Forward and Inverse Kinematics

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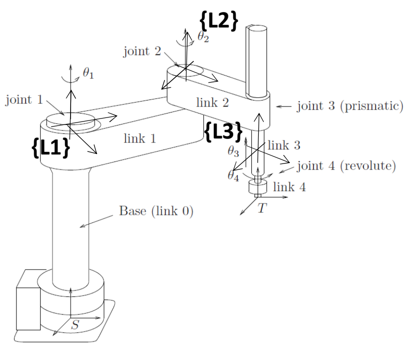
Forward Kinematics

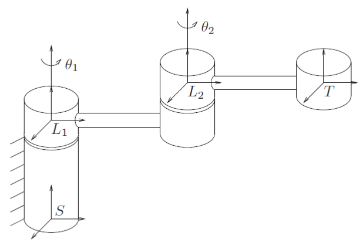
- Consider robotic manipulator. Attach base frame $\{S\} = \{L_o\}$ and tool frame $\{T\} = \{L_{n+1}\}$. **Forward kinematics problem** is then to find a mapping

$$g_{st} : Q \to SE(3)$$

i.e., given joint variables $(\theta_1, \theta_2, ..\theta_n) \in Q$, what is $g_{st}(\theta_1, \theta_2, ..\theta_n) \in \text{SE}(3)$.

- $\theta_i \in [0, 2\pi)$ (for revolute) or $\theta_i \in [d_{\min}, d_{\max}]$ (for prismatic).
- Attach $\{L_i\}$ on **actuation axis** of θ_i. This $\{L_i\}$ moves with link i by θ_i.





Forward Kinematics

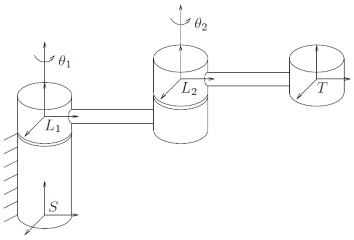
- For 2-DOF robotic manipulator, we have

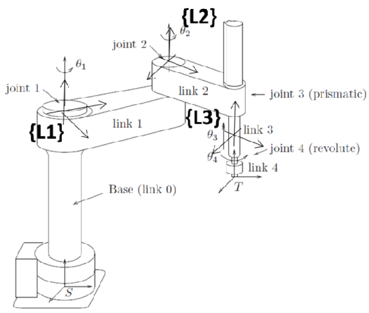
$$\bar{g}_{st}(\theta_1, \theta_2) = \bar{g}_{sl_1}(\theta_1) \cdot \bar{g}_{l_1 l_2}(\theta_2) \cdot g_{l_2 t}$$

- For a general n-DOF robotic manipulator,

$$\bar{g}_{st}(\theta_1, \theta_2, ..., \theta_n) = \bar{g}_{sl_1}(\theta_1) \cdot \bar{g}_{l_1 l_2}(\theta_2) ... \bar{g}_{l_{n-1} l_n}(\theta_n) \cdot \bar{g}_{l_n t}$$

where $g_{l_{i-1} l_i}$ represents rigid motion of $\{L_i\}$ relative to $\{L_{i-1}\}$ expressed in $\{L_{i-1}\}$ (i.e., composition of body-frame SE(3) motion).





Example: SCARA

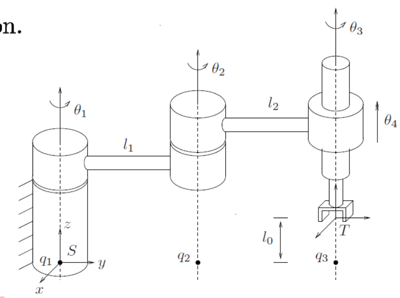
- With l_o as an offset along z-axis from $\{S\}$, we have

$$g_{sl_1}(\theta_1) = (R_z(\theta_1), [0;0;0]), \quad g_{l_1 l_2}(\theta_2) = (R_z(\theta_2), [0; l_1; 0])$$
$$g_{l_2 l_3}(\theta_3) = (R_z(\theta_3), [0; l_2; 0]), \quad g_{l_3 t}(\theta_4) = (I, [0; 0; l_o + \theta_4])$$

