Ch 2. Equations of fluid motion

- Continuum $\quad K_n \equiv \lambda / l \ll 1$ $\quad$ λ : mean free path 6×10^{-8} m

  Knudsen number $\quad$ l : flow length scale

- Eulerian & Lagrangian fields

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \underline{U} \cdot \nabla = \frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x_1} + U_2 \frac{\partial}{\partial x_2} + U_3 \frac{\partial}{\partial x_3}$$

material derivative $\quad$ local time derivative $\quad$ convection by $\underline{U}$

$$= \frac{\partial}{\partial t} + \sum_{i=1}^{3} U_i \frac{\partial}{\partial x_i}$$

$$= \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i} \quad \text{tensor notation}$$

- Continuity equation (mass conserv.)

$$\frac{dm}{dt} = 0$$

$$\rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{U}) = 0 \rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i) = 0$$

$\nabla \cdot \underline{U}$: dilatation $\quad$ U_i : instantaneous velocity

For const ρ, $\underline{\nabla \cdot \underline{U} = 0} \rightarrow \frac{\partial U_i}{\partial x_i} = 0$

↳ $\underline{U}$ is solenoidal or divergence free.

- Momentum equation

τ_{ij} : stress tensor , $\tau_{ij} = \tau_{ji}$: Symmetric tensor

$$\underset{\sim}{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

$i, j = 1, 2, 3$

$$\rho \frac{DU_j}{Dt} = \frac{\partial \tau_{ij}}{\partial x_i} - \rho g_j$$

$$\left(\frac{\partial U_j}{\partial t} + U_i \frac{\partial U_j}{\partial x_i} \right)$$

$$\frac{\partial}{\partial x_i} \frac{\partial U_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{\partial U_i}{\partial x_i} \right)^{\nearrow 0} = 0$$

$$\frac{\partial}{\partial x_i} \frac{\partial U_j}{\partial x_i} = \frac{\partial^2 U_j}{\partial x_i \partial x_i} \rightarrow \nabla^2 U_j$$

For constant-property Newtonian fluids,

$$\tau_{ij} = -\underset{\substack{\uparrow \\ \text{pressure}}}{p}\, \delta_{ij} + \underset{\substack{\uparrow \\ \text{viscosity}}}{\mu} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$$

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & i \neq j \end{cases}$$

$$\frac{\partial}{\partial x_i}(-p\delta_{ij}) = \frac{\partial p}{\partial x_j}$$

$$\rho \frac{DU_j}{Dt} = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 U_j}{\partial x_i \partial x_i} - \rho g_j$$

$\underline{g} = -\nabla\psi \qquad \psi$: gravitational potential

$P = p - \rho\psi$

$$\rightarrow \boxed{\rho\frac{DU_j}{Dt} = -\frac{\partial P}{\partial x_j} + \mu\frac{\partial^2 U_j}{\partial x_i \partial x_i}} \qquad \rho\frac{D\underline{U}}{Dt} = -\nabla p + \mu\nabla^2\underline{U}$$

$$\rho\left(\frac{\partial U_j}{\partial t} + U_i\frac{\partial U_j}{\partial x_i}\right) = -\frac{\partial P}{\partial x_j} + \mu\frac{\partial^2 U_j}{\partial x_i \partial x_i} \qquad \boxed{\frac{D\underline{U}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{U}} \leftarrow$$

Navier–Stokes eq. $\qquad$ kinematic viscosity

For inviscid flow ($\mu = 0$), $\tau_{ij} = -p\delta_{ij}$

$$\rightarrow \boxed{\frac{D\underline{U}}{Dt} = -\frac{1}{\rho}\nabla p} \quad \text{Euler equation}$$

- Role of pressure

$$\frac{\partial}{\partial t} + \underline{U}\cdot\nabla$$

$$\nabla\cdot\left[\left(\frac{D}{Dt} - \nu\nabla^2\right)\underline{U} = -\frac{1}{\rho}\nabla p\right] \qquad \nabla\cdot\nabla p = \nabla^2 p$$

