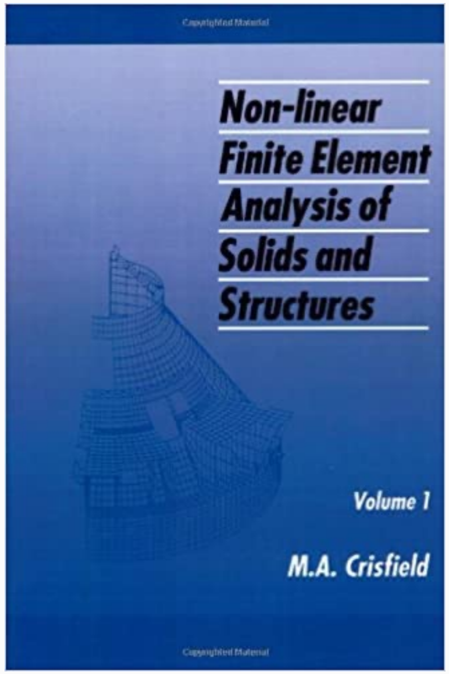
Computational plasticity (전산소성역학)

- Textbook – Nonlinear finite element analysis of solids and structures, Volume 1: essential (First edition)

  M.A. Crisfield



- Presentation slide will be provided
- Basics of numerical method, continuum mechanics, **plasticity theory**, and crystal plasticity (option)
- Preliminary knowledge on plasticity, programming is important!!

Computational plasticity (전산소성역학)

- Video recording will be progressed during class
- Grading: Class attendance (20%), Homework (20%), Midterm exam (30%), Final term project (30%)

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Chapter 1: One-dimensional geometric non-linearity problem, solution method

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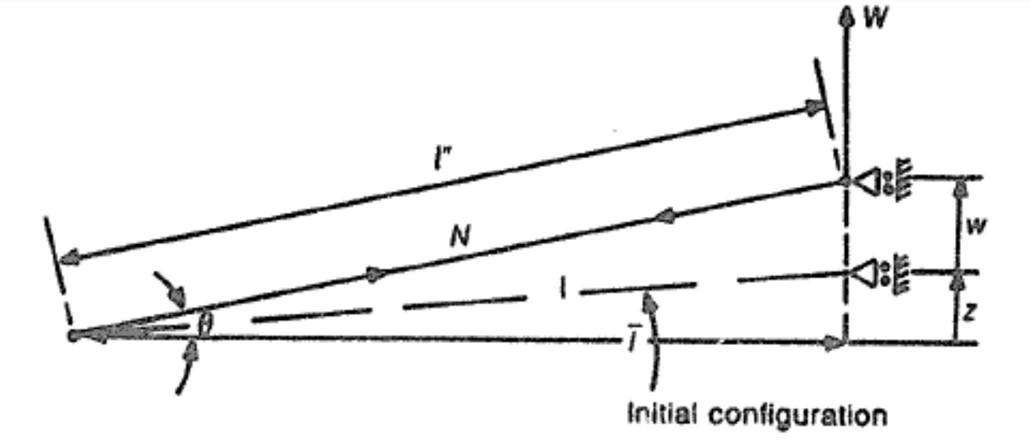
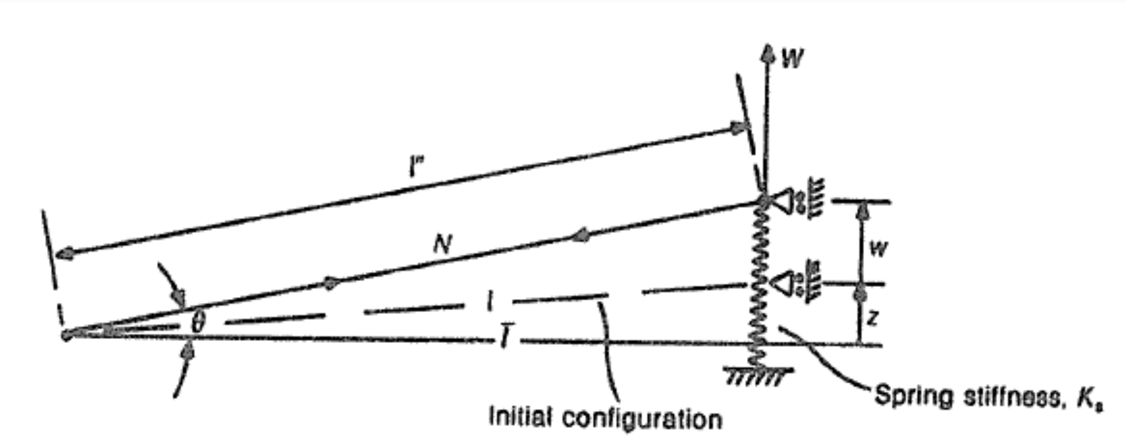
MATERIALS MECHANICS LABORATORY

In this chapter,

- We learn fundamental solution method for simple nonlinear problem with one dimensional bar structure under loading
- Simple example for geometric nonlinear bar with 1 degree of freedom (d.o.f)
- Solution methods
  - Incremental method
  - Iterative solution based on Netwon-Raphson method
  - Combined incremental/NR method, Modified NR, Initial stress method
- Extension to two variable problem
- Other approach
  - Virtual work principle
  - Energy method

- Basic concepts of **geometric nonlinear** problem and its solution method.
- This simple problem will be the basis for more general non-linear finite element (FE) analysis. (c.f. material nonlinearity)