- Forward kinematics map g_{st} is then given by

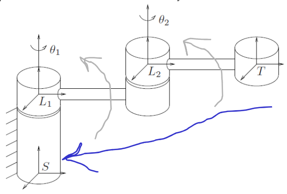
$$\bar{g}_{st}(\theta) = \bar{g}_{sl_1}(\theta_1)\bar{g}_{l_1 l_2}(\theta_2)\bar{g}_{l_2 l_3}(\theta_3)\bar{g}_{l_3 t}(\theta_4) = \begin{bmatrix} R_z(\theta_1+\theta_2+\theta_3) & \begin{pmatrix} -l_1 \, s\theta_1 - l_2 \, s\theta_{12} \\ l_1 \, c\theta_1 + l_2 \, c\theta_{12} \\ l_o + \theta_4 \end{pmatrix} \\ 0 & 1 \end{bmatrix}$$

or can be obtained by a direct observation.



Product of Exponentials: from {T} to {S}

- We may also use ξ and $e^{\hat{\xi}\theta}$ to dscribe forward kinematics: $\{S\} \rightarrow \{T\}$.
- Consider 2-DOF arm, with ξ_1^s and ξ_2^s representing joint motions of θ_1 and θ_2 expressed in $\{S\}$ at its **reference configuration** (i.e., $\theta_1 = \theta_2 = 0$).
- Fix θ_1 and move only θ_2 along ξ_2. Then, we have



$$\bar{g}_{st}(\theta_2) = e^{\hat{\xi}_2\theta_2}\bar{g}_{st}(0)$$

where $e^{\hat{\xi}_2^s\theta_2} = \bar{g}^s_{t(0)t(\theta_2)}$.

- Move this $\{T(\theta_2)\}$ further by θ_1 along ξ_1. Then, we have

$$\bar{g}_{sl}(\theta_1,\theta_2) = e^{\hat{\xi}_1\theta_1}\bar{g}_{sl}(\theta_2) = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}g_{sl}(0)$$

where $e^{\hat{\xi}_1\theta_1} = \bar{g}^s_{t(\theta_2)t(\theta_1,\theta_2)}$.

- ξ_1^s, ξ_2^s represent θ_1, θ_2 motions expressed in $\{S\}$ at **reference** configuration: total motion can then track from $\{T\}$ to $\{S\}$.

Product of Exponentials (POE)

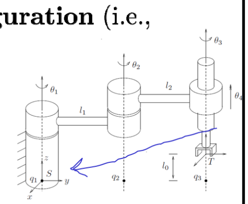
- Consider a n-DOF robotic arm with joints $1, ..., n$ sequentially from $\{S\}$ to $\{T\}$. Its forward kinematics is then given by the following POE:

$$g_{st}(\theta) = e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\dots e^{\hat{\xi}_{n-1}\theta_{n-1}}e^{\hat{\xi}_n\theta_n}g_{st}(0)$$

<- inertial frame composition

where ξ_i represents θ_i joint motion at the **reference configuration** (i.e., $\theta_i = 0$) **expressed in** $\{S\}$.

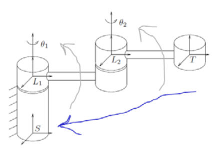
- ξ_i for revolute and prismatic joints are then given by



$$\hat{\xi}_i^s = \begin{bmatrix} \hat{w}_i^s & -w_i \times q_i \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \hat{\xi}_i = \begin{bmatrix} 0 & v_i \\ 0 & 0 \end{bmatrix}$$

where q_i is a point on rotation axis; and with joint variable θ,

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})q \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad e^{\hat{\xi}\theta} = \begin{bmatrix} I & \theta v \\ 0 & 1 \end{bmatrix}$$

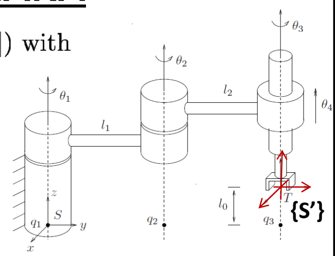


with $||w|| = 1$ or $||v|| = 1$.

