Rigid Body Transformation and SE(3)

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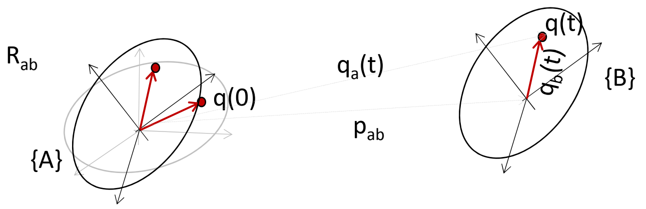
Rigid Body Transformation

- Consider rigid body motion, where the object **rotates first** by $R^a_{ab} \in SO(3)$, **then translates** by $p^a_{ab} \in \Re^3$, all expressed in $\{A\}$.
- Then, for a point q rigidly-attached to the object, we have

$$q_a(t) = R^a_{ab} q_a(0) + p^a_{ab} = g_{ab}(q_a(0))$$

where g_{ab} is the **rigid transformation map**.

- This rigid transformation g_{ab} can serve as:
  1. Configuration of the rigid-body motion.
  2. Coordinate transform btw $\{A\}$ and $\{B\}$: $q_a(t) = g_{ab}(q_a(0)) = g_{ab}(q_b(t))$.
  3. Rigid-body transformation operator: $q_a(t) = g_{ab}(q_a(0))$.



Rigid Transformation

- **Rigid body transformation** g_{ab}:

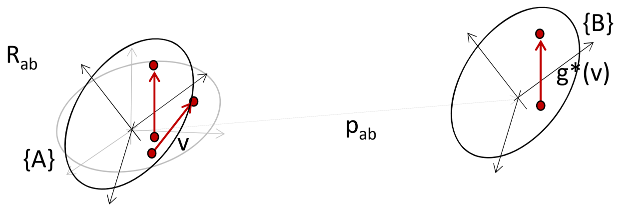
$$q_a(t) = R^a_{ab} q_a(0) + p^a_{ab} = g_{ab}(q_a(0))$$

  1. Configuration of the rigid-body motion.
  2. Coordinate transform btw $\{A\}$ and $\{B\}$: $q_a(t) = g_{ab}(q_a(0)) = g_{ab}(q_b(t))$.
  3. Rigid-body transformation operator: $q_a(t) = g_{ab}(q_a(0))$.

- Rigid transformation action g_{ab*} on a free-vector $v = s - r$:

$$g_{ab*}(v) := g_{ab}(s) - g_{ab}(r) = R_{ab}s + p_{ab} - R_{ab}r - p_{ab} = R_{ab}v$$

  i.e., g_{ab*} simply rotates a free-vector v by R_{ab} in $\{A\}$.



Homogeneous Representation

Homogeneous Rigid Transformation $\bar{g}_{ab}$

$$\bar{q}_a = \begin{pmatrix} q_a \\ 1 \end{pmatrix} = \begin{bmatrix} R^a_{ab} & p^a_{ab} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} q_b \\ 1 \end{pmatrix} = \bar{g}_{ab}\bar{q}_b$$

- Homogeneous representation of a **point** $q \in \Re^3$ and a **free-vector** $v \in \Re^3$ defined by

$$\bar{q} := [q_1; q_2; q_3; 1], \quad \bar{v} = [v_1; v_2; v_3; 0] \in \Re^4$$

- This clearly manifests difference between point vector and free vector: that is, can do $\bar{q}_1 - \bar{q}_2 = v$, $\bar{q}_1 + \bar{v} = \bar{q}_2$, $\bar{v}_1 + \bar{v}_2 = \bar{v}_3$, but, not $\bar{q}_1 + \bar{q}_2$.
- $\bar{g}_{ab} \in \Re^{4\times 4}$ is homogeneous representation of the rigid body transformation $g_{ab} = (p_{ab}, R_{ab}) : \Re^3 \to \Re^3$ from $\{A\}$ to $\{B\}$.
- With homogeneous representation, we have a linear relation $\bar{q}_a = \bar{g}_{ab}\bar{q}_b$ in SE(3) similar to $q_a = R_{ab}q_b$ in SO(3).
- Sometimes, we simply use g_{ab} to denote $\bar{g}^a_{ab}$.

