Rigid Body Rotation and SO(3)

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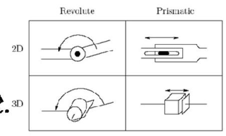
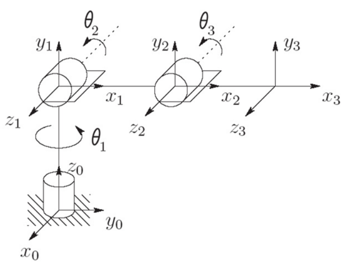
Attaching Coordinate Frames

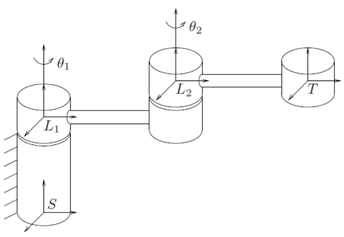
- Robot = rigid links w/ inertia + joints (relative motion w/ actuation or not)
- Typical joints =

$$\begin{cases} \text{revolute joints} & \theta_i \in [0, 2\pi) \approx S; \text{ or} \\ \text{prismatic joints} & \theta_i \in [d_{\min}, d_{\max}] =: D \in \Re. \end{cases}$$

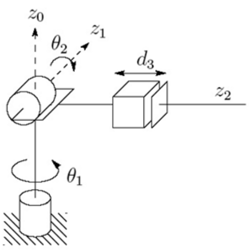


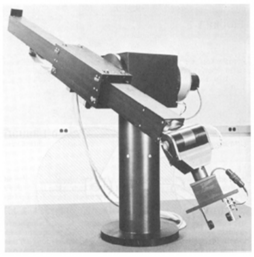
- To describe robot configuration, attach coordinate frame $\{i\}$ on the link i.
- Link 0 starts from the fixed base.
- The i-th joint θ_i between link $i-1$ and link i.
- Link i and $\{i\}$ move together with θ_i.
- θ_i actuation axis along z_{i-1}.

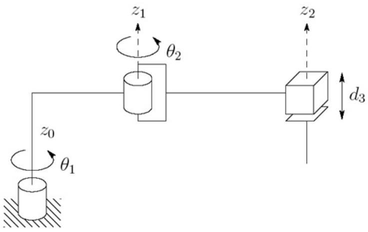




Coordinate Frames: Examples

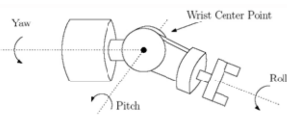


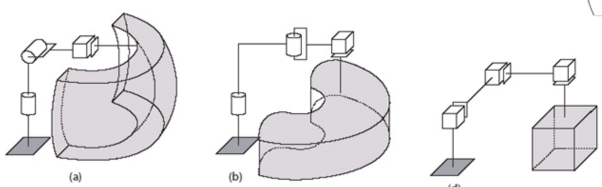




Joint Space Q and Workspace W

1. Joint variable $q := (\theta_1, \theta_2, ..., \theta_n)$
2. **Joint space** $Q := \{q\}$ (e.g., $Q = S \times S \times R$ for SCARA).
3. End-effector: gripper, hand, tool, etc. (typically last joint with $\{E\}$).
4. Wrist: joint between the end-effector and the preceding link.
5. **Workspace** $W \in$ SE(3): set of all permissible pose of EF.
   - Reachable WS $W_R \in E(3)$: set of EF position reachable with some joint angles.
   - Dexterous WS $W_S \in E(3)$: set of EF position reachable with arbitrary EF orientation.
   - $W_D \subset W_R$, $W_D = W_R$ with spherical wrist.

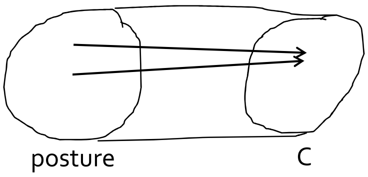


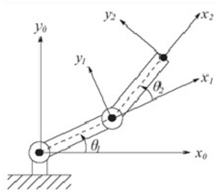


(a) (b) (d)

Configuration Space C

1. Configuration: a set of certain variables that can *completely* specify the location of all the points of the robot (i.e., posture of the robot).
2. A space C is **configuration space** if:
   (a) Every $x \in C$ corresponds to a valid configuration of the system (i.e., onto/surjective with posture set as domain and C as range); and
   (b) Every system configuration can be identified with a unique $x \in C$ (i.e., one-to-one/injective).
3. Joint space Q is a configuration space; Workspace W may or may not be a configuration space.
4. **degree-of-freedom** (DOF) $= \dim(C) = \dim(Q)$.

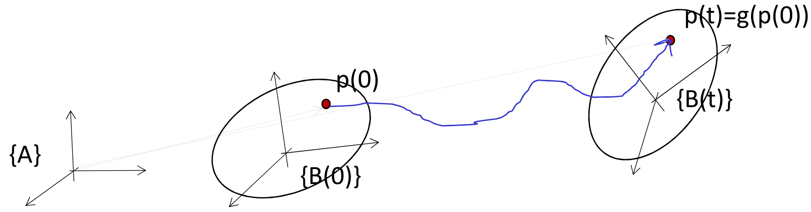




Rigid Body Transformation

In this course, robot consists of rigid links. How to describe rigid body motion?