Example 3.1, 3.3: SCARA

- At reference configuration, $g_{st}(0) = (I, [0; l_1 + l_2; l_o])$ with

$$\xi_1^s = ([0;0;0],[0;0;1])$$
$$\xi_2^s = ([l_1;0;0],[0;0;1])$$
$$\xi_3^s = ([l_1+l_2;0;0],[0;0;1])$$
$$\xi_4^s = ([0;0;1],[0;0;0])$$



- We can then obtain $\bar{g}_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} g_{st}(0)$.

- If we want to control EF relative to its initial pose, we can choose $\{S'\}$ to be coincident with $\{T\}$ at reference configuration. Then, $g_{st}(0) = I$ with

$$\xi_1^{s'} = ([-l_1 - l_2;0;0],[0;0;1]), \quad \xi_2^{s'} = ([-l_2;0;0],[0;0;1])$$
$$\xi_3^{s'} = ([0;0;0],[0;0;1]), \quad \xi_4^{s'} = ([0;0;1],[0;0;0])$$

- We can then compute $\bar{g}_{st}(\theta) = e^{\hat{\xi}_1^{s'}\theta_1} e^{\hat{\xi}_2^{s'}\theta_2} e^{\hat{\xi}_3^{s'}\theta_3} e^{\hat{\xi}_4^{s'}\theta_4}$, where $\xi_i^{s'} \neq \xi_i^s$, since $\{S\} \neq \{S'\}$.

Product of Exponentials: from {S} to {T}

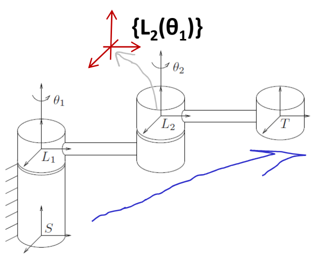
- We can recover the same POE expression $\bar{g}_{st}(\theta_1, \theta_2) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} g_{st}(0)$ even if we start from $\{S\}$, where ξ_1, ξ_2 are associated with θ_1, θ_2 motion at **reference configuration expressed in** $\{S\}$.

- First, move θ_1 along ξ_1^s with θ_2 fixed. Then,

$$g_{st}(\theta_1) = e^{\hat{\xi}_1\theta_1} g_{st}(0)$$

- At this moment, we cannot use ξ_2^s to describe θ_2 motion anymore, since the axis of θ_2 motion has moved from the reference configuration.

- Instead, we find $\xi_2^{\prime s}$ to describe θ_2 motion in $\{S\}$ with its axis moved by θ_1 motion.



Product of Exponentials: from {S} to {T}

- We can represent θ_2 motion with its axis moved to $\{L_2(\theta_1)\}$ by

$$\xi_2'^s = \mathrm{Ad}_{g_{sl_2(\theta_1)}} \mathrm{Ad}_{g^{-1}_{sl_2(0)}} \xi_2^s = \mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1}} \xi_2^s$$

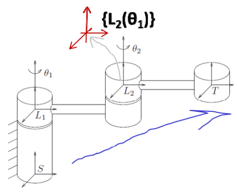
i.e., first map ξ_2^s from $\{S\}$ to ξ_2^b in $\{L_2(0)\}$, rotate θ_1 with $\xi_2^b = \xi_2'^b$, then map back $\xi_2'^b$ from $\{L_2(\theta_1)\}$ to $\xi_2'^s$ in $\{S\}$.

  - $g_{sl_2(\theta_1)} = e^{\hat{\xi}_1^s \theta_1} g_{sl_2(0)}$ with $e^{\hat{\xi}_1^s \theta_1} = g^s_{l_2(0)l_2(\theta_1)}$.
  - $\mathrm{Ad}_{g_1} \mathrm{Ad}_{g_2} = \mathrm{Ad}_{g_1 g_2}$ and $\mathrm{Ad}_g^{-1} = \mathrm{Ad}_{g^{-1}}$.
  - $\xi_2'^b = \mathrm{Ad}_{g^{-1}_{sl_2(0)}} \xi_2^s$ $(= [0,0,0,0,0,1])$ is same for $\{L_2(0)\}$ and $\{L_2(\theta_1)\}$.

