- Rates of strain and rotation

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + \Omega_{ij}$$

vel. grad. tensor — strain-rate tensor — rotation-rate tensor

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \rightarrow S_{ij} = S_{ji} \quad \text{symmetric tensor}$$

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \rightarrow \Omega_{ij} = -\Omega_{ji} \quad \text{anti-symmetric tensor}$$

$$\tau_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = -p\,\delta_{ij} + 2\mu S_{ij}$$

$$\underline{\omega} = \nabla \times \underline{u}$$

vorticity

ϵ_{ijk} : alternating tensor

$$\begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1 \\ \text{all others are zero} \end{cases}$$

$\underline{a} \times \underline{b}$: $\epsilon_{ijk} a_j b_k$

$\underline{\omega} = \nabla \times \underline{U}$: $\omega_i = \epsilon_{ijk} \dfrac{\partial U_k}{\partial x_j} = \epsilon_{ijk} (S_{kj} + \Omega_{kj})$

$$= \underbrace{\underbrace{\epsilon_{ijk}}_{\substack{= \\ -\epsilon_{ikj} \\ \text{anti-sym. tensor}}} \underbrace{S_{kj}}_{\text{sym. ten.}}}_{=0} + \epsilon_{ijk} \underbrace{\Omega_{kj}}_{\substack{= \\ -\Omega_{jk}}}$$

$$\therefore \boxed{\omega_i = -\epsilon_{ijk} \Omega_{jk}}$$

$\Omega_{ij} = -\frac{1}{2} \epsilon_{ijk} \omega_k$

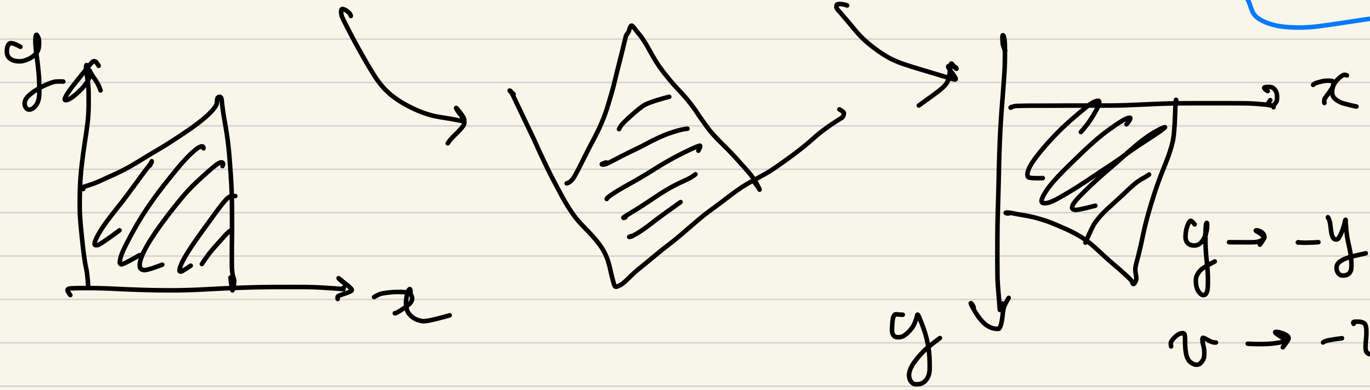
⊙ Transformation properties : important properties of fluid flows

- Reynolds number similarity

$$\frac{\partial U_i}{\partial x_i} = 0$$

$$\frac{\partial U_j}{\partial t} + U_i \frac{\partial U_j}{\partial x_i} = -\frac{\partial p}{\partial x_j} + \frac{1}{Re} \nabla^2 U_j$$

- Rotational and reflectional invariance → transformed N-S eqs w/ rotation and reflection are the same.



$y \to -y$

$v \to -v$

- time reversal

$\xrightarrow{} t$ $\qquad \hat{t} = -t$

$\hat{t} \xleftarrow{}$ $\qquad \hat{u}_j = -u_j$

$$\frac{\partial u_j}{\partial t} = \boxed{\cdots\cdots} + \frac{1}{Re}\nabla^2 u_j$$

same

$$\frac{\partial \hat{u}_j}{\partial \hat{t}} = \boxed{\cdots\cdots} - \frac{1}{Re}\nabla^2 \hat{u}_j$$

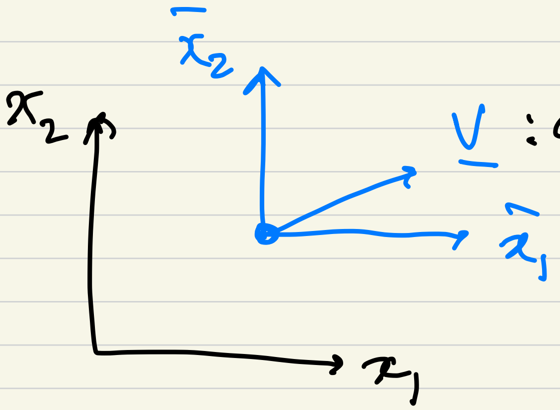
$\therefore$ no invariance due to the viscous term

Euler eqs. are invariant under a time reversal.

