- PDF $\quad \underline{f(V) \equiv \dfrac{dF(V)}{dV}}$ $\qquad \underset{\text{CDF}}{F(V)} \equiv P\{U<V\}$

- Mean (or expectation) of U

$$\langle U \rangle \equiv \int_{-\infty}^{\infty} V\, f(V)\, dV$$

more generally, $\langle Q(U) \rangle = \int_{-\infty}^{\infty} Q(V)\, f(V)\, dV$

$$\langle\langle U \rangle\rangle = \langle U \rangle$$

- fluctuation of U : $\quad u \equiv U - \langle U \rangle$
- variance (mean-square fluctuation)

$$\mathrm{var}(U) \equiv \langle u^2 \rangle = \int_{-\infty}^{\infty} (V - \langle U \rangle)^2 f(V)\, dV$$

- standard deviation (root-mean-square fluctuation)

$$\mathrm{sdev}(U) = \sqrt{\mathrm{var}(U)} = \langle u^2 \rangle^{\frac{1}{2}} \overset{\text{rms}}{\equiv} u' \ (\text{or } \sigma_u)$$

- n^{th} central moment

$$\mu_n \equiv \langle u^n \rangle = \int_{-\infty}^{\infty} (V - \langle U \rangle)^n f(V)\, dV$$

$$\mu_0 = \int_{-\infty}^{\infty} f(V)\, dV = 1 \,, \quad \mu_1 = \int_{-\infty}^{\infty} (V - \langle U \rangle) f(V) dV = \langle U \rangle - \langle U \rangle = 0$$

$$\mu_2 = \sigma_u^2 \,, \quad \cdots$$

- standardization : zero mean and unit variance

$$\hat{U} \equiv (U - \langle U \rangle) / \sigma_u$$

standardized random variable

$$\boxed{\hat{f}(\hat{U}) = \sigma_u f(\langle U \rangle + \sigma_u \hat{U}) = \sigma_u f(U)}$$

standardized PDF of U

pf. $\mathrm{var}(\hat{U}) = 1$

$\mathrm{var}(U) = \sigma_u^2$

$$\widehat{var}(\hat{u}) = \int_{-\infty}^{\infty} \hat{u}^2 \hat{f}(\hat{u})\, d\hat{u} = 1 \qquad \sigma_u\, d\hat{u}$$

$$var(u) = \int_{-\infty}^{\infty} (\underbrace{u - \langle u \rangle}_{= \sigma_u \hat{u}})^2 f(u)\, du = \sigma_u^2$$

$$= \int_{-\infty}^{\infty} \sigma_u^{\cancel{3}}\, \hat{u}^2 f(u)\, d\hat{u} = \cancel{\sigma_u^2}$$

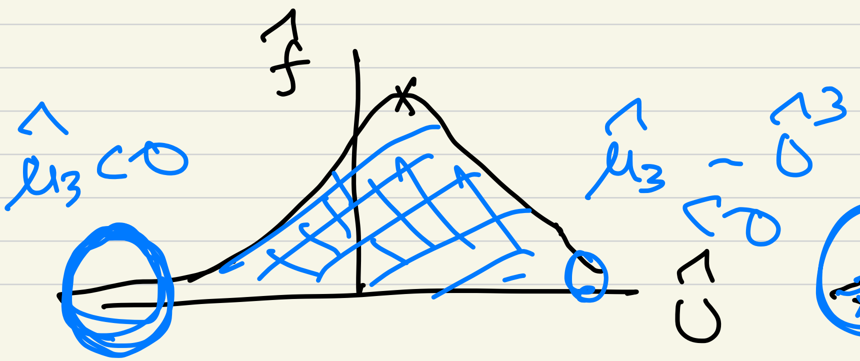
$$\rightarrow \int_{-\infty}^{\infty} \hat{u}^2 \sigma_u f(u)\, d\hat{u} = 1$$