- Then, from $\xi_2'^s = \mathrm{Ad}_{e^{\hat{\xi}_1 \theta_1}} \xi_2^s$ with the definition of Ad, we have $\hat{\xi}_2' = e^{\hat{\xi}_1 \theta_1} \hat{\xi}_2 e^{-\hat{\xi}_1 \theta_1}$ and $e^{\hat{\xi}_2' \theta_2} = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} e^{-\hat{\xi}_1 \theta_1}$. Therefore,

$$\begin{aligned} g_{st}(\theta_1, \theta_2) &= e^{\hat{\xi}_2' \theta_2} g_{st}(\theta_1) \\ &= e^{\hat{\xi}_2' \theta_2} e^{\hat{\xi}_1 \theta_1} g_{st}(0) \\ &= e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} g_{st}(0) \end{aligned}$$

<- inertial frame composition



Denavit-Hartenberg Convention

- For $g_{st}(\theta_1, \theta_2) = g_{sl_1}(\theta_1) g_{l_1 l_2}(\theta_2) g_{l_2 t}$, how to choose frames $\{L_i\}$? A **consistent** way to assign $\{L_i\}$ is to utilize DH-convention:

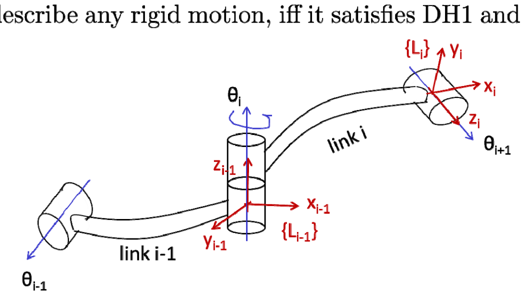
$$g_{l_{i-1} l_i} = \mathrm{Rot}_{z,\theta_i} \mathrm{Trans}_{z,d_i} \mathrm{Trans}_{x,a_i} \mathrm{Rot}_{x,\alpha_i}$$

-> body frame composition

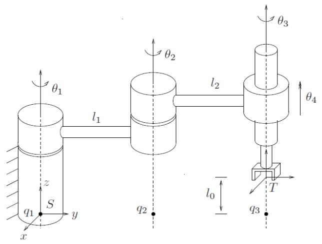
where $\theta_i, d_i, a_i, \alpha_i$ are joint angle, link offset, length, and twist.

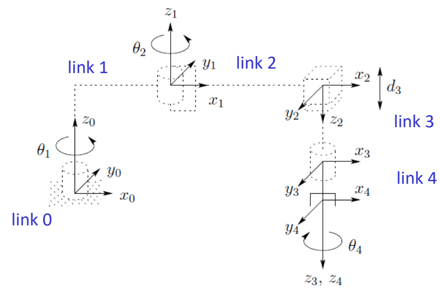
  1. $\{L_i\}$ attached on link i with z_i axis along $i+1$ joint.
  2. (DH1) x_i-axis perpendicular to z_{i-1}-axis.
  3. (DH2) x_i-axis intersects z_{i-1}-axis.

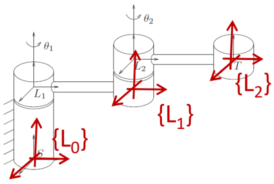
- DH parameters can describe any rigid motion, iff it satisfies DH1 and DH2 conditions.



DH Example: SCARA







link	α	a	d	θ
link 1	0	a_1	d_1	θ_1^*
link 2	π	a_2	0	θ_2^*
link 3	0	0	d_3^*	0
link 4	0	0	d_4	θ_4^*

- DH-convention $\{L_i\}$ is different from the previous way to choose $\{L_i\}$.
- POE in fact doesn't require to attach $\{L_i\}$.
- DH-convention is widely-used in industry. Industrial robots are sometimes communicated with their DH-parameters.