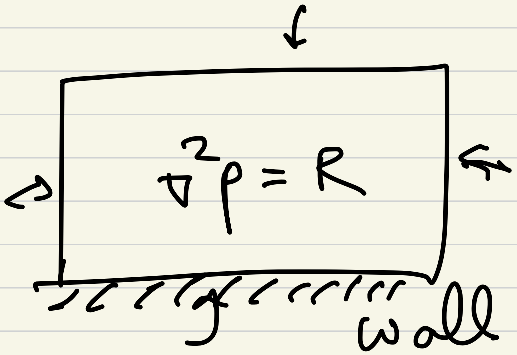
$$\rightarrow \left(\frac{D}{Dt} - \nu\nabla^2\right)\underbrace{(\nabla\cdot\underline{U})}_{0} + \left[\nabla\cdot\left(\frac{D}{Dt} - \nu\nabla^2\right)\underline{U}\right] = -\frac{1}{\rho}\nabla^2 p$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial t}+u_i\frac{\partial}{\partial x_i}\right)u_j = \frac{\partial}{\partial x_j}\left(u_i\frac{\partial}{\partial x_i}\right)u_j$$

$$= \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

$$\rightarrow 0 = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_i} - \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} \rightarrow \boxed{\underbrace{\frac{\partial^2 p}{\partial x_i \partial x_i}}_{\nabla^2 p} = -\rho\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}} \leftarrow \text{incomp. flow}$$

Poisson eq.

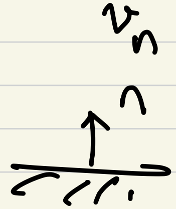


@ wall, $\underline{u} = 0$

$$\cancel{\frac{\partial \underline{u}}{\partial t}}^{\,0} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$$

@ wall, $\underline{u} = 0$, $(\underline{u}\cdot\nabla)\underline{u} = 0$

$$\boxed{\left.\frac{\partial p}{\partial n}\right|_{wall} = \mu\left.\frac{\partial^2 v_n}{\partial n^2}\right|_{wall}} = O(\varepsilon)$$



If p is a sol., $p + c$ is also a sol.

Comp. flow: ρ, $\underline{u}$, I, P ← state eq.

I ← energy eq.

Using Green ft.,

$$p(\underline{x},t) = p^{(h)}(\underline{x},t) + \frac{\rho}{4\pi}\iiint_V \frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}\Big|_{\underline{y},t}\frac{d\underline{y}}{|\underline{x}-\underline{y}|}$$

homo. sol.

- Conserved passive scalars

$\phi(\underline{x},t)$: conserved passive scalar

In a constant-property flow,

$$\frac{D\phi}{Dt} = \Gamma\nabla^2\phi \qquad \Gamma : (\text{const \& uniform}) \text{ diffusivi-}$$

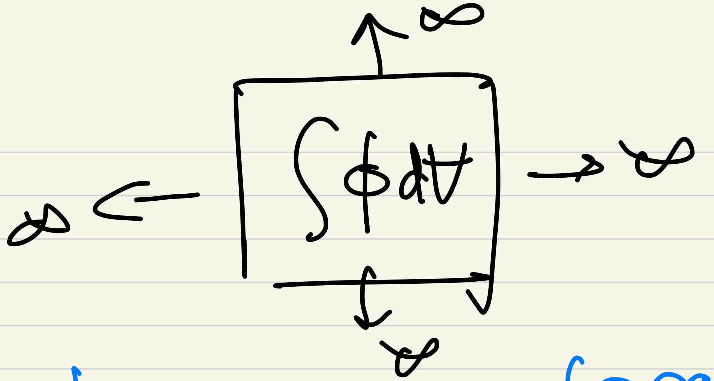
$$\frac{\partial}{\partial x_i}\left(\Gamma\frac{\partial\phi}{\partial x_i}\right) \leftarrow$$

$$\Downarrow$$

$$\frac{\partial\phi}{\partial t} + (\underline{U}\cdot\nabla)\phi$$

$$\frac{\partial\phi}{\partial t} + U_i\frac{\partial}{\partial x_i}\phi = \frac{\partial\phi}{\partial t} + \frac{\partial}{\partial x_i}(U_i\phi) - \phi\cancel{\frac{\partial U_i}{\partial x_i}}^{\,0} \Rightarrow \frac{\partial\phi}{\partial t} + \frac{\partial}{\partial x_i}(U_i\phi) = \frac{\partial}{\partial x_i}\left(\Gamma\frac{\partial\phi}{\partial x_i}\right)$$

ϕ is conserved because there is no source/sink in this eq.



$$\frac{D}{Dt}\int \phi \, d\forall = 0$$

ϕ is <u>passive</u> because it does not change the flow.

