is the scalar plane component of the vorticity vector ($\underline{\omega} \equiv \omega\, \underline{k}$). All non-dimensional quantities are defined based on the characteristic length of a body (e.g., the diameter of a circular cylinder (D) for our test problems) and the velocity of oncoming inflows (q_∞).

The system of Eqs. (6.42), (6.43) and (6.44) must be solved in the fluid domain with a boundary, being subject to the boundary conditions for the velocity, the vorticity and the pressure on the surface (C_B) of a solid body. The no-slip velocity condition states that the velocity of the fluid ($\underline{q}$) is equal to the velocity of the body ($\underline{U}_B$) at the surface points ($\underline{x}_B$) of the body:

$$\underline{q}(\underline{x}_B, t) = \underline{U}_B \text{ on } C_B. \tag{6.45}$$

The two-dimensional version of Eq. (6.33) is represented by

$$\frac{\partial (\mu\, \underline{\omega})}{\partial n} = \underline{n} \times (\rho\, \underline{a}) + \underline{n} \times \nabla p \tag{6.46}$$

or equivalently

$$\nu \frac{\partial \underline{\omega}}{\partial n} = \underline{s} \cdot \frac{\partial \underline{q}}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial s} \tag{6.47}$$

where $\underline{s}$ is a tangential vector and ν is the kinematic viscosity. This represents an explicit expression of the process of vorticity production described only verbally by Lighthill (1963). This quantity of the vorticity flux diffuses into the fluid from the body surface.

The boundary vorticity flux (σ) at the solid body for two-dimensional incompressible flow is

$$\sigma \equiv -\frac{1}{Re}\frac{\partial \omega}{\partial n} = -\underline{k} \cdot \left\{ \underline{n} \times \frac{d\underline{U}_B}{dt} + \underline{n} \times \nabla \left(\frac{p}{\rho} \right) \right\} \quad \text{on } C_B. \tag{6.48}$$

This essential boundary condition for the vorticity at the solid surface can be derived by taking the cross product of the N-S equations with $\underline{n}$, with use of the velocity adherence condition. It represents an explicit expression of the process of vorticity production described only verbally by Lighthill (1963). This quantity of the vorticity flux diffuses into the fluid from the body surface. The above expression applies for $t = 0^+$ as well, and is therefore applicable immediately after a solid body is accelerated impulsively. Similarly, the scalar product of the N-S equations with $\underline{n}$ gives an expression for $\partial H/\partial n$ as:

$$\frac{\partial H}{\partial n} = -\underline{n} \cdot \frac{\partial \underline{q}}{\partial t} + \underline{n} \cdot (\underline{q} \times \underline{\omega}) - \frac{1}{Re}\underline{n} \cdot (\nabla \times \underline{\omega}) \quad \text{on } C_B. \tag{6.49}$$

It is seen from Eqs. (6.48) and (6.49) that the boundary conditions for the vorticity and the pressure are coupled. A more rigorous and extensive analysis on these pressure and vorticity conditions for two- or three-dimensional incompressible or compressible flows was given by Wu & Wu (1993).

6.2.7 Stream function approach: VIC method

The velocity field can be decomposed into

$$\underline{q} = \underline{\mathrm{U}}_\infty + \underline{u}_\omega + \underline{u}_\phi \tag{6.50}$$

where $\underline{\mathrm{U}}_\infty$ is incoming velocity, $\underline{u}_\omega$ represents rotational field, and $\underline{v}_\phi$ represents solenoidal field. The velocity vector can also be expressed according to the Helmholtz decomposition due to the incompressibility,

$$\underline{q} = \underline{\mathrm{U}}_\infty + \nabla \times \underline{\psi} + \nabla \phi. \tag{6.51}$$

The vector potential $\underline{\psi}$ and the scalar potential ϕ should vanish in the far field so that the velocity field recover the free stream velocity.