DH Convention and POE

- DH-convention relies on body-frame composition of $g^{l_i\ 1}_{l_{i-1}l_i(\theta_i)}$; POE uses inertial-frame composition of $e^{\hat{\xi}^s_i\theta_i}$ with ξ^s_i expressed in $\{S\}$.
- We can compute twist $\xi^{l_{i-1}}_{i-1,i}$ in $\{L_{i-1}\}$ to describe θ_i motion s.t., $e^{\hat{\xi}_{l_{i-1}l_i}\theta_i} = \bar{g}^{l_{i-1}}_{l_i(0)l_i(\theta_1)}$ by using

$$\bar{g}_{l_{i-1}l_i}(\theta_i) = e^{\hat{\xi}_{l_{i-1}l_i}\theta_i}\bar{g}_{l_{i-1}l_i}(0)$$

- With $\{L_o\} = \{S\}$ and $\{L_n\} = \{T\}$,

$$\bar{g}_{st}(\theta) = e^{\hat{\xi}_{0,1}\theta_1}\bar{g}_{l_0l_1}(0)\cdot e^{\hat{\xi}_{1,2}\theta_2}\bar{g}_{l_1l_2}(0)\cdot\ldots\cdot e^{\hat{\xi}_{n-1,n}\theta_n}\bar{g}_{l_{n-1}t}(0)$$

- We can then rewrite this expression by

$$\bar{g}_{st}(\theta) = e^{\hat{\xi}^0_{0,1}\theta_1}\cdot e^{(\mathrm{Ad}_{g_{l_0,l_1}(0)}\xi^1_{1,2})^\wedge\theta_2}\cdot\ldots\cdot e^{(\mathrm{Ad}_{g_{l_0,l_{n-1}}(0)}\xi^{n-1}_{n-1,n})^\wedge\theta_n}\bar{g}_{st}(0)$$

that is, $\xi^s_i = \mathrm{Ad}_{g_{l_0,l_{i-1}}(0)}\xi_{i-1,i}$, implying that ξ^s_i is $\xi^{i-1}_{i-1,i}$ mapped from $\{L_{i-1}\}$ to $\{S\}$ at the reference configuration.

Inverse Kinematics

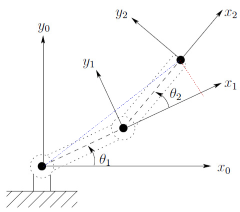
- Given tool-frame pose $g_d \in SE(3)$, find joint variables $\theta_1, \theta_2, ..., \theta_n$ s.t.,

$$g_{st}(\theta_1, \theta_2, ..., \theta_n) = g_d$$

which may have multiple or no solutions.

- Example: 2-DOF robot. With (x, y) given, (θ_1, θ_2) is given by:

$$\theta_2 = \pi - \alpha \quad \text{(elbow-down)} \quad \text{or} \quad \theta_2 = -\pi + \alpha \quad \text{(elbow-up)}$$
$$\theta_1 = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{l_2 \sin\theta_2}{l_1 + l_2 \cos\theta_2}$$



where $\alpha = \cos^{-1}\left(\frac{l_1^2 + l_2^2 - r^2}{2l_1 l_2}\right)$ and $r^2 = \sqrt{x^2 + y^2}$.

- Recall the product of exponential formula

$$\bar{g}_{st}(\theta) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} .....e^{\hat{\xi}_n\theta_n} \bar{g}_{st}(0)$$

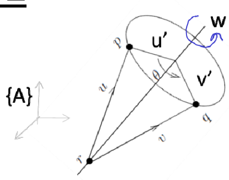
We decompose IK-problem into **subproblems** with closed-form solution.

- Most industrial manipulators have closed-form IK-solutions.

Paden-Kahan Sub-Problem 1

- Given zero-pitch unit twist ξ and $p, q \in \Re^3$, find $\theta \in \Re$ s.t.

$$e^{\hat{\xi}\theta}\bar{p} = \bar{q} \quad (\leftarrow \ \bar{p}_s(\theta) = \bar{g}^s_{b(0)b(\theta)}\bar{p}_s(0) = e^{\hat{\xi}^s\theta}\bar{p}_s(0))$$



i.e., rotation angle θ of point p about ξ to match q.