- Rigid body motion description of O during $[0, t)$
  - Consider a rigid object O in Euclidean space $\Re^3$.
  - Attach a coordinate frame $\{B(0)\}$ at a point on O at $t = 0$.
  - Keep track the pose of $\{B(t)\}$
- This rigid body motion can be thought of as *rigid body transformation* map $g : \Re^3 \rightarrow \Re^3$, s.t.,
  $$g(p(0)) = p(t)$$
  where $p(t) \in \Re^3$, $t \geq 0$, is a point p of $O(t)$.



Free Vector

- For the positions $p, q \in \Re^3$, define

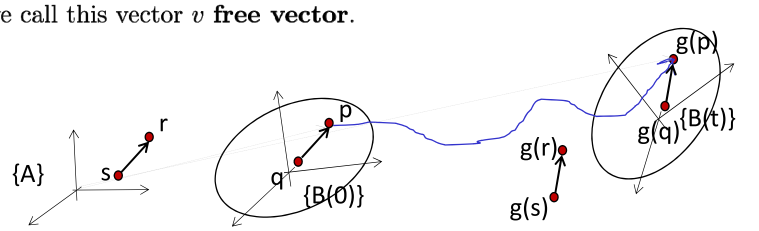
$$v := p - q \in \Re^3$$

  Although $v \in \Re^3$ appears similar to $p, q \in \Re^3$, it is conceptually different.

- The mapped $g(p)$ can change its length, yet, it shouldn't happen with the mapping of v under rigid body motion g.
- Action g_* of rigid transformation g defined s.t., with $v = p - q = r - s$,

$$g_*(v) := g(p) - g(q) = g(r) - g(s)$$

  Note v and $g_*(v)$ are free to float from where it starts. Due to this reason, we call this vector v **free vector**.



Rigid Transformation: Definition

Definition 1 *A mapping $g : \Re^3 \to \Re^3$ is rigid body transformation if*

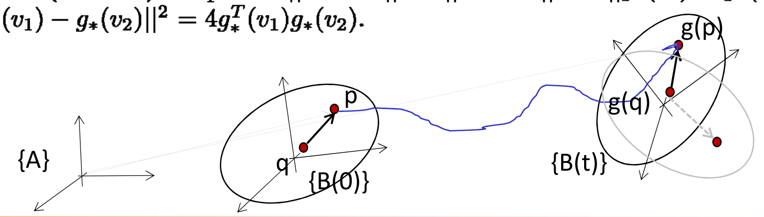
1. $||g(p) - g(q)|| = ||p - q|| \; \forall p, q \in \Re^3$ *(i.e., distance preserving);*
2. $g_*(v \times w) = g_*(v) \times g_*(w)$ *(e.g., no mirroring).*

where $g_*(v) := g(p) - g(q)$, $||x||^2 := x^T x$ and $\times$ is the cross product.

Properties of g_*:

1. $||g_*(v)|| = ||v||$ (norm preserving)
2. $g_*(av) = a g_*(v)$, $g_*(v_1 + v_2) = g_*(v_1) + g_*(v_2)$ (linearity)
3. $g_*^T(v_1) g_*(v_2) = v_1^T v_2$ (isometry)

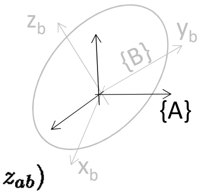
Proof (Item 2): $4 v_1^T v_2 = ||v_1 + v_2||^2 - ||v_1 - v_2||^2 = ||g_*(v_1) + g_*(v_2)||^2 - ||g_*(v_1) - g_*(v_2)||^2 = 4 g_*^T(v_1) g_*(v_2)$.



Rotation Matrix R

- Rigid body motion = rotation + translation. Consider first rotation.
- Rotation of a rigid body can be described by the body-frame $\{B\}$ attached to the object relative to $\{A\}$.
- This rotation of $\{B\}$ relative to $\{A\}$ can be be written as **rotation matrix**

$$R_{ab} = \left[\begin{array}{ccc} x_b^a & y_b^a & z_b^a \end{array}\right] \in \Re^{3\times 3}$$



where

$$x_b^a = [x_b^T x_a; x_b^T y_a; x_b^T z_a] \in \Re^3 \quad \text{(similar for } y_{ab}, z_{ab})$$

where $x_b, y_b, z_b \in \Re^3$ and $x_a, y_a, z_a \in \Re^3$ are the orthonormal principle-axis basis vectors of $\{B\}$ and $\{A\}$; and x_{ab}, y_{ab}, z_{ab} are x_b, y_b, z_b represented in the inertial frame $\{A\}$.

- Note that $R_{ab}[1;0;0] = x_b^a$, $R_{ab}[0;1;0] = y_b^a$, $R_{ab}[0;0;1] = z_b^a$ with $x_b^b = [1;0;0]$, $y_b^b = [0;1;0]$, $z_b^b = [0;0;1]$ (i.e., each representing principle axis of $\{B\}$ represented in $\{A\}$).