- Galilean invariance

moving frame w/ constant velocity V

A quantity that is the same in different inertial frames is said to be Galilean invariant.



$\underline{V}$: const

$$\bar{x}_i = x_i - V_i t \ , \quad \bar{t} = t$$

$$\bar{U}_i(\bar{x}_i, \bar{t}) = U_i(x_i, t) - V_i \ : \ \text{Galilean invar.}$$

Vel. is not Galilean invar.

$$\frac{\partial \bar{U}_i}{\partial \bar{x}_j} = \frac{\partial \bar{U}_i}{\partial x_j}\frac{\partial x_j}{\partial \bar{x}_j} = \frac{\partial U_i}{\partial x_j} \ : \ \text{Galilean invar.}$$

$$\frac{\partial \bar{U}_i}{\partial \bar{t}} = \frac{\partial \bar{U}_i}{\partial t}\frac{\partial t}{\partial \bar{t}} + \frac{\partial \bar{U}_i}{\partial x_j}\frac{\partial x_j}{\partial \bar{t}} = \frac{\partial}{\partial t}(U_i - V_i) + \frac{\partial}{\partial x_j}(U_i - V_i)\cdot V_j$$

$$= \frac{\partial U_i}{\partial t} + V_j \frac{\partial U_i}{\partial x_j}$$

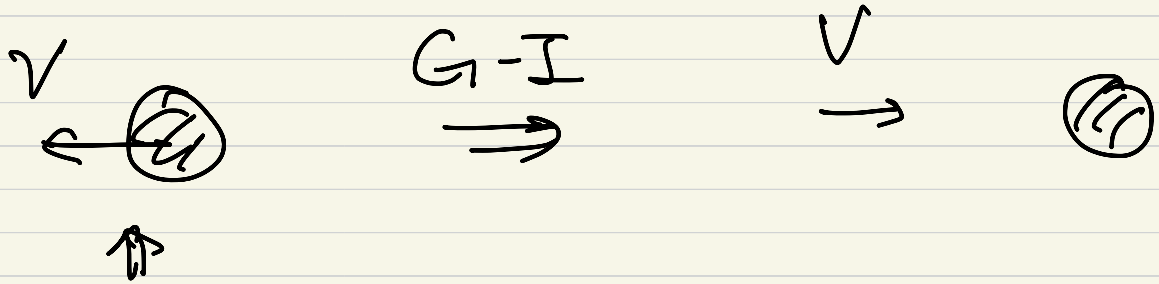
partial time derivative is Not Galilean invariant.

fluid acceleration

$$\frac{D\bar{U}_i}{D\bar{t}} = \cdots = \frac{DU_i}{Dt}$$: material derivative is Galilean invar.

N-S $\longrightarrow$ $\overline{N-S}$ $\Rightarrow$ same

$\therefore$ N-S eq. is Galilean invariant in different inertial frames. HW 1 (1) prove this!



• N-S. eq. is invariant under rectilinear acceleration of frame.

$$\Rightarrow \frac{\partial \bar{U}_j}{\partial \bar{t}} + \bar{U}_i \frac{\partial \bar{U}_j}{\partial \bar{x}_i} = \nu \nabla^2 \bar{U}_j \underbrace{- \frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{x}_j} - \frac{dV_j}{dt}}_{= -\frac{1}{\rho}\frac{\partial \bar{p}'}{\partial \bar{x}_j}} \qquad A_j = \frac{dV_j}{dt} = \text{const}$$

$$\bar{p}' \equiv \bar{p} + \rho \bar{x}_i A_i \qquad \frac{\partial \bar{x}_i}{\partial \bar{x}_j} = \delta_{ij}$$

$\Rightarrow$ extended form of Galilean invariance

• N-S eq is NOT invariant under rotational acceleration.

$\underline{\Omega} \times (\underline{\Omega} \times \underline{r})$ $\qquad$ $2\underline{\Omega} \times \underline{v}$

generates centrifugal force, Coriolis force and

can be absorbed into press. term $\quad -\bar{x}_i \bar{\Omega}_{ik} \bar{\Omega}_{kj}$ $\qquad$ $-2\bar{v}_i \bar{\Omega}_{ij}$

angular acceleration force

$\frac{d\underline{\Omega}}{dt} \times \underline{r}$ $\qquad$ $-\bar{x}_i \frac{d\bar{\Omega}_{ij}}{dt}$ $\qquad$ cannot be absorbed into press. term.

For 2-D flow and steady rotation, these terms go to zero.

→ N-S eq. is invariant w/ the steady rotation of the frame for 2D flows.

HW 1 2) 2-1 derive the gov. eq.