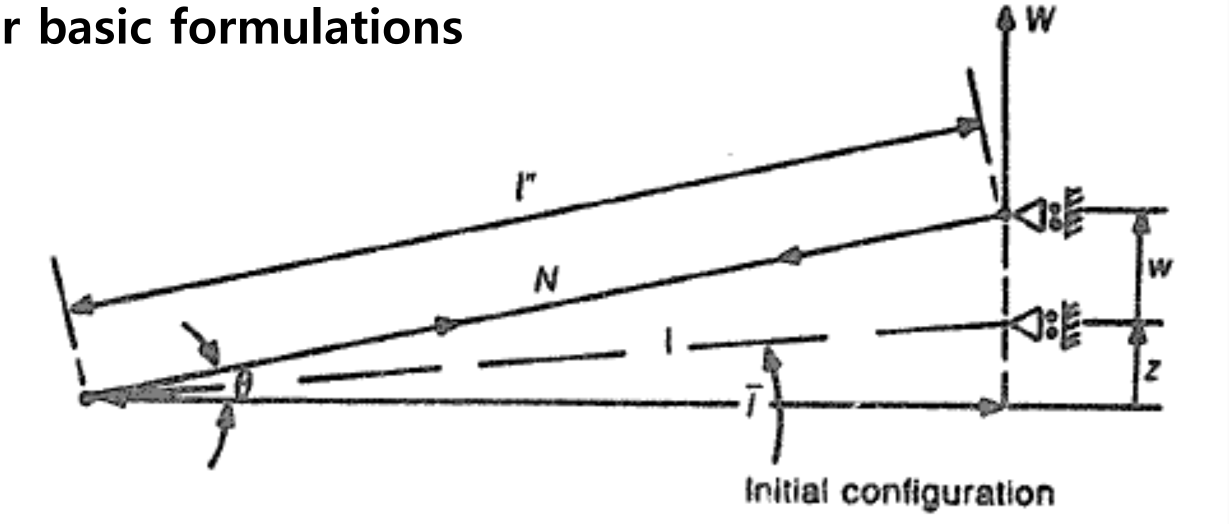




[Fig 1.1 Simple problem with one degree of freedom]

- **Out-of-balance force vector**
- **Tangent stiffness matrix**
- **Solution procedures**
  - **Incremental approach**
  - **Iterative approach (NR)**
  - **Principal of virtual work (PVW)**

- **Derivation for basic formulations**



When θ is small, equilibrium states,

Area =A, Young's modulus = E, Load = W
Displacement (vertical) = w

$$W = N\sin\theta \simeq \frac{N(z+w)}{l"} \simeq \frac{N(z+w)}{l}$$ **[eq. 1.1]**

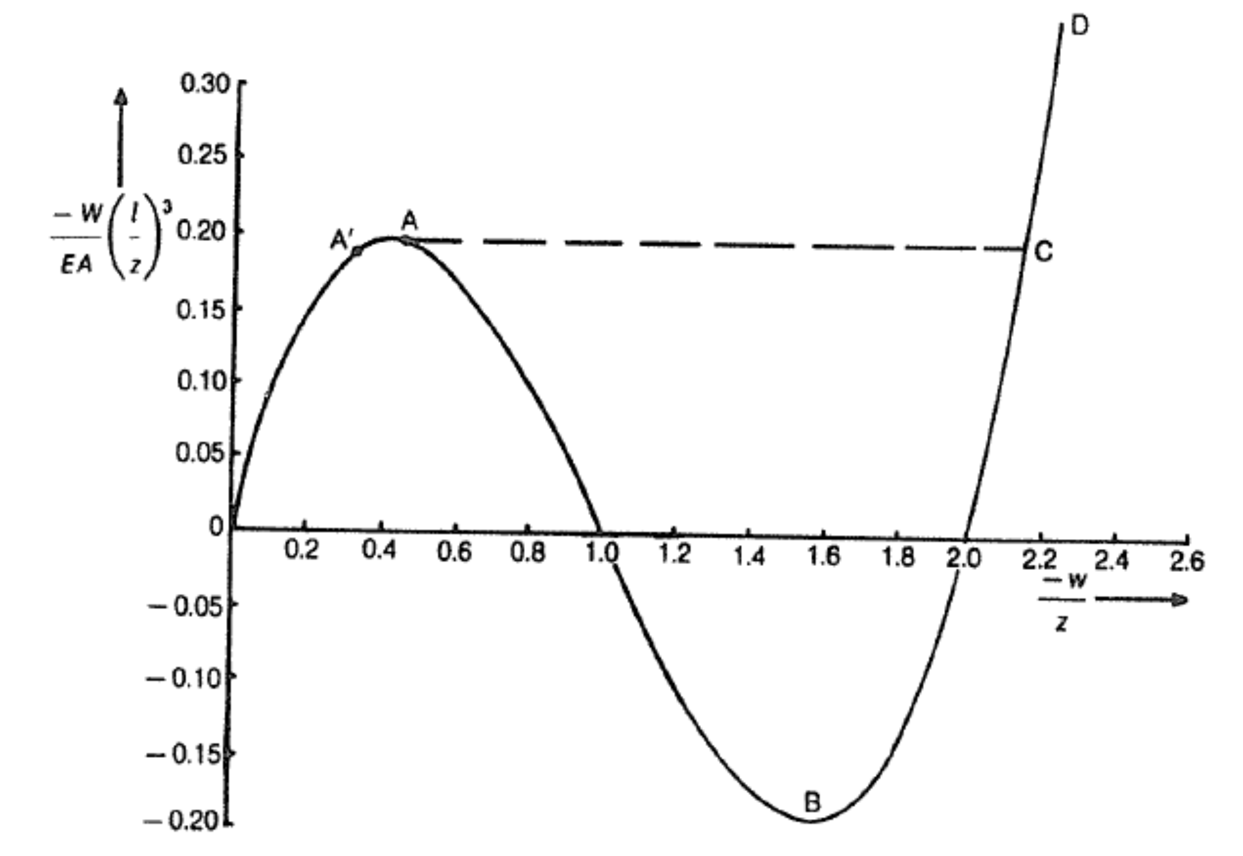
$$\varepsilon = \frac{\left((z+w)^2+\bar{l}^2\right)^{1/2} - \left(z^2+\bar{l}^2\right)^{1/2}}{\left(z^2+\bar{l}^2\right)^{1/2}}$$ **[eq. 1.2-1.5]**

$$\simeq \left(\frac{z}{\bar{l}}\right)\left(\frac{w}{\bar{l}}\right)+\frac{1}{2}\left(\frac{w}{\bar{l}}\right)^2 \simeq \left(\frac{z}{l}\right)\left(\frac{w}{l}\right)+\frac{1}{2}\left(\frac{w}{l}\right)^2$$

$$N = EA\varepsilon = EA\left(\left(\frac{z}{l}\right)\left(\frac{w}{l}\right)+\frac{1}{2}\left(\frac{w}{l}\right)^2\right)$$ **[eq. 1.6]**

Combining eq.1.1 and eq.1.6,

$$W = \frac{EA}{l^3}\left(z^2 w + \frac{3}{2} z w^2 + \frac{1}{2} w^3\right) \quad \text{[eq. 1.7]}$$



[Fig 1.2(a) Load/deflection relationships for simple one-dimensional problem. Response for bar alone]