Three Roles of R

1. R serves as **configuration** for rotation motion: $\{B\}$ relative to $\{A\}$.
2. R serves as **coordinate transformation**: $q \in \Re^3$ can be expressed in $\{A\}$ or $\{B\}$ s.t.

$$q = q_x^a x_a + q_y^a y_a + q_z^a z_a = q_x^b x_b + q_y^b y_b + q_z^b z_b$$

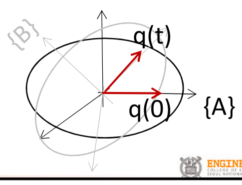
with $q_a = [q_x^a; q_y^a; q_z^a]$ (in $\{A\}$) and $q_b = [q_x^b; q_y^b; q_z^b]$ (in $\{B\}$). Then,

$$\begin{pmatrix} q_x^a \\ q_y^a \\ q_z^a \end{pmatrix} = \begin{bmatrix} x_a^T x_b & x_a^T y_b & x_a^T z_b \\ y_a^T x_b & y_a^T y_b & y_a^T z_b \\ z_a^T x_b & z_a^T y_b & z_a^T z_b \end{bmatrix} \begin{pmatrix} q_x^b \\ q_y^b \\ q_z^b \end{pmatrix}, \quad \text{i.e.,} \quad q_a = R_{ab} q_b$$

3. R serves as **rotation operator**: if q rigidly-attached on the object and the object rotates from $\{A\}$ to $\{B\}$ during $[0,t]$, $q_b(t) = q_a(0)$ $\forall t \geq 0$. Then,

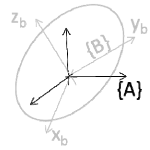
$$q_a(t) = R_{ab} q_b(t) = R_{ab} q_a(0)$$

that is, $R : q_a(0) \mapsto q_a(t)$.



Properties of R

- $R^TR = RR^T = I,\ R^{-1} = R^T$
- $\det R = +1.$
- $R = R_*$, i.e., the action of R for free-vector is also R.



(Proof):

- (Item 1) If we write

$$R_{ab} = \begin{bmatrix} x_a^T x_b & x_a^T y_b & x_a^T z_b \\ y_a^T x_b & y_a^T y_b & y_a^T z_b \\ z_a^T x_b & z_a^T y_b & z_a^T z_b \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$$

$r_1 = x_b^a, r_2 = y_b^a, r_3 = z_b^a$ are the principle axis, therefore,

$$r_i^T r_j = 0 \ \text{ if } i \neq j; \quad r_i^T r_j = 1 \ \text{ if } i = j$$

- (Item 2) $\det R = r_1^T(r_2 \times r_3) = +1$, since (r_1, r_2, r_3) are right-handed.
- (Item 3) with $v = p - q$,

$$R_*(v_b) = R(p_b) - R(q_b) = p_a - q_a = v_a = R(v_b)$$

Properties of R

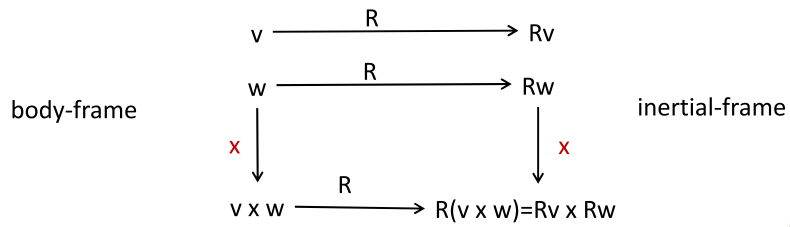
- For $a = [a_1; a_2; a_3] \in \Re^3$, define

$$(a)^\wedge := \hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in \Re^{3\times 3} \quad \text{s.t.} \quad a \times b = (a)^\wedge b$$

This skew-symmetric matrix defines Lie algebra for SO(3), constituting vector space of rotational velocity.

Lemma 1 $R(v \times w) = (Rv) \times (Rw), \quad R(w)^\wedge R^T = (Rw)^\wedge$

(Proof): First shows that $\times$ and R commute.



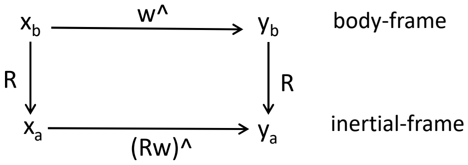
Properties of R

Lemma 1 $R(v \times w) = (Rv) \times (Rw), \quad R(w)^\wedge R^T = (Rw)^\wedge$

Second shows $(w)^\wedge$ as a map in $\{B\}$ is given in $\{A\}$ by $(Rw)^\wedge = R(w)^\wedge R^T$. Consider $y_b = (w_b)^\wedge x_b$ in $\{B\}$. This can then be represented in $\{A\}$ by

$$y_a = Ry_b = R(w_b)^\wedge x_b = R(w_b)^\wedge R^T x_a = R(w_b \times x_b) = (Rw_b)^\wedge x_a$$

that is, $(Rw_b)^\wedge = R(w_b)^\wedge R^T$.