1. Find r a point on ξ-axis and compute $u = p - r$ and $v = q - r$. Then,

$$e^{\hat{\xi}\theta}(\bar{p} - \bar{r}) = \bar{q} - \bar{r} \quad \Rightarrow \quad e^{\hat{w}\theta}u = v \quad \text{with} \quad e^{\hat{\xi}\theta}\bar{r} = \bar{r}$$

2. Define projections of u, v onto the plane perpendicular to ξ:

$$u' = u - w^T uw, \quad v' = v - w^T vw$$

3. Necessary condition for solution: $w^T u = w^T v$, $||u'|| = ||v'||$.
4. Under this condition, if $u' = 0$, infinitely-many solutions; if $u' \neq 0$,

$$\theta = \text{atan2}(w^T(u' \times v'), u' \cdot v')$$

from $u' \times v' = w||u'|| \cdot ||v'|| \sin\theta$ and $u' \cdot v' = ||u'|| \cdot ||v'|| \cos\theta$.

Paden-Kahan Sub-Problem 2

- Let ξ_1, ξ_2 be zero pitch unit twists (given), with their screw axes intersecting at $r \in \Re^3$. Given $p, q \in \Re^3$, find $\theta_1, \theta_2 \in \Re$ s.t.

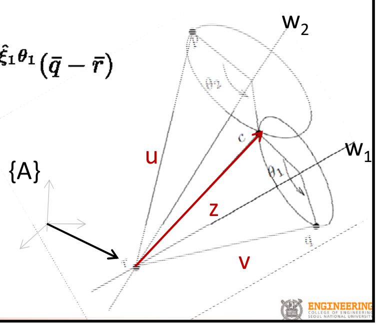
$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \bar{p} = \bar{q}$$

i.e., p rotates about ξ_2 by θ_2, then, about ξ_1 by θ_1 to match q.

1. If $w_1 \times w_2 = 0$, ξ_1, ξ_2 aligned with infinitely-many solutions $\theta_1 + \theta_2$.
2. If $w_1 \times w_2 \neq 0$, with intermediate point c (not known *a prior*)

$$e^{\hat{\xi}_2\theta_2}\bar{p} = \bar{c} = e^{-\hat{\xi}_1\theta_1}\bar{q}$$
$$e^{\hat{\xi}_2\theta_2}(\bar{p} - \bar{r}) = \bar{c} - \bar{r} = e^{-\hat{\xi}_1\theta_1}(\bar{q} - \bar{r})$$
$$e^{\hat{w}_2\theta_2}u = z = e^{-\hat{w}_1\theta_1}v$$

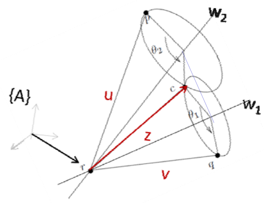
3. Write $z = \alpha w_1 + \beta w_2 + \gamma(w_1 \times w_2)$



Paden-Kahan Sub-Problem 2

- Let ξ_1, ξ_2 be zero pitch unit twists (given), with their screw axes intersecting at $r \in \Re^3$. Given $p, q \in \Re^3$, find $\theta_1, \theta_2 \in \Re$ s.t.

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \bar{p} = \bar{q}$$



1. Write $z = \alpha w_1 + \beta w_2 + \gamma(w_1 \times w_2)$.
2. From $w_2^T z = w_2^T u$ and $w_1^T v = w_1^T z \Rightarrow w_2^T u = \alpha w_2^T w_1 + \beta$ and $w_1^T v = \alpha + \beta w_1^T w_2 \Rightarrow (\alpha, \beta)$.
3. From $||z||^2 = \alpha^2 + \beta^2 + 2\alpha\beta w_1^T w_2 + \gamma^2 ||w_1 \times w_2||^2$ with $||u||^2 = ||z||^2$,

$$\gamma^2 = \frac{||u||^2 - \alpha^2 - \beta^2 - 2\alpha\beta w_1^T w_2}{||w_1 \times w_2||^2}$$

4. Two solutions exist with $\pm\gamma$ if circles intersect at two points; or only one solution with $\gamma = 0$ if intersect at one point.
5. With c identified, apply Subproblem 1 to $e^{\hat{\xi}_2\theta_2}\bar{p} = \bar{c} = e^{-\hat{\xi}_1\theta_1}\bar{q}$ to compute θ_1 and θ_2.