Special Euclidean Group SE(3)

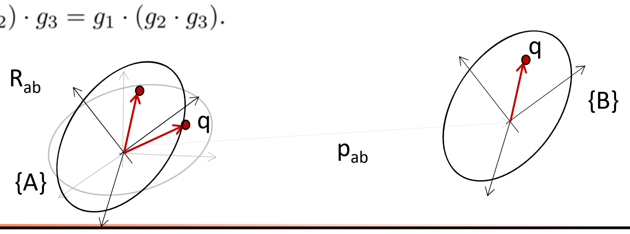
Def. 1 (Special Euclidean Group SE(n))

$$SE(n) = \Re^n \times SO(n)$$
$$SE(3) = \Re^3 \times SO(3) = \{(p, R) \mid p \in \Re^3, R \in SO(3)\}$$

- SE(3) represents rigid body motion, as SO(3) rotation motion.
- SE(3) identified by $\bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \Re^{4\times 4}$.
- SE(3) is a group (i.e., $\bar{g}$ is a group under matrix multiplication):
  1. If $g_1, g_2 \in SE(3)$, $g_1 \cdot g_2 \in SE(3)$:
  $$\bar{g}_1 \cdot \bar{g}_2 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 p_2 + p_1 \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3)$$
  2. Identity $I_{4\times 4} \in SE(3)$.

Special Euclidean Group SE(3)

- SE(3) identified by $\bar{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \Re^{4\times 4}$.
- SE(3) is a group (i.e., $\bar{g}$ is a group under matrix multiplication):
  1. If $g_1, g_2 \in SE(3)$, $g_1 \cdot g_2 \in SE(3)$.
  2. Identify $I_{4\times 4} \in \mathrm{SE}(3)$.
  3. Given $g \in SE(3)$, g^{-1} is also in $SE(3)$, as given by
  $$\bar{g}^{-1} = \begin{bmatrix} R^T & R^T(-p) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ba} & -p^b \\ 0 & 1 \end{bmatrix} \in SE(3)$$
  where $R^T = R_{ba}$ and $p^b = R_{ba} p^a_{ab} = p^b_{ab}$.
  4. $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.



Properties of SE(3)

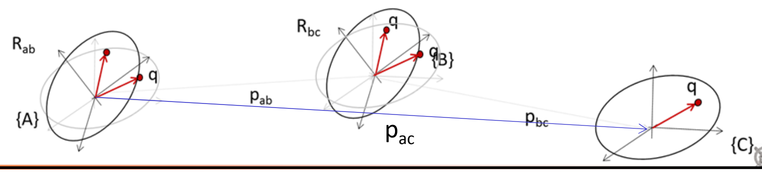
- Homogeneous representation of g_*: for $v = s - r$ with $\bar{v} = [v; 0]$,

$$\bar{g}_* = \bar{g}(\bar{s}) - \bar{g}(\bar{r}) = \begin{bmatrix} R & p \\ 0 & 0 \end{bmatrix} (\bar{s} - \bar{r}) = \begin{bmatrix} R & p \\ 0 & 0 \end{bmatrix} \bar{v} = \begin{pmatrix} Rv \\ 0 \end{pmatrix}$$

- We can also show that $||\bar{g}(\bar{p}) - \bar{g}(\bar{q})|| = ||R(p - q)||$ and $g_*(v \times w) = g_* v \times g_* w$, i.e., $g \in SE(3)$ defines a rigid body transformation.
- Consider rigid motion $\{A\} \to \{B\}$ via g^a_{ab} and $\{B\} \to \{C\}$ via g^b_{bc}. Then,

$$\begin{aligned} q_b &= R^b_{bc} q_c + p^b_{bc} \\ q_a &= R^a_{ab} q_b + p^a_{ab} \end{aligned} \Rightarrow \bar{q}_a = \begin{bmatrix} R^a_{ab} R^b_{bc} & R^a_{ab} p^b_{bc} + p^a_{ab} \\ 0 & 1 \end{bmatrix} \bar{q}_c = \bar{g}_{ac} \bar{q}_c$$

where $\bar{g}^a_{ac} = (p^a_{ac}, R^a_{ac}) = \bar{g}^a_{ab} \cdot \bar{g}^b_{bc}$ with $R^a_{ac} = R^a_{ab} R^b_{bc}$ and $p^a_{ac} = R^a_{ab} p^b_{bc} + p^a_{ab} = p^a_{ab} + p^a_{bc}$: **composition** of successive body-frame SE(3) motion.