Prop. 1 (2.2) *A rotation R is a rigid-body transformation, i.e.,*

$$||R(p-q)|| = ||p-q||, \quad R(v \times w) = Rv \times Rw$$

First from $(R(p-q))^T R(p-q) = (p-q)^T(p-q) = ||p-q||^2$. Second from Lem. 1.

Composition of Rotations

Consider $R_1 : \{A\} \to \{B\}$ for $[t_0, t_1)$ and $R_2 : \{B\} \to \{C\}$ for $[t_1, t_2)$. The rotation R_2 can be expressed by R_2^a w.r.t. $\{A\}$ or by R_2^b w.r.t. $\{B\}$. Then, the composition of rotations $q_a(t_2) = R_{ac} q_a(t_o)$ is given by

- Successive rotations w.r.t. **body frames**: $R_{ac} = R_1^a R_2^b$
- Successive rotations w.r.t. **inertial frame**: $R_{ac} = R_2^a R_1^a$

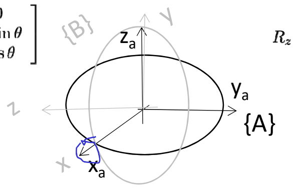
- From $q_a(t_1) = R_1^a q_a(t_o)$ and $q_b(t_2) = R_2^b q_b(t_1)$ with $q_b(t_1) = q_a(t_o)$,

$$q_a(t_2) = R_1^a q_b(t_2) = R_1^a R_2^b q_b(t_1) = R_1^a R_2^b q_a(t_o)$$

- From $q_a(t_1) = R_1^a q_a(t_o)$ and $q_a(t_2) = R_2^a q_a(t_1)$: $q_a(t_2) = R_2^a R_1^a q_a(t_o)$.

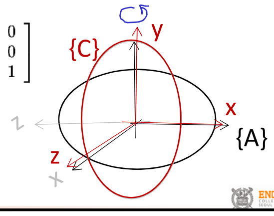
$R_1 : R_1^a = R_x, R_1^b = R_x$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



$R_2 : R_2^a = R_z, R_2^b = R_y$

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Composition of Rotations

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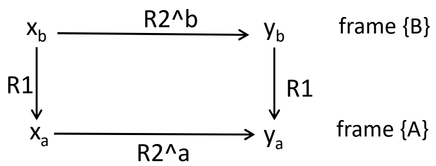
- Successive rotations w.r.t. **body frames**: $R_{ac} = R_1^a R_2^b$
- Successive rotations w.r.t. **inertial frame**: $R_{ac} = R_2^a R_1^a$

- Since these two rotations are the same, we have

$$R_2^b = R_1^T R_2^a R_1$$

which is the mapping R_2^a in $\{A\}$ written in $\{B\}$. Note that both R_2^a and R_2^b represent the same rotation, yet, expressed in different frames.



Special Orthogonal Group SO(3)

Def. 1 (Special orthogonal group)

$$SO(n) := \{R \in \Re^{n \times n} \mid R^T R = I, \det R = +1\}$$

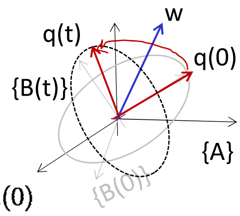
- SO(3) is a Lie group under matrix multiplication.
- Lie group is a group G (i.e., $gh \in G \; \forall g, h \in G$), which is also a smooth manifold (i.e., assumes local smooth coordinate charts) and for which $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.

- (closure) if $R_1, R_2 \in$ SO(3), $R_1 R_2 \in$ SO(3), since $(R_1 R_2)(R_1 R_2)^T = I$ and $\det(R_1 R_2) = \det R_1 \det R_2 = 1$.
- (identity) $R = I \in$ SO(3) is the identity with $RI = IR = R$.
- (inverse) for each R, there exists an unique inverse $R^T \in$ SO(3).
- special with $\det R = +1$; SO(3) represents rotation in 3D; SO(2) rotation in 2D.
- SO(3) not vector space: how does the agular velocity look like?

Exponential of w

Consider a rigid body rotating from $\{B(0)\}$ to $\{B(t)\}$ with *constant* angular velocity w_a expressed in inertial frame $\{A\}$. What's relation between w and R?

- Consider a point q attached to the object. We can then express in $\{A\}$ s.t.,



$$\dot{q}_a = w_a \times q_a = \hat{w}_a q_a$$

  where $\hat{w} = (w)^\wedge$ (e.g., $w_a = [0; 0; w_3]$ in $\{A\}$).

- We can further have

$$q_a(t) = e^{\hat{w}_a t} q_a(0) = R^a_{ab(t)} q_b(t) = R^a_{ab(t)} q_b(0)$$
$$= R^a_{ab(t)} [R^a_{ab(0)}]^T q_a(0) = R^a_{b(0)b(t)} q_a(0)$$

  with $R^a_{ab(t)} = R^a_{b(0)b(t)} R^a_{ab(0)}$, i.e., composition of rotation w.r.t. $\{A\}$.