- Tangent stiffness

linear stiffness matrix

geometric matrix (initial-stress matrix)

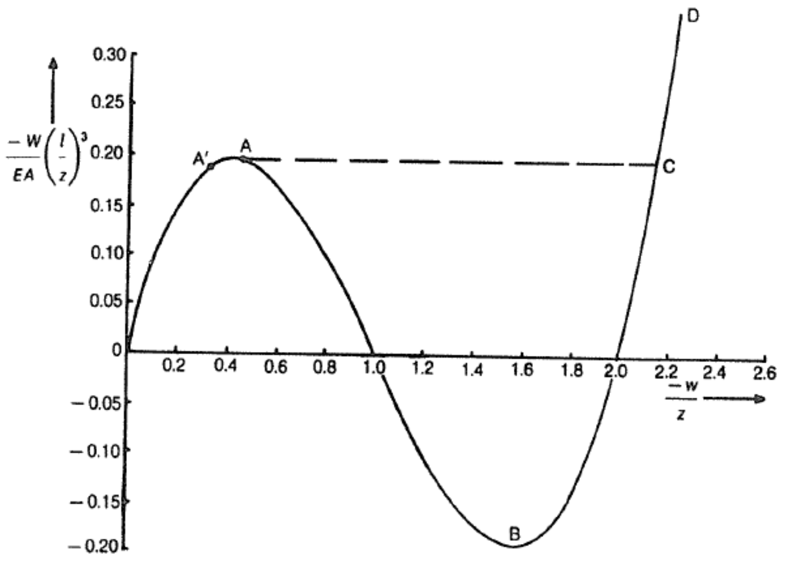
$$K_t = \frac{dW}{dw} = \frac{EA}{l}\left(\frac{z}{l}\right)^2 + \frac{EA}{l}\left(\frac{2zw + w^2}{l^2}\right) + \frac{N}{l}$$

[eq. 1.8-1.10]

initial-displacement matrix (initial-slope matrix)

- **Linear stiffness matrix:**
  - only a function of initial geometry (& material)
- **Initial-displacement matrix (initial-slope matrix):**
  - can be omitted by 'updated coordinate' system
- **Geometric matrix (initial-stress matrix):**
  - changes during deformation in any the coordinate system

- In this case, degree of freedom is 1, so tangent stiffness is scalar value
- In general, if degree of freedom is N, tangent stiffness is (N x N) matrix.



$$W = \frac{EA}{l^3}\left(z^2 w + \frac{3}{2} z w^2 + \frac{1}{2} w^3\right)$$

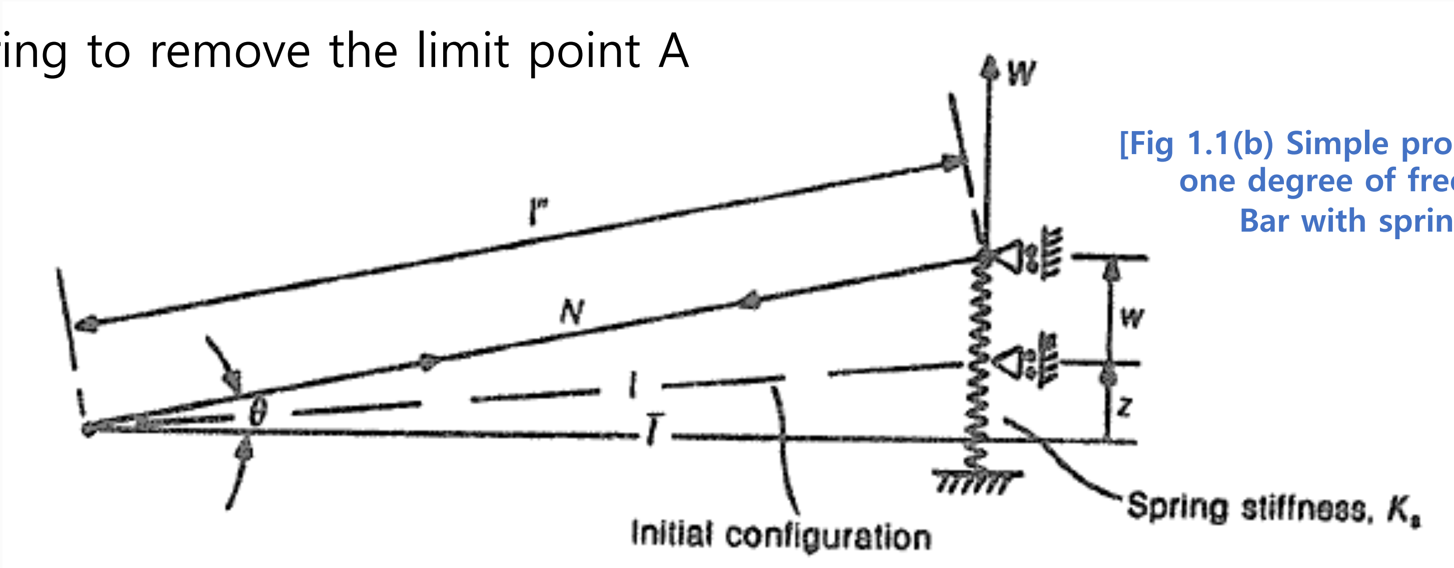
- **Standard** finite element procedures would allow the non-linear equilibrium path to be traced until a point A' just before 'local limit point' A.

- '**Displacement control**' will not fail and would trace the complete equilibrium path OABCD.