Composition of SE(3) Motions

Consider $g_1 : \{A\} \to \{B\}$ for $[t_0, t_1)$ and $g_2 : \{B\} \to \{C\}$ for $[t_1, t_2)$, with a rigidly-attached point q evoles $q(t_o) \to q(t_1) \to q(t_2)$, each can be expressed in $\{A\}$ or $\{B\}$. Then, the composition of motion $\bar{q}_a(t_2) = \bar{g}_{ac} \bar{q}_a(t_o)$ is given by

- Successive rotations w.r.t. **body frames**: $\bar{g}_{ac} = \bar{g}^a_1 \cdot \bar{g}^b_2$
- Successive rotations w.r.t. **inertial frame**: $\bar{g}_{ac} = \bar{g}^a_2 \cdot \bar{g}^a_1$

- From $\bar{q}_a(t_1) = \bar{g}^a_1 \bar{q}_a(t_o)$ and $\bar{q}_b(t_2) = \bar{g}^b_2 \bar{q}_b(t_1)$ with $q_b(t_1) = q_a(t_o)$,

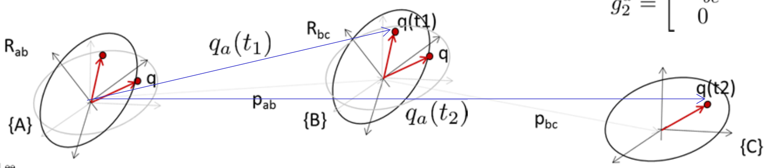
$$\bar{q}_a(t_2) = \bar{g}^a_1 \bar{q}_b(t_2) = \bar{g}^a_1 \cdot \bar{g}^b_2 \bar{q}_b(t_1) = \bar{g}^a_1 \bar{g}^b_2 \bar{q}_a(t_o) = \bar{g}_{ac} \bar{q}_a(t_o)$$

- From $\bar{q}_a(t_1) = \bar{g}^a_1 \bar{q}_a(t_o)$ and $\bar{q}_a(t_2) = \bar{g}^a_2 \bar{q}_a(t_1)$,

$$\bar{q}_a(t_2) = \bar{g}^a_2 \cdot \bar{g}^a_1 \bar{q}_a(t_o) = \bar{g}_{ac} \bar{q}_a(t_o)$$

$$\bar{g}^b_2 = \begin{bmatrix} R^b_{bc} & p^b_{bc} \\ 0 & 1 \end{bmatrix}$$

$$\bar{g}^a_2 = \begin{bmatrix} R^a_{bc} & -R^a_{bc} p^a_{ab} + p^a_{ac} \\ 0 & 1 \end{bmatrix}$$



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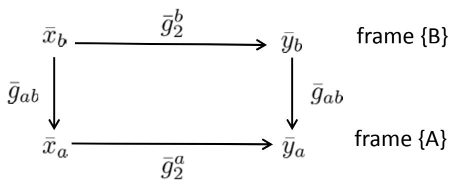
Consider $g_1 : \{A\} \to \{B\}$ for $[t_0, t_1)$ and $g_2 : \{B\} \to \{C\}$ for $[t_1, t_2)$, with a rigidly-attached point q evoles $q(t_o) \to q(t_1) \to q(t_2)$, each can be expressed in $\{A\}$ or $\{B\}$. Then, the composition of motion $\bar{q}_a(t_2) = \bar{g}_{ac}\bar{q}_a(t_o)$ is given by

- Successive rotations w.r.t. **body frames**: $\bar{g}_{ac} = \bar{g}_1^a \cdot \bar{g}_2^b$
- Successive rotations w.r.t. **inertial frame**: $\bar{g}_{ac} = \bar{g}_2^a \cdot \bar{g}_1^a$

- If these two compositions represent the same motion, we have

$$\bar{g}_2^a = \bar{g}_{ab} \cdot \bar{g}_2^b \cdot \bar{g}_{ab}^{-1}$$

where $\bar{g}_2^a$ and $\bar{g}_2^b$ represent the same SE(3) motion from $\{B\} \to \{C\}$, but expressed respectively in $\{A\}$ and $\{B\}$.