- **Exponential of w**

$$e^{\hat{w}_a t} = I + \hat{w}_a t + \frac{(\hat{w}_a t)^2}{2!} + \frac{(\hat{w}_a t)^3}{3!} + ... = R^a_{b(0)b(t)}$$

  represents rotation from $\{B(0)\}$ to $\{B(t)\}$ via w during t expresed in $\{A\}$.

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Angular Velocity and so(3)

Def. 1 (Lie Algebra so(n))

$$so(n) := \{S \in \Re^{n \times n} \mid S^T = -S\}$$

- Any angular velocity $w \in \Re^3$ can be identified by so(3) via $(w)^\wedge$.
- so(3) is the vector space of **angular velocity** with $a\hat{w}_1 + b\hat{w}_2 \in so(3)$, $\forall a, b, \in \Re$.
- In contrast, SO(3) is not a vector space (e.g., $aR_1 + bR_2 \notin SO(3)$).
- so(3) is Lie algebra of SO(3) with the bracket structure

$$[\hat{w}_1, \hat{w}_2] = \hat{w}_1\hat{w}_2 - \hat{w}_2\hat{w}_1 = (w_1 \times w_2)^\wedge$$

  with, $\forall \hat{v}, \hat{w}, \hat{z} \in$ so(3),

$$[\hat{v}, \hat{w}] = -[\hat{w}, \hat{v}], \quad [[\hat{v}, \hat{w}], \hat{z}] + [[\hat{z}, \hat{v}], \hat{w}] + [[\hat{w}, \hat{z}], \hat{v}] = 0$$

  - Lie algebra of Lie group G is the tangent space at identity T_eG with the bracket $[\xi, \eta] := [\xi_L, \eta_L](e)$, where ξ_L is left-invariant vector field s.t., $\xi_L(e) \in T_eG$ and $\xi_L(g \cdot h) = T_h L_g \xi_L(h)$, $\forall g, h \in G$.
  - $L_R[\hat{w}_1, \hat{w}_2] = [R(w_1 \times w_2)]^\wedge = (Rw_1 \times Rw_2)^\wedge = [L_R\hat{w}_1, L_R\hat{w}_2]$, i.e., $[,]$ is left invariant too.

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Rodrigues' Formula

Exponential of w is a infinite series, thus, not practically useful. Rodrigues' formular provides a closed-form expression of $e^{\hat{w}t} \in$SO(3) given (w, θ).

- First, normalize angular velocity by $w\theta$ with $||w|| = 1$ (direction) and $\theta \in \Re$ (duration):

$$e^{\hat{w}\theta} = I + \theta\hat{w} + \frac{\theta^2}{2!}\hat{w}^2 + \frac{\theta^3}{3!}\hat{w}^3 + ...$$

- **Rodrigues' formula**

$$e^{\hat{w}\theta} = I + \hat{w}\sin\theta + \hat{w}^2(1 - \cos\theta)$$

- (Proof): From the following facts:

$$\hat{a}^2 = aa^T - ||a||^2 I, \quad \hat{a}^3 = -||a||^2\hat{a}$$

$\hat{w}^3 = -\hat{w}$, $\hat{w}^4 = -\hat{w}^2$,.... Thus, we have

$$e^{\hat{w}\theta} = I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}...\right)\hat{w} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}...\right)\hat{w}^2$$

Exponential Coordinates for SO(3)

In fact, (w, θ), $||w|| = 1$, can be used as "coordinates" for SO(3).

Prop. 1 (2.4) *For every $\hat{w} \in so(3)$ and $\theta \in \Re$, $e^{\hat{w}\theta} \in SO(3)$.*

(Proof): From matrix exponential property and skew-symmetricity of $\hat{w}$,

$$[e^{\hat{w}\theta}]^{-1} = e^{-\hat{w}\theta} = e^{\hat{w}^T\theta} = [e^{\hat{w}\theta}]^T$$

verifying that $R^T R = I$. Also, $\det(e^{\hat{w}\theta}) = 1$, since $\det(e^{\hat{w}0}) = +1$ and $e^{\hat{w}\theta}$ is a continuous map w.r.t. θ.

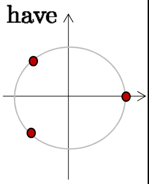
- This shows that, given (w, θ), we can always compute $R = e^{\hat{w}\theta}$, which defines a valid rotation in SO(3). How to find (w, θ) given R then?

Logarithm Map for SO(3)

Prop. 1 (2.5) *For every $R \in SO(3)$, there exist $w \in \Re^3$ ($||w|| = 1$) and $\theta \in \Re$ s.t.*

$$R = e^{\hat{w}\theta}$$

(Proof): Equating R and Rodrigues' formula $e^{\hat{w}\theta}$ componentwise, we have



$$\mathrm{trace}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta$$
$$r_{32} - r_{23} = 2w_1 \, s\,\theta, \; r_{13} - r_{31} = 2w_2 \, s\,\theta, \; r_{21} - r_{12} = 2w_3 \, s\,\theta$$

where $-1 \leq \mathrm{trace}(R) = \sum_{i=1}^{3} \lambda_i(R) \leq 3$ (i.e., solution θ exists), since

$$\det(R) = \lambda_1\lambda_2\lambda_3 = \lambda_1|\lambda_2|^2 = +1$$

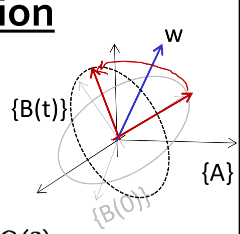
where $\lambda_1 = 1$ since $Rw_1 = \lambda_1 w_1$ should preserve the length. In fact, w is eigenvector of $R = e^{\hat{w}\theta}$ with $\lambda_1 = +1$.