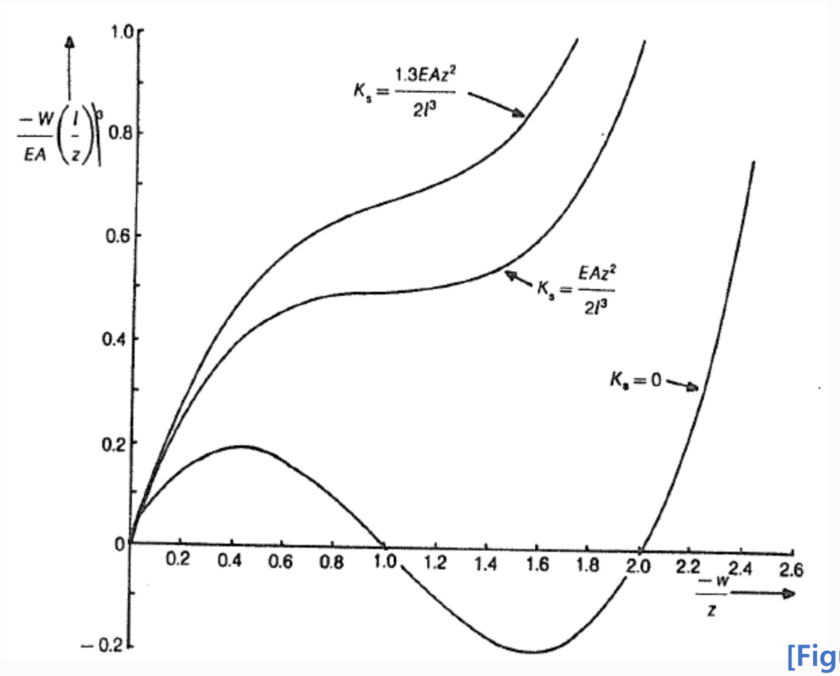
- For many degrees of freedom (in this case, degree of freedom is only 1, w), it is not trivial to adopt displacement control.
  - Constrained displacement can be easily converted to external force, but the opposite is not so trivial (section 2.2.5 in the textbook)

- '**Load control**' is employed in the example

• Added spring to remove the limit point A



[Fig 1.1(b) Simple problem with one degree of freedom. Bar with spring]



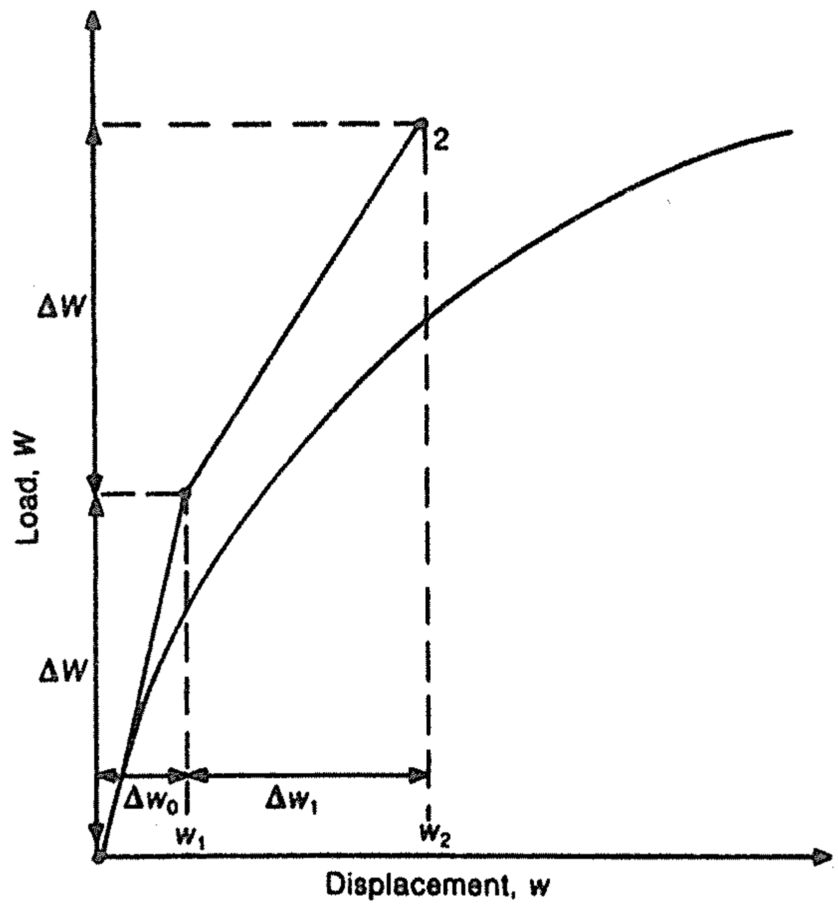
$$W = \frac{EA}{l^3}\left(z^2 w + \frac{3}{2} z w^2 + \frac{1}{2} w^3\right) + K_s w$$

[eq. 1.11]

$$K_t = \frac{dW}{dw} = \frac{EA}{l}\left(\frac{z}{l}\right)^2 + \frac{EA}{l}\left(\frac{2zw + w^2}{l^2}\right) + \frac{N}{l} + K_s$$

[Figure 1.2(b) Load/deflection relationship for simple one-dimensional problem. Set of responses for bar-spring system]

Solution method - An Incremental solution

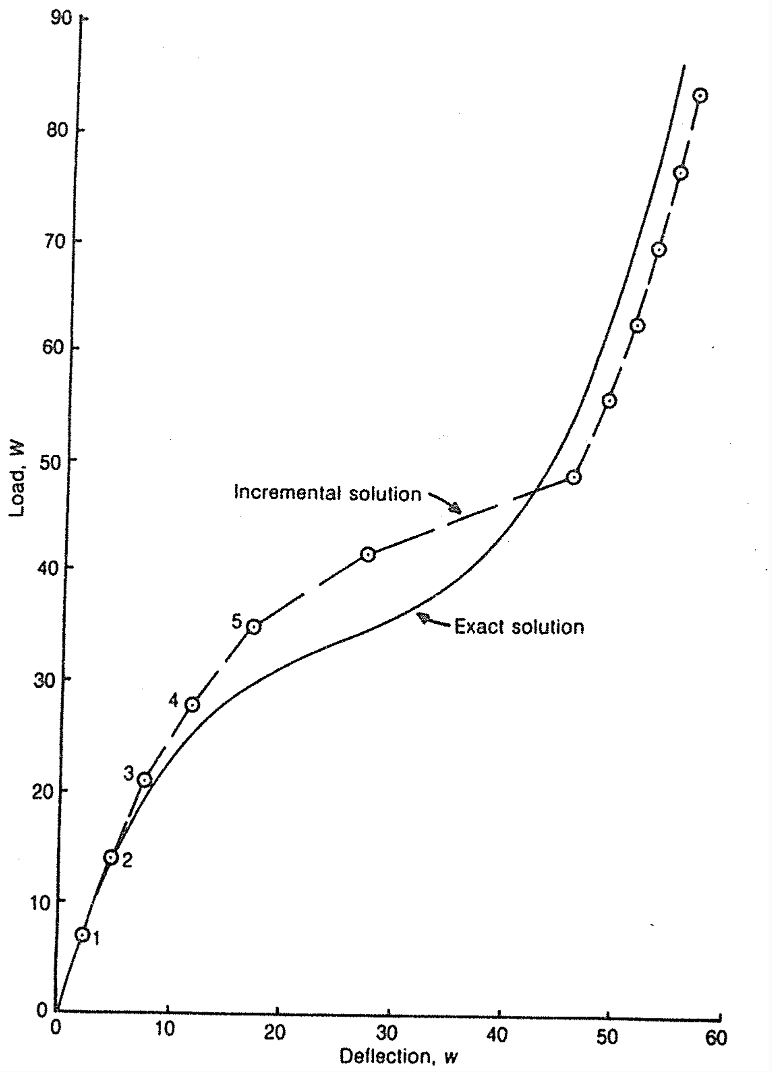


- An incremental (or Euler) solution scheme involves repeated application of:

$$\Delta w = \left(\frac{dW}{dw} \right)^{-1} \Delta W = K_t^{-1} \Delta W$$

[eq. 1.13]