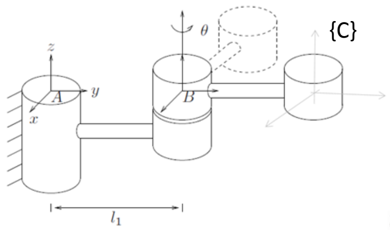


Example 2.1

- Describe rigid body motion given by the rotation about the joint axis.
- Attach the coordinate frames $\{A\}$ and $\{B\}$.
- Note that $\{B\}$ is attached to the joint not at the end-effector.
- Then,

$$R_{ab}^a = \begin{bmatrix} c\,\theta & -s\,\theta & 0 \\ s\,\theta & c\,\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{g}_{ab}^a(\theta) = \begin{bmatrix} R_{ab}(\theta) & \begin{matrix} 0 \\ l_1 \\ 0 \end{matrix} \\ 0 & 1 \end{bmatrix}$$

- $\bar{g}_{ab}(\theta)[0; 0; 0; 1] = [0; l_1; 0; 1]$ (i.e., transformed position of origin of $\{B\}$ expressed in $\{A\}$).
- $\bar{g}_{ab}(\theta)[0; l_2; 0; 1] = [-l_2\,s\,\theta; l_1 + l_2\,c\,\theta; 0; 1]$ (i.e., transformed position of origin of $\{C\}$ expressed in $\{A\}$).



Twist

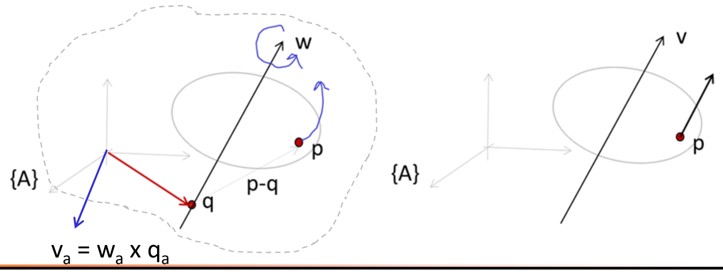
- Consider rigid body rotation about w ($||w|| = 1$) with q being a point on w (e.g., revolute joint). For point p rigidly attached on the object, we then have

$$\dot{p}_a(t) = w_a \times (p_a(t) - q_a)$$

with $p_a - q_a$ being the offset between w-line and origin of $\{A\}$; or

$$\dot{\bar{p}}_a = \begin{pmatrix} \dot{p}_a \\ 0 \end{pmatrix} = \begin{bmatrix} \hat{w}_a & -w_a \times q_a \\ 0 & 0 \end{bmatrix} \begin{pmatrix} p_a \\ 1 \end{pmatrix} = \hat{\xi}_a \bar{p}_a$$

all expressed in $\{A\}$. Note $\hat{\xi}$ contains both w and q informations.



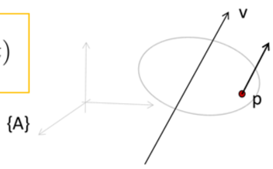
Twist

- For the revolute joint,

$$\dot{\bar{p}}_a = \begin{pmatrix} \dot{p}_a \\ 0 \end{pmatrix} = \begin{bmatrix} \hat{w}_a & -w_a \times q_a \\ 0 & 0 \end{bmatrix} \begin{pmatrix} p_a \\ 1 \end{pmatrix} = \hat{\xi}_a \bar{p}_a$$

- Consider translation along v (e.g., prismatic joint). Then, $\dot{p}_a = v_a$, or

$$\dot{\bar{p}}_a(t) = \begin{pmatrix} \dot{p}_a(t) \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \bar{p}_a = \hat{\xi}_a \bar{p}_a(t)$$