- **Logarithm on SO(3)**: $R = e^{\hat{w}\theta} \Longleftrightarrow \log R = \hat{w}\theta$

$$2\cos\theta + 1 = \mathrm{trace}(R) \quad \text{and} \quad \hat{w} = (R - R^T)/(2\sin\theta), \quad R \neq I$$

Equivalent Axis Representation

- **Rodrigues' formula**



$$e^{\hat{w}\theta} = I + \hat{w}\sin\theta + \hat{w}^2(1 - \cos\theta) = R$$

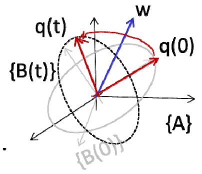
  - (w, θ) can be used as a configuration for SO(3) via $e^{\hat{w}\theta}$
  - Thus, follows the name "exponential coordinates" for SO(3).
  - Given R, we can find rotation axis w and rotation angle θ.
  - This exponential coordinate is many-to-one (i.e., if (w, θ) is a solution, so are $(-\theta, -w)$ and $(\theta \pm 2n\pi, w)$);
  - Singular when $R = I = e^{\hat{w}\theta}$, for which $\theta = 0$ and $w \in \Re^3$ not defined.

Theorem 1 (2.6: Euler) *Any orientation $R \in SO(3)$ is equivalent to a rotation about a fixed axis $w \in \Re^3$ ($||w|| = 1$) by an angle $\theta \in [0, 2\pi)$.*

  - As above, this representation is singular at $R = I$.

Exponential Coordinates in Different Frames

- Consider object rotating from $\{B(0)\}$ to $\{B(t)\}$ with constant w during $[0, t]$.



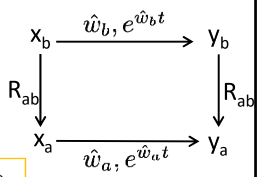
- Then, for a point q rigidly-attached to the object, we have

$$\dot{q}_a(t) = w_a \times q_a(t) \quad \longrightarrow \quad q_a(t) = e^{\hat{w}_a t} q_a(0)$$

where (w_a, t) is exponential coordinates of rotation written in $\{A\}$.

- Now, let us see how this same rotation is expressed by exponential coordinates in $\{B(0)\}$.

- For this, we have



$$q_a(t) = R_{ab(0)} q_{b(0)}(t) = e^{\hat{w}_a t} R_{ab(0)} q_{b(0)}(0)$$

that is,

$$q_{b(0)}(t) = R^T_{ab(0)} e^{\hat{w}_a t} R_{ab(0)} q_{b(0)}(0) = e^{\hat{w}_{b(0)} t} q_{b(0)}(0)$$

since, with $\hat{w}_b = R^T_{ab} \hat{w}_a R_{ab}$,

$$R^T_{ab(0)} [I + \hat{w}_a \, s\, t + \hat{w}_a^2 (1 - c\, t)] R_{ab(0)} = e^{\hat{w}_{b(0)} t}$$

Geometric Meaning of exp(w_at) and exp(w_bt)

Exponential Coordinates in $\{B\}$**:** Given (w_a, t) in $\{A\}$, $q_b(t) = e^{\hat{w}_b t} q_b(0)$ with $e^{\hat{w}_b t} = R^T_{ab} e^{\hat{w}_a t} R_{ab}$, i.e., $(w, t) \approx (w_a, t)$ in $\{A\}$ and $\approx (w_b, t)$ in $\{B\}$.

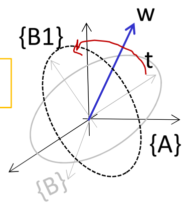
- $e^{\hat{w}_a t}$ in fact represents $R^a_{bb_1}$, i.e., rotation from $\{B\}$ to $\{B_1\}$ represented in $\{A\}$, since, with $q_b(0) = q_{b_1}(t)$ ($\{B\}$ rigidly attached),

$$q_a(t) = R_{ab_1} q_{b_1}(t) = R_{ab_1} q_b(0) = e^{\hat{w}_a t} q_a(0) = e^{\hat{w}_a t} R_{ab} q_b(0)$$

  for all $q_b(0)$, therefore,

$$R_{ab_1} = e^{\hat{w}_a t} R_{ab} = R^a_{bb_1} R_{ab} \quad \to \quad \boxed{e^{\hat{w}_a t} = R^a_{bb_1}}$$

  from the composition of successive rotations w.r.t. $\{A\}$.