ϕ can be temperature $\rightarrow$ Γ : thermal diffusivity

$$Pr = \nu / \Gamma$$

concentration $\rightarrow$ Γ : molecular diffusivity

$$Sc = \nu / \Gamma$$

* boundedness : If the initial and boundary values of ϕ lie within a given range

$$\phi_{min} \le \phi \le \phi_{max},$$

then $\phi(\underline{x}, t)$ for all $(\underline{x}, t)$ also lies in this range.

That is, the values of ϕ greater than ϕ_{max} or less than ϕ_{min} cannot occur.

- Vorticity equation

turbulent flow → rotational → non-zero vorticity $\underline{\omega} \neq 0$

vorticity $\underline{\omega} = \nabla \times \underline{u}$

$\nabla \times$ [N-S eq.] :

vorticity equation

$$\boxed{\frac{D\underline{\omega}}{Dt} = \frac{\partial \underline{\omega}}{\partial t} + (\underline{u}\cdot\nabla)\underline{\omega} = (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega}}$$

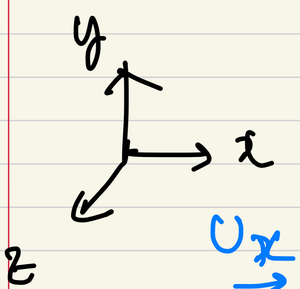
vortex stretching term

⇑

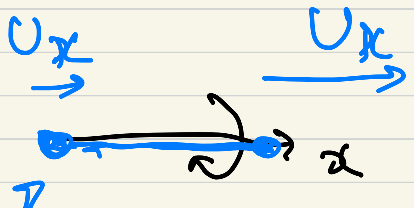
increases $|\underline{\omega}|$

$$\frac{\partial \phi}{\partial t} + (\underline{u}\cdot\nabla)\phi = \nu\nabla^2\phi$$

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$$



$$(\underline{\omega}\cdot\nabla)\underline{u}\Big|_x \rightarrow \omega_x\frac{\partial u_x}{\partial x} + \omega_y\frac{\partial u_x}{\partial y} + \omega_z\frac{\partial u_x}{\partial z}$$



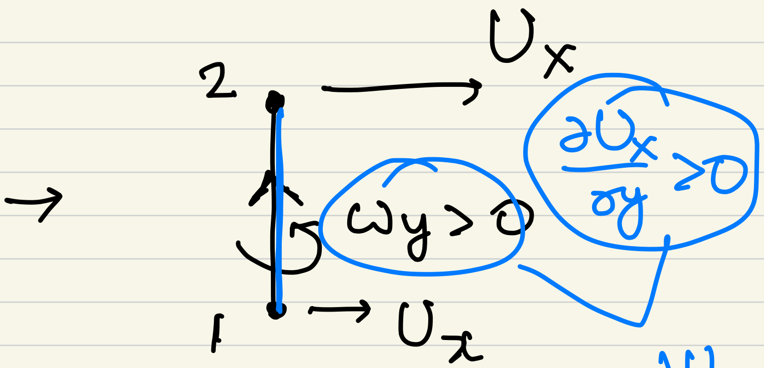
$\Rightarrow \frac{\partial u_x}{\partial x} > 0$

$\omega_x > 0$

$\omega_x \frac{\partial u_x}{\partial x} > 0$ ⇒

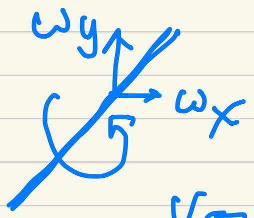
$$\frac{D\omega_x}{Dt} = \underbrace{(\underline{\omega}\cdot\nabla)\underline{u}\Big|_x}_{>0} + \underbrace{\nu\nabla^2\omega_x}_{\text{decaying}}$$

vortex stretching



$\omega_y \frac{\partial U_x}{\partial y} > 0$

tilting occurs



vortex tilting $\Rightarrow$ $\frac{D\omega_x}{Dt} > 0$

In inviscid flow, the vorticity vector behaves in the same way as an infinitesimal material line element.

$\left(\frac{D\underline{\xi}}{Dt} = (\underline{\xi}\cdot\nabla)\underline{u}\right.$ for material line)

2-D flow $(u, v) \rightarrow \omega_z \neq 0 \Rightarrow (\underline{\omega}\cdot\nabla)\underline{u} = 0$ no vortex stretching

u_x, u_y $\qquad \omega_x = \omega_y = 0$ $\qquad \frac{D\underline{\omega}}{Dt} = \nu \nabla^2 \underline{\omega}$

$\hookrightarrow$ one component of $\underline{\omega}$ evolves as a conservative passive scalar.