- Thus, for both cases, we have

$$\dot{\bar{p}}_a(t) = \hat{\xi}_a \bar{p}_a(t) \quad \approx \quad \dot{p}_a(t) = \hat{w}_a p_a(t) \ \text{ in SO(3)}$$

expressed in $\{A\}$, and, if $\hat{\xi}_a$ is constant,

$$\bar{p}_a(t) = e^{\hat{\xi}_a t} \bar{p}_a(0) \quad \approx \quad p_a(t) = e^{\hat{w}_a t} p_a(0) \ \text{ in SO(3)}$$

Twist and se(3)

Definition 1 (se(3))

$$se(3) := \{\xi = (v, \hat{w}) \mid v \in \Re^3, \hat{w} \in so(3)\}$$

- We call an element $\xi \in se(3)$ **twist** or infinitesimal generator of SE(3)
- se(3) is Lie algebra of SE(3).
- $\xi = (v, w) \in$ se(3) is expressed in homogeneous representation by

$$\hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} \in \Re^{4\times 4} \quad \text{with} \quad \dot{\bar{p}}_a(t) = \hat{\xi}_a \bar{p}_a(t)$$

- Given $\xi = (v, w) \in \Re^6$, $\vee$, $\wedge$ defined by $\xi^\wedge = \hat{\xi} \in \Re^{4\times 4}$ and $\hat{\xi}^\vee = \xi \in \Re^6$.
- Given $\xi = (v, w)$, we can interpret w as angular velocity. Note also that $\begin{bmatrix} \hat{w}_a & v_a \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \vec{0} \\ 1 \end{pmatrix}_a = \begin{pmatrix} v_a \\ 0 \end{pmatrix}$, where $[\vec{0}; 1]_a$ is the origin of $\{A\}$, i.e., v_a is velocity of **extended** object observed at origin of $\{A\}$.
- We didn't specify object's size, e.g., for revolute joint, $v_a = -w_a \times q_a$.

Exponential Coordinates for SE(3)

Proposition 1 (2.8) *For every $\hat{\xi} \in se(3)$ and $\theta \in \Re$, $e^{\hat{\xi}\theta} \in SE(3)$.*

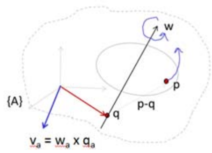
(Proof) First, suppose $w = 0$. Then, $\hat{\xi}^2 = \hat{\xi}^3 = ... = 0$, thus,

$$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \quad \text{with} \quad \bar{p}(t) = [p(t); 1] = [p(0) + v\theta; 1].$$

Second, suppose $w \neq 0$ with $||w|| = 1$ (if not, we can scale θ). Define

$$\bar{g} = \begin{bmatrix} I & w \times v \\ 0 & 1 \end{bmatrix}, \quad \bar{g}^{-1} = \begin{bmatrix} I & -w \times v \\ 0 & 1 \end{bmatrix}$$

and define $\hat{\xi}' = \bar{g}^{-1}\hat{\xi}\bar{g}$. Then, we have



$$\hat{\xi}' = \begin{bmatrix} \hat{w} & \hat{w}(w \times v) + v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{w} & (w^T v)w \\ 0 & 0 \end{bmatrix}$$

where we use $a \times (b \times c) = (a^T c)b - (a^T b)c$.

- This surmounts to find screw expression of $\xi = (w, v)$ (i.e., rotation w + translation along w) and express ξ in new frame $\{A'\}$ with origin on screw axis.

Exponential Coordinates for SE(3)

Proposition 1 (2.8) *For every* $\hat{\xi} \in se(3)$ *and* $\theta \in \Re$, $e^{\xi\theta} \in SE(3)$.