Solution method - An Incremental solution



[Fig 1.4 Incremental solution for bar-spring problem]

$$N_0 = 0$$

$$K_0 = \frac{EA}{l}\left(\frac{z}{l}\right)^2 + K_s$$

$$\Delta w_0 = K_0^{-1}\Delta W$$

$$w_1 = w_0 + \Delta w_0 = \Delta w_0$$

⬇

$$N_1 = EA\left(\left(\frac{z}{l}\right)\left(\frac{w_1}{l}\right) + \frac{1}{2}\left(\frac{w_1}{l}\right)^2\right)$$

$$K_1 = \frac{EA}{l}\left(\frac{z}{l}\right)^2 + \frac{EA}{l}\left(\frac{2zw_1 + w_1^2}{l^2}\right) + \frac{N_1}{l} + K_s$$

$$\Delta w_1 = K_1^{-1}\Delta W$$

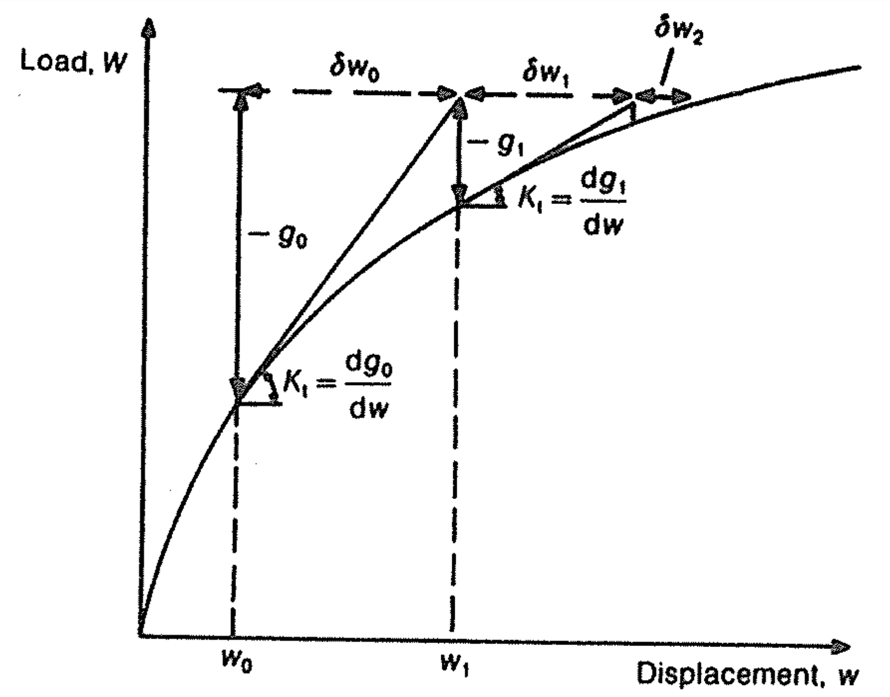
$$w_2 = w_1 + \Delta w_1$$

⬇

$$N_2 = EA\left(\left(\frac{z}{l}\right)\left(\frac{w_2}{l}\right) + \frac{1}{2}\left(\frac{w_2}{l}\right)^2\right)$$

[eq. 1.14-1.20]

Solution method - an Iterative solution by the Newton-Raphson method



[Fig 1.5 The Newton-Raphson method]

- **Newton-Raphson method to obtain w for a given load W:**

$$W = \frac{EA}{l^3}\left(z^2 w + \frac{3}{2} z w^2 + \frac{1}{2} w^3\right)$$

$$g = \frac{EA}{l^3}\left(z^2 w + \frac{3}{2} z w^2 + \frac{1}{2} w^3\right) - W = \frac{N(z+w)}{l} - W = 0$$

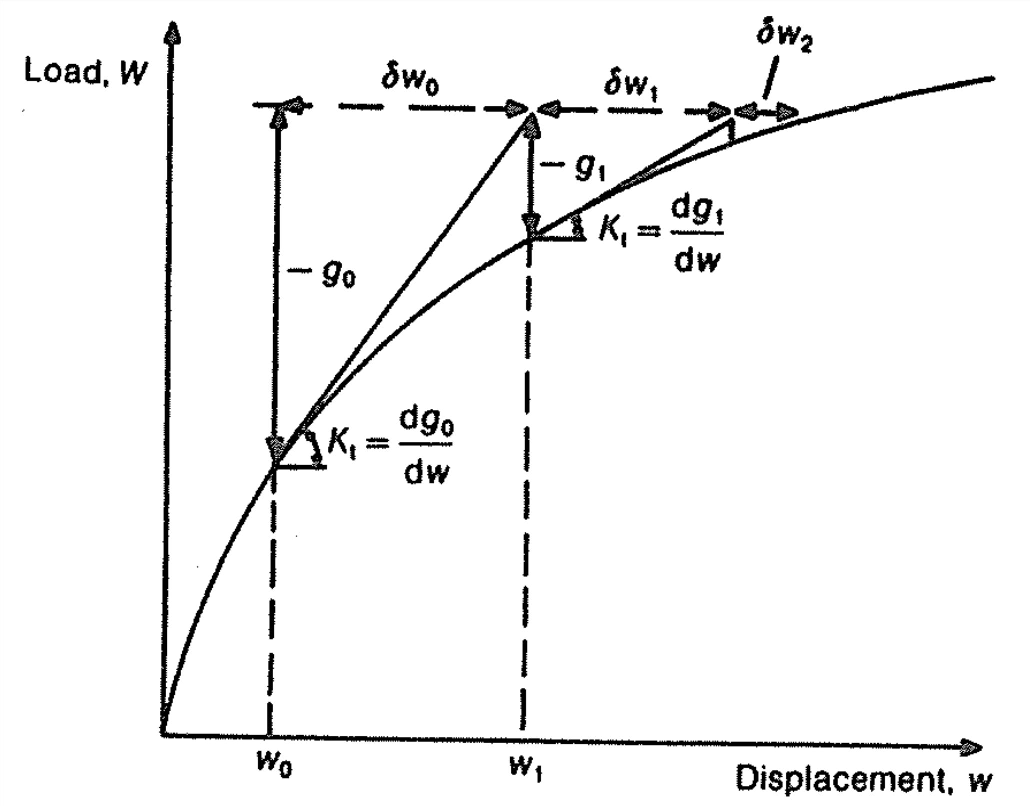
internal force   external force

$$\delta w = -\left(\frac{dg}{dw}\right)^{-1} g$$

Out-of-balance force (vector)