Paden-Kahan Sub-Problem 3

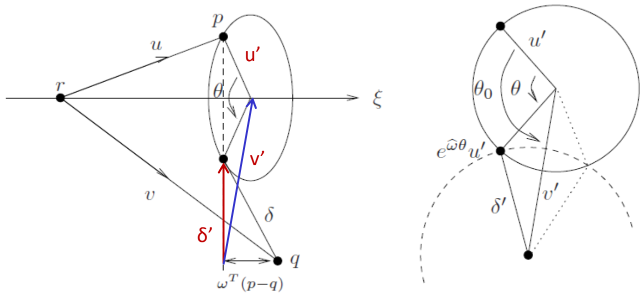
- Let ξ be zero pitch unit twist. Given $p, q \in \Re^3$ and $\delta > 0$, find $\theta \in \Re$ s.t.

$$||e^{\hat{\xi}\theta}\bar{p} - \bar{q}|| = \delta$$

i.e., after rotation θ about ξ, p is at distance δ from q.

1. With a point r on ξ, $u := p - r$ and $v := q - r \Rightarrow ||e^{\hat{w}\theta}u - v||^2 = \delta^2$
2. Define projections of u, v, δ onto the plane perpendicular to ξ s.t.

$$u' = u - w^T uw, \quad v' = v - w^T vw, \quad \delta'^2 = \delta^2 - |w^T(p-q)|^2$$



Paden-Kahan Sub-Problem 3

- Let ξ be zero pitch unit twist. Given $p, q \in \Re^3$ and $\delta > 0$, find $\theta \in \Re$ s.t.

$$||e^{\hat{\xi}\theta}\bar{p} - \bar{q}|| = \delta$$

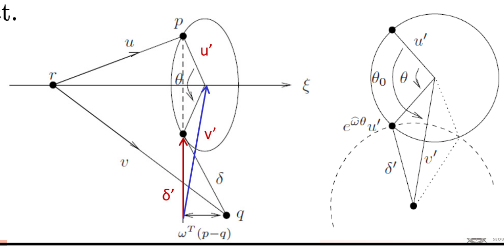
1. Define projections of u, v, δ onto the plane perpendicular to ξ s.t.

$$u' = u - w^T uw, \quad v' = v - w^T vw, \quad \delta'^2 = \delta^2 - |w^T(p-q)|^2$$

2. From the figure, $\theta_o = \text{atan2}(w^T(u' \times v'), u' \cdot v')$. Also, applying cosine law,

$$\theta = \theta_o \pm \cos^{-1}\left(\frac{||u'||^2 + ||v'||^2 - \delta'^2}{2||u'||||v'||}\right)$$

3. Two solutions if δ-sphere intersect circle at two points; one if tangent; no solution if not intersect.



Example 3.5: Elbow Manipulator

- Given $g_d \in \mathrm{SE}(3)$, find $(\theta_1, ..., \theta_6)$ s.t. $g_{st}(\theta) = e^{\hat{\xi}_1\theta_1} ... e^{\hat{\xi}_6\theta_6} g_{st}(0) = g_d$, or

$$e^{\hat{\xi}_1\theta_1} ... e^{\hat{\xi}_6\theta_6} = g_d g_{st}^{-1}(0) =: g_1$$

- Kinematic decoupling: apply this to p_w, which is at intersection of ξ_4, ξ_5, ξ_6,

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \bar{p}_w = g_1 \bar{p}_w \quad \text{(i.e., only for robot-arm)}$$