(Proof) Using $\bar{g} = \begin{bmatrix} I & w \times v \\ 0 & 1 \end{bmatrix}$, define

$$\hat{\xi}' = \bar{g}^{-1}\hat{\xi}\bar{g} = \begin{bmatrix} \hat{w} & \hat{w}(w \times v) + v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{w} & (w^T v)w \\ 0 & 0 \end{bmatrix}$$

we then have

$$(\hat{\xi}')^2 = \begin{bmatrix} \hat{w}^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad (\hat{\xi}')^2 = \begin{bmatrix} \hat{w}^3 & 0 \\ 0 & 0 \end{bmatrix} \cdots \quad \Rightarrow \quad e^{\hat{\xi}'\theta} = \begin{bmatrix} e^{\hat{w}\theta} & (w^T v)w\theta \\ 0 & 1 \end{bmatrix}$$

Then, using $e^{\hat{\xi}'\theta} = \bar{g}^{-1}e^{\hat{\xi}\theta}\bar{g}$, we can obtain

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})(w \times v) + (w^T v)w\theta \\ 0 & 1 \end{bmatrix} \tag{1}$$

showing that $e^{\hat{\xi}\theta} \in \text{SE}(3)$ with $e^{\hat{w}\theta} \in \text{SO}(3)$ and $(I - e^{\hat{w}\theta})(w \times v) + (w^T v)w\theta \in \Re^3$.

- This $e^{\hat{\xi}\theta}$ provides closed-form expression similar to Rodrigues' formula.

Exponential Coordinates for SE(3)

Proposition 1 (2.9) $\forall\ g = (p, R) \in SE(3)$, $\exists\ \hat{\xi} \in se(3)$, $\theta \in \Re$ *s.t.* $\bar{g} = e^{\hat{\xi}\theta}$.

1. If $g = (I, 0)$, $\theta = 0$ and $\xi = (\hat{w}, v)$ can be arbitrary.
2. If $R = I$, no rotation w/ pure translation p during θ. Thus,

$$\hat{\xi} = \begin{bmatrix} 0 & p/||p|| \\ 0 & 0 \end{bmatrix}, \quad \theta = ||p|| \ \Rightarrow\ e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} = \bar{g}$$

3. If $R \neq I$, we can first solve (w, θ) via $R = e^{\hat{w}\theta}$. Given this $\hat{w}$ and $0 < \theta < 2\pi$, we can obtain v by using

$$p = (I - e^{\hat{w}\theta})(w \times v) + \theta w w^T v = [(I - e^{\hat{w}\theta})\hat{w} + \theta w w^T]v = Av$$

which assumes unique solution v, since $\text{null}(A) = \emptyset$, because:
- First term of A has nullspace $\{x \in \Re^3 \mid x = \alpha w\}$.
- Second term of A has nullspace $\{x \in \Re^3 \mid w^T x = 0\}$.
- These two nullspaces are orthogonal with each other.

- We call (ξ, θ) **exponential coordinates** of $g \in \text{SE}(3)$ with $\bar{g} = e^{\hat{\xi}\theta}$, which is many-to-one map, as so is $e^{\hat{w}\theta} = R$.

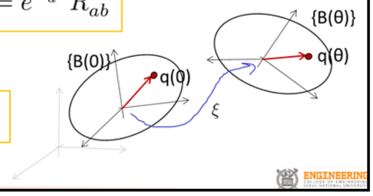
Geometric Meaning of exp(x_at)

- Consider rigid motion given by constant twist ξ, which sends $\{B(0)\}$ to $\{B(\theta)\}$. Denote rigid transformation from $\{A\}$ to $\{B(\theta)\}$ by $g_{ab}(\theta) = g_{ab}(\theta)$, with $g_{ab}(0) = g_{ab(0)}$.
- Then, $e^{\hat{\xi}_a\theta} = \bar{g}^a_{b(0)b(\theta)}$, i.e., represents rigid motion from $\{B(0)\}$ to $\{B(\theta)\}$ via ξ expressed in $\{A\}$, similar to $e^{\hat{w}_a\theta} = R^a_{b(0)b(\theta)}$ in SO(3).
- First, from $\dot{\bar{q}}_a = \hat{\xi}_a\bar{q}_a$, we have $\bar{q}_a(\theta) = e^{\hat{\xi}_a\theta}\bar{q}_a(0)$ similar to $q_a(\theta) = e^{\hat{w}_a\theta}q_a(0)$ in SO(3).
- We also have $\bar{q}_a(\theta) = \bar{g}^a_{ab(\theta)}\bar{q}_{b(\theta)}(\theta) = \bar{g}^a_{ab}(\theta)\bar{q}_{b(0)}(0)$.
- Then, using $\bar{q}_a(0) = g^a_{ab(0)}\bar{q}_{b(0)}(0)$, $\bar{q}_a(\theta) = \bar{g}^a_{ab}(\theta)\bar{q}_{b(0)}(0) = e^{\hat{\xi}_a\theta}\bar{g}^a_{ab}(0)q_{b(0)}(0)$, i.e.,