$$\underline{q} \rightarrow \underline{\mathrm{U}}_{\infty} \text{ as } |\underline{x}| \rightarrow \infty \tag{6.52}$$

where $\underline{x}$ is the spatial coordinate. The vector potential is related to a stream function in two dimension. If we take the curl of the equation (6.51),

$$\underline{\omega} = \nabla \times (\nabla \times \underline{\psi}) = -\nabla^2 \underline{\psi} + \nabla(\nabla \cdot \underline{\psi}) \tag{6.53}$$

If we enforce $\nabla \cdot \underline{\psi} = 0$, the equation results in Poisson equation,

$$\nabla^2 \underline{\psi} = -\underline{\omega} \tag{6.54}$$

and its solution is

$$\underline{\psi} = \frac{1}{4\pi} \int_V \frac{\underline{\omega}}{r} \, dV \tag{6.55}$$

Fially the rotational velocity field is $\underline{u}_{\omega} = \nabla \times \underline{\psi}$,

$$\underline{u}_{\omega} = -\frac{1}{4\pi} \int_V \underline{\omega} \times \nabla \left(\frac{1}{r} \right) \, dV, \tag{6.56}$$

where r is the distance from the volume element dV to the field point. This equation commonly referred to the Biot-Savart formula.

The rotational velocity field can be evaluated using the Biot-Savart law (6.56). But, the direct calculation involves $O(N^2)$ cost for N elements. This is computationally intensive so that fast evaluation method such as multipole expansion has been developed in order to cut down the cost. The VIC method reduces the computational cost to $O(N \log N)$ by employing grid based fast Poisson solvers. The VIC method is composed of three basic steps. First, the vorticity field is projected to the grid using the interpolation kernel. The Poisson equation for vector potential (6.54) is solved on the grid with the boundary value of $\underline{\psi}$. The velocity on the grid is computed from the definition $\underline{u}_{\omega} = \nabla \times \underline{\psi}$ with the finite difference formula, and then the velocity is interpolated back to the particles.

6.2.8 Particle method in solving the vorticity transport equation

Let us start with a simplified conservative form of the vorticity transport equation, $L\omega = f$ with a suitable differential operator L:

$$L\,\omega = \frac{\partial \omega}{\partial t} + \nabla \cdot (\underline{q}\,\omega) + c_0\,\omega = f \tag{6.57}$$

For a material volume $V(t)$, the integral form would be

$$\frac{d}{dt}\int_{V(t)} \omega\,dV + \int_{V(t)} c_0\,\omega\,dV = \int_{V(t)} f\,dV \tag{6.58}$$

where we have used the Reynolds transport theorem. Here we introduce the idea of particle methods in which mass on points is concentrated:

$$\omega(\underline{x}, t) = \alpha(t)\,\delta(\underline{x} - \underline{x}_p(t)) \tag{6.59}$$

With such particle representation, the above integral becomes discrete values, and then the vorticity transport equation reduces to a set of ordinary differential equations. As example, for the homogeneous equation $L\omega = 0$, we have the general solution form:

$$\frac{d\alpha}{dt} + c_0(\underline{x}_p(t), t)\,\alpha = 0, \qquad \text{with} \quad \frac{d\underline{x}_p}{dt} = \underline{q}(\underline{x}_p, t) \tag{6.60}$$

Now, the extension of this concept to the vorticity transport equation in 3-D gives us following govering equations in the vortex particle methods:

$$\underline{\omega} = \sum_p \underline{\alpha}_p\,\delta(\underline{x} - \underline{x}_p(t)) \tag{6.61}$$

$$\frac{d\underline{x}_p}{dt} = \underline{q}(\underline{x}_p, t) \tag{6.62}$$

$$\frac{d\underline{\alpha}_p}{dt} = \nabla\underline{q}(\underline{x}_p, t)\;\underline{\alpha}_p + \text{diffusion term} \tag{6.63}$$

The effects of the diffusion term can be employed by the PSE(Particle Strength Exchange) scheme and the integral formula for the wall no-slip condition. The overall insights on the PSE scheme and the wall viscous diffusion will be explined in Chapter 8.