- On the other hand, we have

$$R_{ab_1} = e^{\hat{w}_a t} R_{ab} = R_{ab} e^{\hat{w}_b t} R^T_{ab} R_{ab} = R_{ab} e^{\hat{w}_b t} \quad \to \quad \boxed{e^{\hat{w}_b t} = R^b_{bb_1}}$$

  i.e., rotation of $\{B_1\}$ relative to $\{B\}$ written in $\{B\}$.

- Note that $e^{\hat{w}_a t}$ and $e^{\hat{w}_b t}$ represent the *same* rotation given by (w, t), yet, expressed in $\{A\}$ and $\{B\}$!

Rotational Velocity

Consider rotation of rigid-body via angular velocity w_{ab} relative to $\{A\}$ with a point q and frame $\{B(t)\}$ rigidly attached on it. Then, $q_a(t) = R_{ab}q_b(t)$ with $q_b(t)$ constant.

- Spatial angular velocity: angular velocity w_{ab} expressed in $\{A\}$:

$$\dot{q}_a = w^s_{ab} \times q_a(t) = \dot{R}_{ab}q_b(t) = \dot{R}_{ab}R^T_{ab}q_a(t) \;\rightarrow\; \hat{w}^s_{ab} := \dot{R}_{ab}R^T_{ab}$$

- Body angular velocity: angular velocity w_{ab} expressed in $\{B\}$:

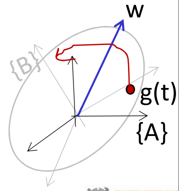
$$w^b_{ab} = R^T_{ab}w^s_{ab} \;\rightarrow\; \hat{w}^b_{ab} = R^T_{ab}\hat{w}^s_{ab}R_{ab} = R^T_{ab}\dot{R}_{ab}$$

from $R(w^\wedge)R^T = (Rw)^\wedge$. Further, we have

$$v_{q_b} = R^T_{ab}\dot{q}_a(t) = R^T_{ab}(w^s_{ab} \times q_a(t)) = w^b_{ab} \times q_b(t)$$

from $R(v \times w) = Rv \times Rw$ (note that $v_b = R^T_{ab}\dot{q}_a \neq \dot{q}_b = 0$).

- $\dot{R}_{ab} = \hat{w}^s_{ab}R_{ab} = R_{ab}\hat{w}^b_{ab}$.

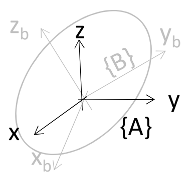


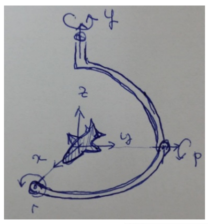
Parameterization of SO(3): Euler Angle

- **Euler angles**
  - ZYZ Euler angles: $R = R_z(\alpha)R_y(\beta)R_z(\gamma)$
  - Roll/pitch/yaw: $R_{rpy} = R_z(y)R_y(p)R_x(r)$
- R_x, R_y, R_z are basic rotation matrices: with $x := [1;0;0], y = [0;1;0]$,

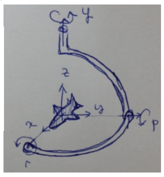
$$R_x(\theta) = e^{\hat{x}\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathrm{c}\,\theta & -\mathrm{s}\,\theta \\ 0 & \mathrm{s}\,\theta & \mathrm{c}\,\theta \end{bmatrix}, \quad R_y(\theta) = e^{\hat{y}\theta} = \begin{bmatrix} \mathrm{c}\,\theta & 0 & \mathrm{s}\,\theta \\ 0 & 1 & 0 \\ -\mathrm{s}\,\theta & 0 & \mathrm{c}\,\theta \end{bmatrix}$$

- Roll $= R_x(r)$, pitch $= R_y(p)$, yaw $= R_z(y)$ (ZYX Euler angles):
  - $y \rightarrow p \rightarrow r$ w.r.t. body frame; or
  - $r \rightarrow p \rightarrow y$ w.r.t. inertial frame.





Parameterization of SO(3): Euler Angle



- **Euler angles**
  - ZYZ Euler angles: $R = R_z(\alpha)R_y(\beta)R_z(\gamma)$
  - Roll/pitch/yaw: $R_{rpy} = R_z(y)R_y(p)R_x(r)$
- Given R, we can find r, p, y or α, β, γ (for closed-form, see book).
- R_{ZYZ} and R_{rpy} both have singularity (only local coordinates of SO(3)):
  - $R_{ZYZ}(\alpha, 0, -\alpha) = I$ (singular at $R = I$);
  - $R_{rpy}(r, -\pi/2, y) = R_{rpy}(r + \alpha, -\pi/2, y + \alpha)$ for any α (i.e., singular at $p = \pm\pi/2$) (gimbal lock with roll = yaw axis).
- Differential relation (Jacobian): from $\hat{w}^b = R^T \dot{R}$ and $w^a = Rw^b$,

$$w^b = \begin{bmatrix} 1 & 0 & -sp \\ 0 & cr & cpsr \\ 0 & -sr & cpcr \end{bmatrix} \begin{pmatrix} \dot{r} \\ \dot{p} \\ \dot{y} \end{pmatrix}, \quad w^a = \begin{bmatrix} cpcy & -sy & 0 \\ cpsy & cy & 0 \\ -sp & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{r} \\ \dot{p} \\ \dot{y} \end{pmatrix}$$

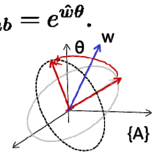
where note that $w_z^b \neq \dot{y}$.