$$\bar{g}^a_{ab}(\theta) = e^{\hat{\xi}_a\theta}\bar{g}^a_{ab}(0) \quad \approx \quad R^a_{ab_1} = e^{\hat{w}_a\theta}R^a_{ab}$$

implying that $e^{\hat{\xi}_a\theta} = \bar{g}^a_{b(0)b(\theta)}$.

- Twist coordinate transformation $\hat{\xi}_a = \bar{g}_{ab}\hat{\xi}_b\bar{g}^{-1}_{ab}$



Example 2.2

- Given joint rotation α, compute twist coordinate (ξ_a, θ) expressed in $\{A\}$.
- Attach $\{B\}$ to end-effector. Then, $g_{ab}(\alpha) = \begin{bmatrix} c\,\alpha & -s\,\alpha & 0 & -l_2\,s\,\alpha \\ s\,\alpha & c\,\alpha & 0 & l_1 + l_2\,c\,\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
- Recall

$$g^a_{ab}(\alpha) = e^{\hat{\xi}_a\theta}g^a_{ab}(0), \quad \text{i.e.,} \quad e^{\hat{\xi}_a\theta} = g^a_{b(0)b(\alpha)}$$

- We can then compute $e^{\hat{\xi}_a\theta}$ by using $e^{\hat{\xi}_a\theta} = g^a_{ab}(\alpha)[g^a_{ab}(0)]^{-1}$.
- Or, can use observation of $\xi_a = (\hat{w}_a, v_a)$ s.t.

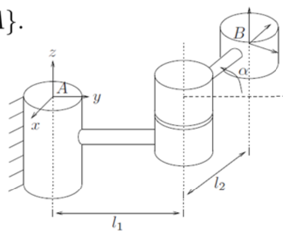
$$w_a = [0;0;1], \quad v_a = [l_1;0;0]$$

$$e^{\hat{\xi}_a\alpha} = \begin{bmatrix} c\alpha & -s\alpha & 0 & l_1 s\alpha \\ s\alpha & c\alpha & 0 & l_1(1-c\alpha) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where v_a is velocity of extended rigid body at origin of $\{A\}$.
- Assume $\{A(\theta)\}$ moves with α. Then,

$$g_{aa(\theta)} = e^{\hat{\xi}_a\theta}g_{aa(0)} = e^{\hat{\xi}_a\theta}$$

$e^{\hat{\xi}_a\theta}$ represents α-motion for any point expressed in $\{A\}$.



Screw Motion

Consider rigid-body motion, where the object first rotates about w by θ with q on w ($||w|| = 1$) and translates by d along w.

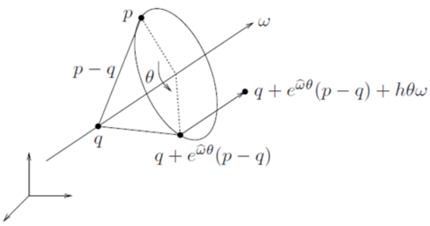
Definition 1 *screw motion:=* $\{$*axis* l, *pitch* h, *magnitude* $\theta\}$

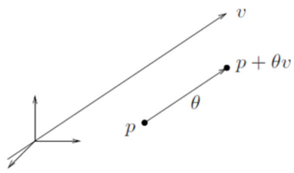
- If $\theta \neq 0$, define pitch $h := d/\theta$. Then, for a rigidly-attached point p,

$$p_a(\theta) = e^{\hat{w}_a \theta}(p_a(0) - q_a) + q_a + h\theta w$$

where q is on the axis $l := \{q + \lambda w \mid \lambda \in \Re\}$, or,

$$\bar{p}_a(\theta) = \begin{bmatrix} e^{\hat{w}_a \theta} & (I - e^{\hat{w}_a \theta})q_a + h\theta w \\ 0 & 1 \end{bmatrix} \bar{p}_a(0) = \bar{g}_{\text{screw}} \bar{p}_a(0)$$