[eq.1.23,1.28]

Solution method - an Iterative solution by the Newton-Raphson method



$$\delta w_0 = -\left(\frac{dg_0}{dw}\right)^{-1} g_0(w_0)$$

$$w_1 = w_0 + \delta w_0$$

Repeat the process or iteration

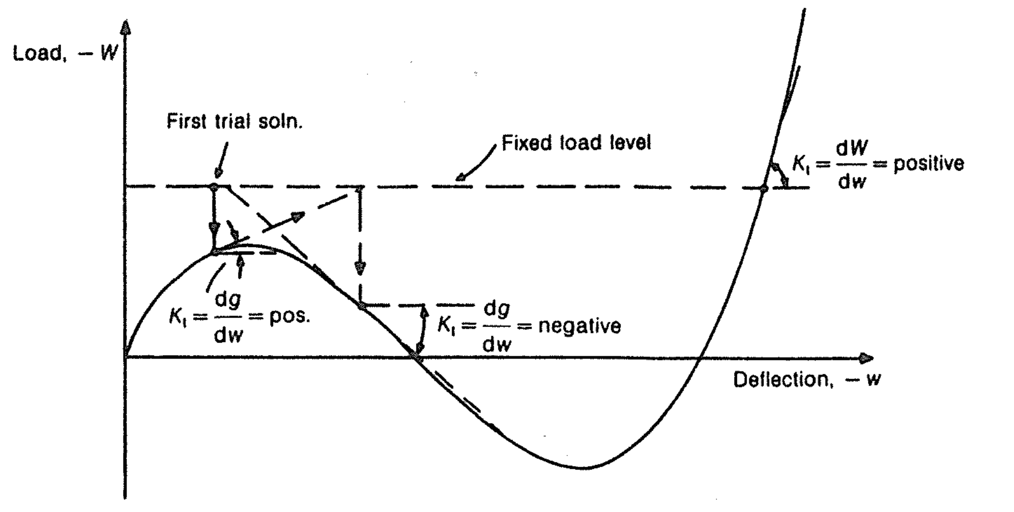
$$\delta w_1 = -\left(\frac{dg_1}{dw}\right)^{-1} g_1(w_1)$$

$$w_2 = w_1 + \delta w_1$$

Solution method - an Iterative solution by the Newton-Raphson method

$$g=\frac{EA}{l^3}\left(z^2w+\frac{3}{2}zw^2+\frac{1}{2}w^3\right)-W=\frac{N(z+w)}{l}-W=0$$

[eq. 1.23, 1.28]



[Fig 1.6 Positive and negative tangent stiffness]

$$\frac{dg}{dw}=\frac{(z+w)}{l}\frac{dN}{dw}+\frac{N}{l}=\frac{EA}{l}\left(\frac{z}{l}\right)^2+\frac{EA}{l}\left(\frac{2zw+w^2}{l^2}\right)+\frac{N}{l}=K_t$$

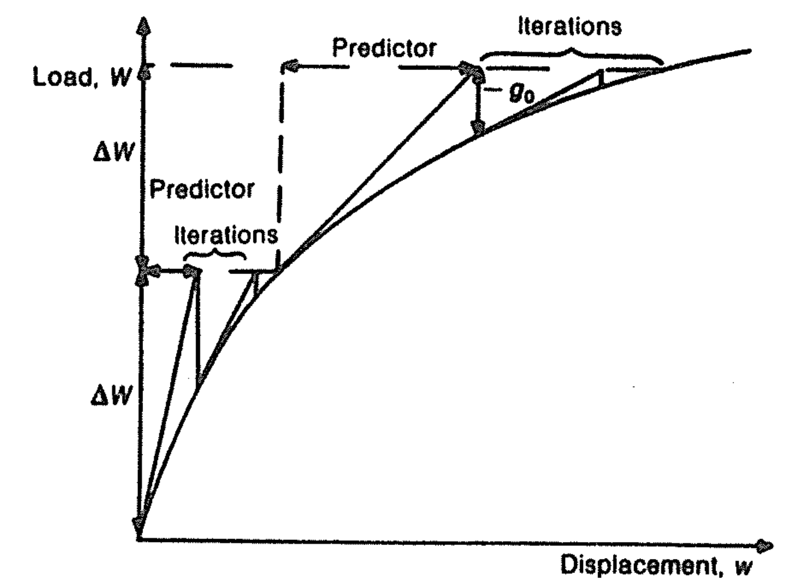
[eq. 1.29]

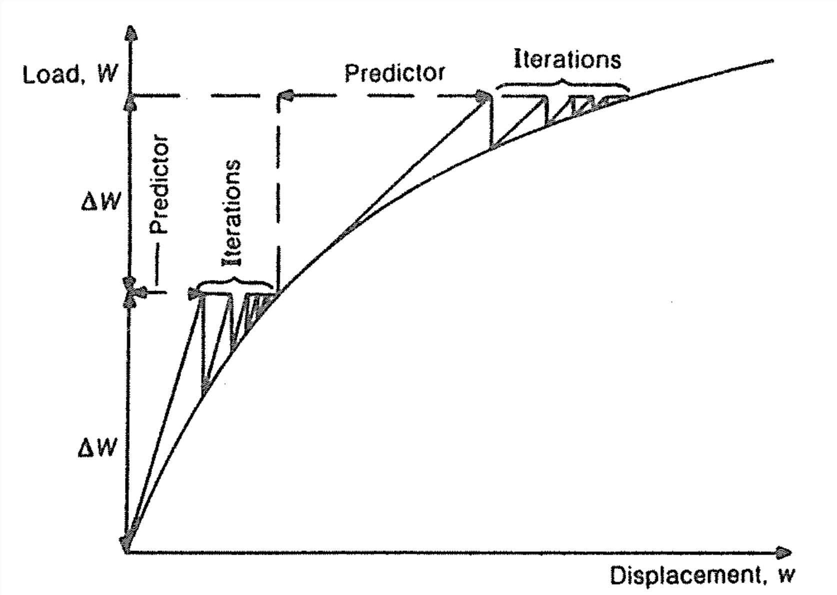
$$K_t=\frac{dW}{dw}=\frac{EA}{l}\left(\frac{z}{l}\right)^2+\frac{EA}{l}\left(\frac{2zw+w^2}{l^2}\right)+\frac{N}{l}$$