6.2.9 Hydrodynamic Forces

The force exerted by the fluid on the body can be separated into the hydrostatic force and the hydrodynamic force. The hydrodynamic force $\underline{F}$ on the body due to the motion is defined as

$$\frac{\underline{F}}{\rho} = -\frac{d\underline{I}}{dt} \tag{6.64}$$

The quantity $\underline{I}$ is called the *hydrodynamic impulse* that needs to be applied to the body to set it in motion against the inertia of the fluid (Lamb 1932). Thus,

$$\underline{I} = \frac{1}{d-1}\int_V \underline{x} \times \underline{\omega}\, dV \tag{6.65}$$

with d the dimension of the space ($d = 3$ in 3-D, $d = 2$ in 2-D). [2] In two-dimensional case, the position $(\tilde{x}, \tilde{y})$ of vorticity are related to the components (I_x, I_y) of hydrodynamics impulse

$$\begin{aligned} I_x &= \int y\, \underline{\omega}\, dS \approx \sum_i y_i\, \Gamma_i \\ I_y &= -\int x\, \underline{\omega}\, dS \approx -\sum_i x_i\, \Gamma_i \end{aligned} \tag{6.67}$$

and then the components of the force (F_x, F_y) is

$$F_x = -\rho\, \frac{dI_x}{dt}\,, \qquad F_y = -\rho\, \frac{dI_y}{dt} \tag{6.68}$$

where $\dfrac{d\underline{I}}{dt} = \dfrac{\underline{I}\,(t+\triangle t) - \underline{I}\,(t-\triangle t)}{2\triangle t}$. The x-component of the hydrodynamic force is called the *drag* and the y-component is the *lift*.

[2] In Eq. (1.113), we set $\underline{f} = \underline{q}$ to find

$$\underline{I} \equiv \int_V \underline{q}\, dV = \frac{1}{d-1}\int_V \underline{x} \times (\nabla \times \underline{q})\, dV - \frac{1}{d-1}\oint_S \underline{x} \times (\underline{n} \times \underline{q})\, dS \tag{6.66}$$

and then the second integral term would vanish from the no-slip boundary condition ($\underline{q} = \underline{U}_B$) for a stationary body (also for steadily moving bodies) and the far-field boundary condition($\underline{q} = \underline{U}_\infty$). Accordingly, the second term on the right-hand side of Eq. (6.66) does not contribute the hydrodynamic forces.

Extensively, if we take $\underline{q} = \nabla p$ in Eq. (6.66) and the divergence theorem for the first volume integral, then the pressure forces can be written as, since the second volume integral term must vanish (i.e., identically $\nabla \times \nabla p = 0$),

$$\underline{F}_p \equiv -\oint_{S_B} p\, \underline{n}\, dS = \frac{1}{d-1} \oint_{S_B} \underline{x} \times (\underline{n} \times \nabla p)\, dS \tag{6.69}$$

Using Eq. (6.46), the 2-D version of the pressure forces is represented by, in terms of the vorticity flux on body surface and the body acceleration,

$$\underline{F}_p^{(2D)} \equiv -\oint_{S_B} p\, \underline{n}\, dS = \oint_{S_B} \underline{x} \times \left\{ \frac{\partial(\mu\, \underline{\omega})}{\partial n} - \underline{n} \times (\rho\, \underline{\dot{U}}_B) \right\} dS \tag{6.70}$$

For the 2-D case of an impulsively started body, the result reduces to, in terms of vorticity flux distribution on the body surface,

$$\underline{F}_p^{(2D)} = -\oint_{S_B} p\, \underline{n}\, dS = \oint_{S_B} \underline{x} \times \frac{\partial(\mu\, \underline{\omega})}{\partial n}\, dS \tag{6.71}$$

In derived Eq. (1.113), we have noted that the left-hand side of Eq. (1.112) and Eq. (1.113) is independent of the choice of the origin of $\underline{x}$, so must be the right-hand side. Namely, if we remove $\underline{x}$ from the right-hand side of these equations, the remaining integrals must vanish.