(b) pure translation

Screw Motion

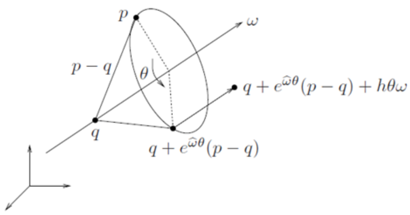
Consider rigid-body motion, where the object first rotates about w by θ with q on w ($||w|| = 1$) and translates by d along w.

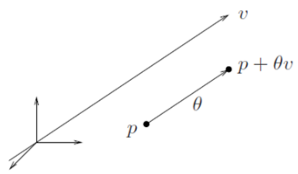
Definition 1 *screw motion:=* $\{$*axis* l, *pitch* h, *magnitude* $\theta\}$

- For pure translation, we set $h = \infty$, $w = 0$, and axis $l := \{\lambda v\}$ with

$$p_a(\theta) = p_a(0) + v\theta, \quad \text{or} \quad \bar{p}_a(\theta) = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \bar{p}_a(0) = \bar{g}_{\text{screw}} \bar{p}_a(0)$$

where $v\theta$ is the translation velocity with $||v|| = 1$.

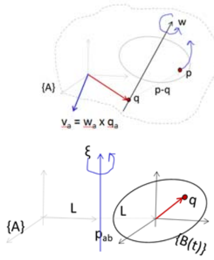




(b) pure translation

Screw and Twist

Proposition 1 (2.10) *Given a screw* (l, h, θ) $\exists$ *an unit twist* ξ *(i.e.,* $||w|| = 1$ *if* $w \neq 0$*; or* $||v|| = 1$ *if* $w = 0$*) s.t.* $e^{\hat{\xi}\theta} = \bar{g}_{screw}$, *and vice versa.*



- Given screw (l, h, θ), we can infer twist ξ s.t.,

$$\xi = (v, w) = \begin{cases} (-w \times q + hw, w) & \text{if } h \neq \infty \\ (v, 0) & \text{if } h = \infty \end{cases}$$

- Given twist ξ, we can find screw (l, h, θ) by equating

$$\begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})(w \times v) + ww^T v\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})q + h\theta w \\ 0 & 1 \end{bmatrix}$$

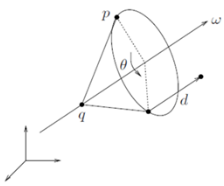
  i.e., $q = w \times v$, $l = \{q + \lambda w | \lambda \in \Re\}$, $h = w^T v$, $\theta = \theta$ (or $l = \{\lambda v\}$ if $w = 0$.

- The last derivation can also be obtained by finding q for $\xi = (w, v)$ s.t. $v_q = w \times q + v = hw$ (i.e., ξ expressed with q as origin becomes screw motion).

Chasles Theorem

Theorem 1 (2.11:Chasles) *Every rigid body motion can be realized by a rotation about an axis with a translation parallel to that axis.*

- Screw motion is independent on how to choose q' on $l := \{q + \lambda w\}$.
- For a point rigidly-attached p, we have



$$\bar{p}_a(\theta) = e^{\bar{\xi}_a \theta} \bar{p}_a(0)$$

  Also, if the screw motion drives $\{B(0)\}$ to $\{B(\theta)\}$,

$$\bar{g}_{ab}(\theta) = e^{\hat{\xi}_a \theta} \bar{g}_{ab}(0) = \bar{g}^a_{b(0)b(\theta)} \bar{g}_{ab}(0)$$

  i.e., $e^{\hat{\xi}_a \theta}$ represents rigid-body motion created by the screw sending $\{B(0)\}$ to $\{B(\theta)\}$ expressed in $\{A\}$.

- If we choose $p_a(0) = q + \lambda w$, $p_a(\theta) = p_a(0) + h\theta w_a$.