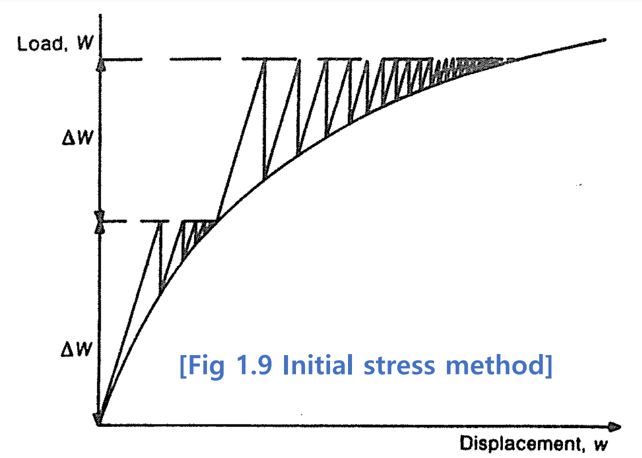
[eq. 1.8-1.10]

- K_t=dW/dw is the value of equilibrium state, so it should be always positive. Otherwise, object will move toward the opposite direction of force. In general (DOF ≥ 1), K_t=dW/dw is **positive definite**. On the other hand, K_t=dg/dw do not have to follow this rule, since it is just a computational value.

Solution method - combined Incremental/Iterative solutions (full or modified Newton-Raphson or the Initial-stress method)





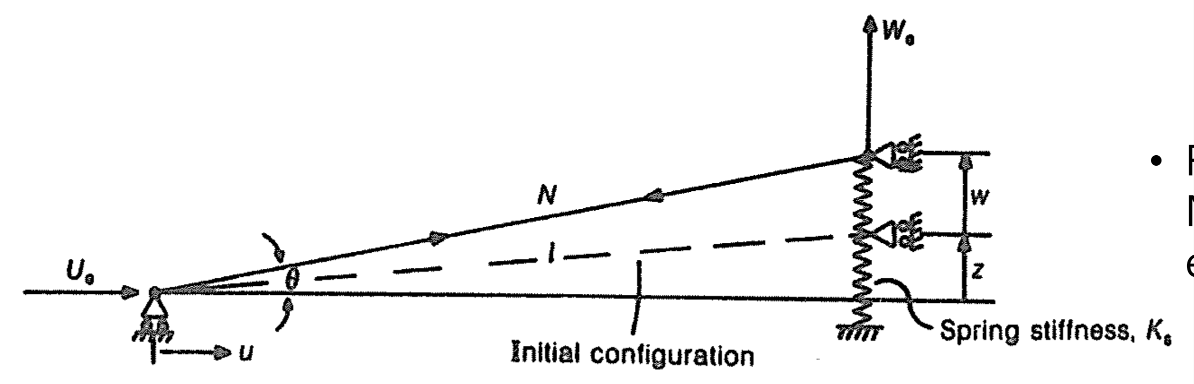


[Fig 1.9 Initial stress method]

- Full Newton-Raphson method provides 'quadratic convergence'

$$\frac{e_1}{e_0^2} \simeq \frac{e_2}{e_1^2} \simeq \cdots$$

- Modified Newton-Raphson method provides 'linear convergence' $\frac{e_1}{e_0} \simeq \frac{e_2}{e_1} \simeq \cdots$



- For two degrees of freedom, Newton-Raphson method is expanded as:

[Fig 1.10 Simple problem with two degrees of freedom]

$$\mathbf{p}^T = (u, \mathrm{w})$$ [eq. 1.50]

$$\varepsilon = -\frac{u}{l} + \left(\frac{z}{l}\right)\left(\frac{w}{l}\right) + \frac{1}{2}\left(\frac{w}{l}\right)^2$$ [eq. 1.51]

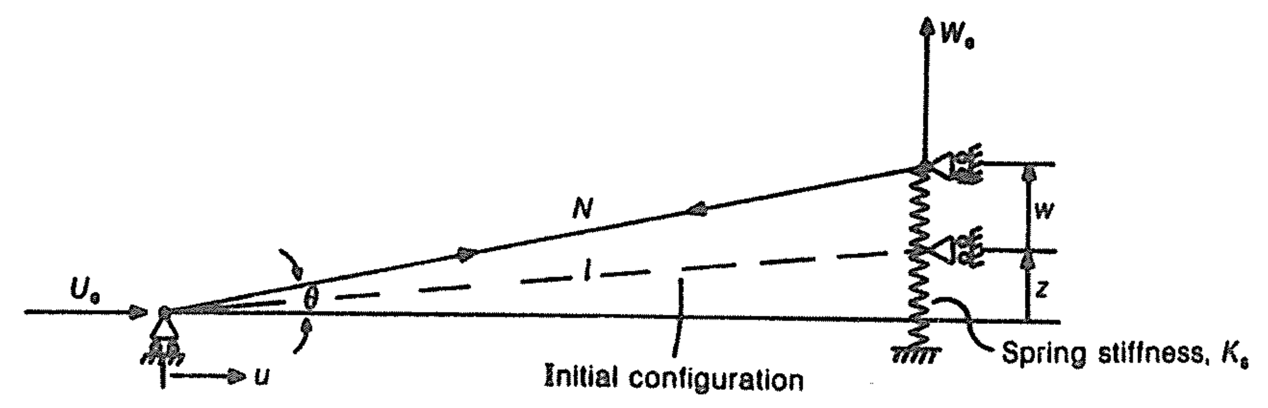
$$N = EA\varepsilon = EA\left(\left(\frac{z}{l}\right)\left(\frac{w}{l}\right) + \frac{1}{2}\left(\frac{w}{l}\right)^2\right)$$ [eq. 1.6]

$$U_e + N\cos\theta \simeq U_e + N = 0$$ [eq. 1.52]

$$W_e = N\sin\theta + K_s w \simeq \frac{N(z+w)}{l} + K_s w$$ [eq. 1.53]

$$\mathbf{g} = \mathbf{q}_i - \mathbf{q}_e$$
$$= \frac{N}{l}\begin{pmatrix} -l \\ z+w \end{pmatrix} + \begin{pmatrix} 0 \\ K_s w \end{pmatrix} - \begin{pmatrix} U_e \\ W_e \end{pmatrix}$$
$$= \begin{pmatrix} U_i \\ W_i \end{pmatrix} - \begin{pmatrix} U_e \\ W_e \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

internal force
external force
[eq. 1.54]