2-2. prove for 2-D and steady rotation

$\overline{\omega}_i = \epsilon_{ijk}\, \partial \overline{U}_k / \partial \overline{x}_j$

Coriolis force　　angular acceleration

$$\frac{\partial \overline{\omega}_i}{\partial \overline{t}} + \overline{U}_j \frac{\partial \overline{\omega}_i}{\partial \overline{x}_j} = \overline{\omega}_j \frac{\partial \overline{U}_i}{\partial \overline{x}_j} + \nu \frac{\partial^2 \overline{\omega}_i}{\partial \overline{x}_j \partial \overline{x}_j} - 2\epsilon_{ijk} \frac{\partial \overline{U}_\ell}{\partial \overline{x}_j} \overline{\Omega}_{\ell k} - \epsilon_{ijk} \frac{d\overline{\Omega}_{jk}}{dt}.$$

vorticity eq. is NOT Galilean invariant under rotational acceleration.

For 2-D flow (U_1, U_2)
ω_3
steady rotation

two terms are zero.

In this special case, vorticity is unaffected by frame rotation.

HW 1)

3-1 Derive the vorticity eq.

3-2 Prove for 2-D.

Due date : April 1 → ① send an e-mail of scanned homework

② come to TA and hand in.

Ch. 3. Statistical description of turbulent flows

⊙ Random nature of turbulence

Turbulence is a random phenomenon

$\downarrow$

neither deterministic nor impossible

⊙ Characterization of random variables

- U is random ⇐ its value is inherently unpredictable.

So, theory aims at predicting the probability of events.

$\downarrow$

probability density function (PDF)

- event

$$B \equiv \{ U < V_b \}$$

$$C \equiv \{ V_a \leq U < V_b \} \quad \text{for } V_b > V_a$$

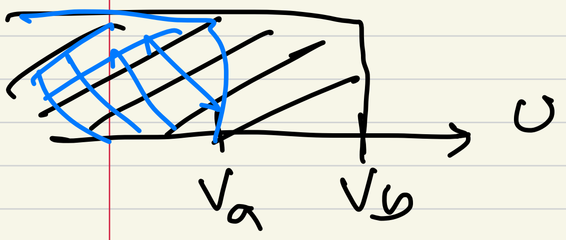
probability of the event B is

$$p = P(B) = P\{ U < V_b \}$$

Cumulative distribution function (CDF)

$$F(V) \equiv P\{U < V\}$$

then, $P(B) = P\{U < V_b\} = F(V_b)$



$$P(C) = P\{V_a \leq U < V_b\} = P\{U < V_b\} - P\{U < V_a\}$$
$$= F(V_b) - F(V_a)$$

$F(V_b) \geq F(V_a)$ for $V_b > V_a$: F is a non-decreasing ft.

$$F(-\infty) = 0, \quad F(\infty) = 1$$

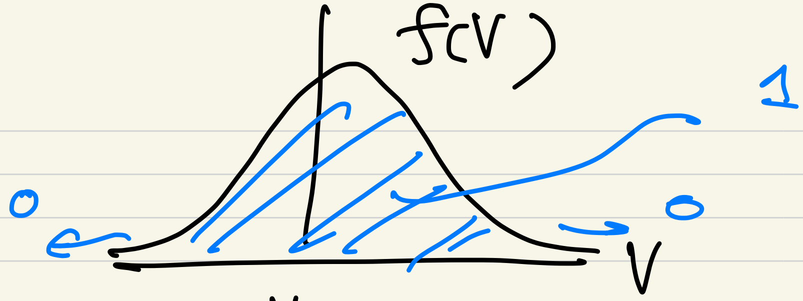
- Probability density function (PDF)

$$f(V) \equiv \frac{dF(V)}{dV} = \lim_{\Delta V \to 0} \frac{F(V+\Delta V) - F(V)}{\Delta V} \geq 0$$

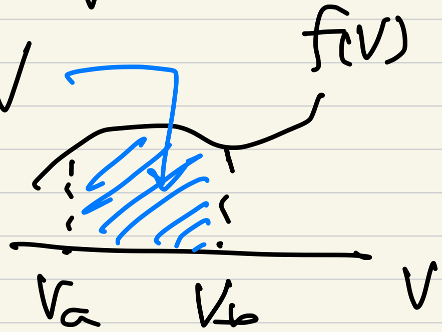
$\rightarrow \quad f(V) \geq 0$

$$\int_{-\infty}^{\infty} f(V)\, dV = \int_{F(-\infty)}^{F(\infty)} dF(V) = F(\infty) - F(-\infty) = 1$$: normalization condition

$$f(\infty) = f(-\infty) = 0$$



$$P\{V_a \le \mathcal{V} < V_b\} = F(V_b) - F(V_a) = \int_{V_a}^{V_b} f(V)\,dV$$



$$P\{V \le \mathcal{V} < V + dV\} = F(V+dV) - F(V)$$
$$= f(V)\,dV$$

$\Rightarrow$ PDF $f(V)$ is the probability per unit distance

$\rightarrow$ called probability density function.

- Two or more random variables have the same PDF

$\rightarrow$ they are said to be identically distributed or statistically indentical.