- Subtracting p_b, which is at intersection of ξ_1, ξ_2,

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \bar{p}_w - \bar{p}_b = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} (e^{\hat{\xi}_3\theta_3} \bar{p}_w - \bar{p}_b) = g_1 \bar{p}_w - \bar{p}_b$$

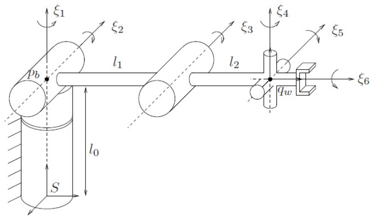
i.e., $||e^{\hat{\xi}_3\theta_3} p_w - p_b|| = ||g_1 p_w - p_b||$

$\Rightarrow$ Subproblem 3 for θ_3.

- With θ_3 known, we have

$$e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} \bar{p}_w = g_1 \bar{p}_w$$

which is Subproblem 2 for θ_1, θ_2.



Example 3.5: Elbow Manipulator

- From $e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5} e^{\hat{\xi}_6\theta_6} = g_1$, we then have,

$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5} e^{\hat{\xi}_6\theta_6} = e^{-\hat{\xi}_3\theta_3} e^{-\hat{\xi}_2\theta_2} e^{-\hat{\xi}_1\theta_1} g_1 =: g_2 \quad \text{(i.e., only for the wrist)}$$

- Choose p_6 on ξ_6-axis, but not on ξ_4, ξ_5. Then,

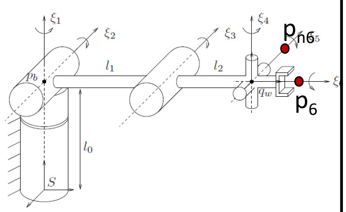
$$e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5} e^{\hat{\xi}_6\theta_6} p_6 = e^{\hat{\xi}_4\theta_4} e^{\hat{\xi}_5} p_6 = g_2 p_6$$

$\Rightarrow$ Subproblem 2 for θ_4, θ_5.

- Finally, choose p_{n6} not on ξ_6-axis. Then,

$$e^{\hat{\xi}_6\theta_6} \bar{p}_{n6} = [e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} ... e^{\hat{\xi}_5\theta_5}]^{-1} g_1 \bar{p}_{n6}$$

which is Subproblem 1 for θ_6.



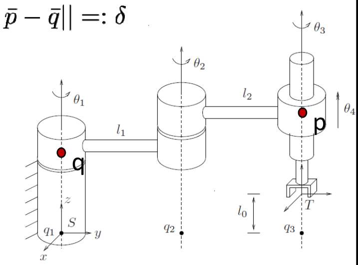
- Maximum 8 solutions are possible due to the presence of 2 Subproblems 2 and and 1 Subproblem 3.
- Here, we also have **kinematic decoupling**: inverse kinematics problem decoupled into that of wrist and that of robot-arm.

Example 3.6: SCARA

- Desired tool pose $g_d = (R_z(\phi), [x; y; z])$, $z = \theta_4 + l_o$: can compute first θ_4.
- Given θ_4, we have $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3} = g_d g_{st}^{-1}(0)e^{-\hat{\xi}_4\theta_4} =: g_1$
- To obtain θ_1, θ_2, choose p on ξ_3-axis and q on ξ_1-axis. Then,

$$||e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}\bar{p} - \bar{q}|| = ||e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\bar{p} - \bar{q}|| = ||e^{\hat{\xi}_1\theta_1}(e^{\hat{\xi}_2\theta_2}p - q)||$$
$$= ||e^{\hat{\xi}_2\theta_2}\bar{p} - \bar{q}|| = ||g_1\bar{p} - \bar{q}|| =: \delta$$

  which is Subproblem 3 for θ_2.



- Also, for p_3 on ξ_3-axis, we have

$$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\bar{p}_3 = g_1\bar{p}_3$$

  which is Subproblem 1 for θ_1.

- Finally, $e^{\hat{\xi}_3\theta_3} = [e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}]^{-1}g_1$, which is Subproblem 1 for θ_3.
- Maximum 2 solutions possible for SCARA robot, due to the presence of 1 Subproblem 3.