Parameterization of SO(3): Quaternions

Unit quaternions: SO(3) rotation can be represented by

$$\bar{q}_{ab} = q_o + q_1\vec{i} + q_2\vec{j} + q_3\vec{k} = (q_o, \vec{q}) = (\cos\tfrac{\theta}{2}, \vec{w}\sin\tfrac{\theta}{2})$$

where $\vec{w}, \theta$ are the equivalent rotation (unit) axis and angle with $R_{ab} = e^{\hat{w}\theta}$.



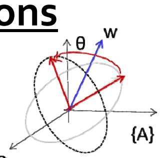
- $||\bar{q}||^2 = q_o^2 + \vec{q}^T\vec{q} = 1$ (unit quaternion).
- Quaternion product $\otimes$ defined by: with Hamilton convention,

$$\bar{q} \otimes \bar{p} = (q_o p_o - \vec{q}^T\vec{p}, q_o\vec{p} + p_o\vec{q} + \vec{q} \times \vec{p}) = [\bar{q}]_L \bar{p} \qquad [\bar{q}]_L = q_o I_4 + \begin{bmatrix} 0 & -\vec{q}^T \\ \vec{q} & [\vec{q}]^\wedge \end{bmatrix}$$

  with $\vec{v} \otimes \vec{w} = \vec{v} \times \vec{w} - \vec{v}^T\vec{w}$ for pure quaternions.
- $\bar{e} = 1 = (1, \vec{0})$ is identity of $\otimes$; inverse of $\bar{q} = q_o + \vec{q}$ is $\bar{q}^{-1} = q_o - \vec{q}$.
- Singularity of exponential coordinate (w, θ) at $\theta = 0$ is now avoided: at $R = I$, $\bar{q} = 1$ uniquely defined.
- $\bar{q} = (\theta, \vec{w})$ and $-\bar{q} = (2\pi - \theta, -\vec{w})$ represent the same rotation though: provides a singularity-free SO(3) parameterization up to the two-to-one mapping (e.g., restrict $q_o \geq 0$).

Parameterization of SO(3): Quaternions

Unit quaternions



$$\bar{q}_{ab} = q_o + \vec{q} = \cos\frac{\theta}{2} + \vec{w}\sin\frac{\theta}{2}, \quad ||\bar{q}|| = 1, \; ||\vec{w}|| = 1$$

- Given R_{ab}, we can compute $\bar{q}_{ab}$ using $R_{ab} = e^{\hat{w}\theta}$: with $q_o > 0$,

$$q_o = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}, \quad \vec{q} = \frac{1}{4q_o}[r_{32} - r_{23}, r_{13} - r_{31}, r_{21} - r_{12}]^T$$

- Given $\bar{q}_{ab}$, we can compute $R_{ab}(\bar{q})$ using $R_{ab} = e^{\hat{w}\theta}$ s.t.,

$$R_{ab}(\bar{q}) = (q_o^2 - \vec{q}^T\vec{q})I + 2\vec{q}\vec{q}^T + 2q_o[\vec{q}\times]$$

- Rotation of vector $\vec{v}$ by $\bar{q}_{ab}$: using $\bar{q}_{ab} = \mathrm{c}\frac{\theta}{2} + \vec{w}\,\mathrm{s}\frac{\theta}{2}$ and matching with $\vec{v}' = R_{ab}\vec{v}$,

$$\bar{v}' = 0 + \vec{v}' = \bar{q}_{ab} \otimes \bar{v} \otimes \bar{q}_{ab}^{-1} \quad \approx \quad v' = R_{ab}v$$

- Rotate $\vec{v}$ by $\bar{p} \otimes \bar{q} \Rightarrow \vec{v}' = \bar{p} \otimes \bar{q} \otimes \bar{v} \otimes (\bar{p} \otimes \bar{q})^{-1} = \bar{p} \otimes (\bar{q} \otimes \bar{v} \otimes \bar{q}^{-1}) \otimes \bar{p}^{-1}$
$\Rightarrow \bar{p} \otimes \bar{q}$ is body-frame composition of $\bar{p}$ and $\bar{q}$, $\Rightarrow$ $R(\bar{p} \otimes \bar{q}) = R(\bar{p})R(\bar{q})$.
- Differential relation with $w \in \mathrm{so}(3)$:

$$\dot{\bar{q}} = \frac{1}{2}\bar{w}_a \otimes \bar{q} \;\approx\; \dot{R} = S(w_a)R, \qquad \dot{\bar{q}} = \frac{1}{2}\bar{q} \otimes \bar{w}_b \;\approx\; \dot{R} = RS(w_b)$$