- **For two degrees of freedom, incremental method and the Newton-Raphson method are expanded as:**

$$\Delta \mathbf{p} = \mathbf{K}_t^{-1} \Delta \mathbf{q}_e$$

$$\mathbf{K}_t = \frac{\partial \mathbf{g}}{\partial \mathbf{p}} = \frac{\partial \mathbf{q}_i}{\partial \mathbf{p}}$$

$$\mathbf{p}^T = (u, \mathrm{w})$$

$$\begin{pmatrix} \Delta U_e \\ \Delta W_e \end{pmatrix} = \mathbf{K}_t \begin{pmatrix} \Delta u \\ \Delta w \end{pmatrix}$$

$$\beta = \frac{z + w}{l}$$

$$\delta \mathbf{p} = -\left(\frac{\partial \mathbf{g}}{\partial \mathbf{p}}\right)^{-1} \mathbf{g}_0 = -\mathbf{K}_t^{-1} \mathbf{g}$$

$$\mathbf{K}_t = \begin{pmatrix} \frac{\partial q_{i1}}{\partial p_1} & \frac{\partial q_{i1}}{\partial p_2} \\ \frac{\partial q_{i2}}{\partial p_1} & \frac{\partial q_{i2}}{\partial p_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial U_i}{\partial u} & \frac{\partial U_i}{\partial w} \\ \frac{\partial W_i}{\partial u} & \frac{\partial W_i}{\partial w} \end{pmatrix} = \frac{EA}{l} \begin{pmatrix} 1 & -\beta \\ -\beta & \beta^2 + K_s l / EA \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & N / l \end{pmatrix}$$

1.3.2 The use of virtual work

- It is hard to use equilibrium equation directly in finite element method.
- Instead of equilibrium equation, out-of-balance force vector **g** and tangent stiffness $\mathbf{K_t}$ can be derived from **principle of virtual work (PVW).**
- **PVW = virtual work should be zero for any arbitrary virtual displacements**

$$\varepsilon = -\frac{u}{l} + \left(\frac{z}{l}\right)\left(\frac{w}{l}\right) + \frac{1}{2}\left(\frac{w}{l}\right)^2$$ [eq. 1.51]

[eq. 1.75,1.76]

$$\delta\varepsilon_v = -\frac{\delta u_v}{l} + \left(\frac{z+w}{l}\right)\left(\frac{\delta w_v}{l}\right) + \frac{1}{2}\left(\frac{\delta w_v}{l}\right)^2$$

"V=virtual work undertaken by internal and external forces"

[eq. 1.78, 1.79]

$$V = \int \sigma\delta\varepsilon_v dV + K_s w\delta w_v - U_e\delta u_v - W_e\delta w_v = Nl\delta\varepsilon_v + K_s w\delta_v - U_e\delta u_v - W_e\delta w_v$$
$$= \mathbf{g}^T\delta\mathbf{p}_v$$
$$= (\mathbf{q}_i - \mathbf{q}_e)^T\delta\mathbf{p}_v$$

$$\Rightarrow \quad \mathbf{g} = \mathbf{q}_i - \mathbf{q}_e = \frac{N}{l}\begin{pmatrix} -l \\ z+w \end{pmatrix} + \begin{pmatrix} 0 \\ K_s w \end{pmatrix} - \begin{pmatrix} U_e \\ W_e \end{pmatrix}$$

$$\delta V = \mathbf{g}^T\delta\mathbf{p}_v = \delta\mathbf{p}_v^T\delta\mathbf{g} = \delta\mathbf{p}_v^T\frac{\partial\mathbf{g}}{\partial\mathbf{p}}\delta\mathbf{p} = \delta\mathbf{p}_v^T\mathbf{K}_t\delta\mathbf{p}$$ [eq. 1.80]

$$\Rightarrow \quad \mathbf{K}_t = \frac{EA}{l}\begin{pmatrix} 1 & -\beta \\ -\beta & \beta^2 + K_s l/EA \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & N/l \end{pmatrix} \qquad \beta = \frac{z+w}{l}$$

1.3.3 An energy method

- Out-of-balance force vector **g** and tangent stiffness $\mathbf{K}_t$ can also be derived from **principle of stationary potential energy.**

$$\phi = \int \frac{1}{2} E\varepsilon^2 dV + \frac{1}{2} K_s w^2 - \mathbf{q}_e^T \mathbf{p}$$

$$= \frac{1}{2} EAl \left(-\frac{u}{l} + \left(\frac{z}{l}\right)\left(\frac{w}{l}\right) + \frac{1}{2}\left(\frac{w}{l}\right)^2 \right)^2 + \frac{1}{2} K_s w^2 - U_e u - W_e w$$

[eq. 1.81, 1.82]

$$\mathbf{g} = \frac{\partial \phi}{\partial \mathbf{p}} \quad (\textit{differentiation done vertically})$$

[eq. 1.83]

$$\Rightarrow \quad \mathbf{g} = \mathbf{q}_i - \mathbf{q}_e = \frac{N}{l}\begin{pmatrix} -l \\ z + w \end{pmatrix} + \begin{pmatrix} 0 \\ K_s w \end{pmatrix} - \begin{pmatrix} U_e \\ W_e \end{pmatrix}$$

$$\mathbf{K}_t = \frac{\partial^2 \phi}{\partial \mathbf{p}^2}$$ "Hessian"

$$\Rightarrow \quad \mathbf{K}_t = \frac{EA}{l}\begin{pmatrix} 1 & -\beta \\ -\beta & \beta^2 + K_s l / EA \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & N/l \end{pmatrix}$$

Thank you!