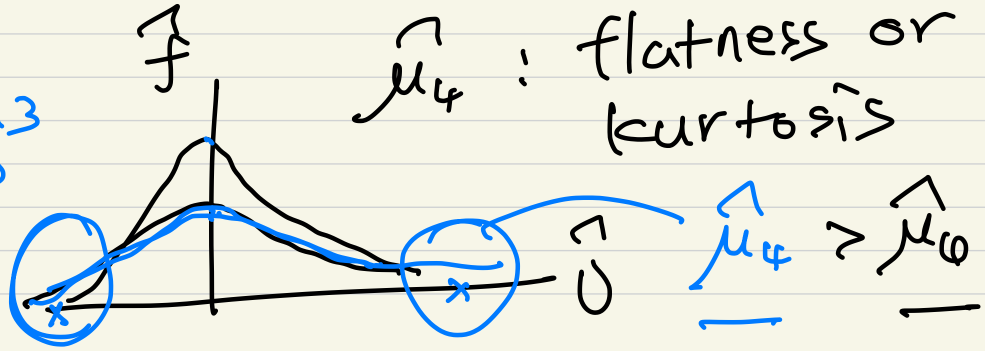
$$\boxed{\hat{f}(\hat{u}) = \sigma_u f(u)} \quad \& \quad var(\hat{u}) = 1$$

$$\hat{\mu}_n = \frac{\langle u^n \rangle}{\sigma_u^n} = \frac{\mu_n}{\sigma_u^n} = \int_{-\infty}^{\infty} \hat{v}^n \hat{f}(\hat{v})\, d\hat{v}$$

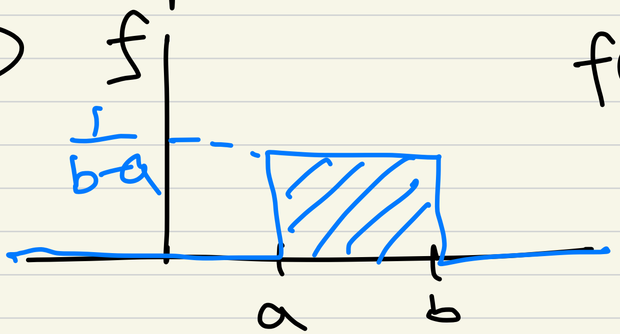
$\hat{\mu}_0 = 1$, $\hat{\mu}_1 = 0$, $\hat{\mu}_2 = 1$, $\hat{\mu}_3$: skewness

$\hat{\mu}_4$: flatness or kurtosis





- Examples of probability distributions

① 

$f(v)$ is uniform in $a \le v < b$

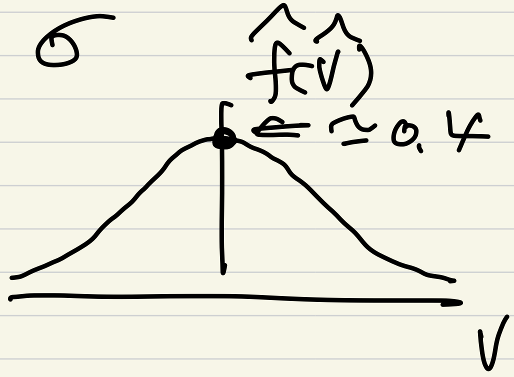
$$f(v) = \begin{cases} \frac{1}{b-a} & \text{for } a \le v < b \\ 0 & \text{for } v < a \text{ and } v \ge b \end{cases}$$

$$\to \langle v \rangle = \int_{-\infty}^{\infty} v f(v)\, dv = \int_a^b v \cdot \frac{1}{b-a}\, dv = \frac{1}{2}(a+b)$$

$$\mathrm{var}(v) = \int_{-\infty}^{\infty} (v - \langle v \rangle)^2 f(v)\, dv = \cdots = \frac{1}{12}(b-a)^2$$

$$\hat{\mu}_3 = 0, \quad \hat{\mu}_4 = 9/5 .$$

② v is normally (or Gaussian) distributed with mean μ and standard deviation σ



$$\to f(v) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}(v-\mu)^2/\sigma^2\right]$$

$$\hat{v} = \frac{v-\mu}{\sigma} \to \hat{f}(\hat{v}) = \frac{1}{\sqrt{2\pi}} e^{-\hat{v}^2/2}$$

$\hat{\mu}_3 = 0$, $\boxed{\hat{\mu}_4 = 3}$, $\hat{\mu}_5 = 0$, $\hat{\mu}_6 = 15$, - - - .

- Joint random variables

  velocity (U_1, U_2, U_3)

  · CDF of the joint random variables (U_1, U_2)

$$F_{12}(V_1, V_2) \equiv P\{U_1 < V_1, U_2 < V_2\}$$

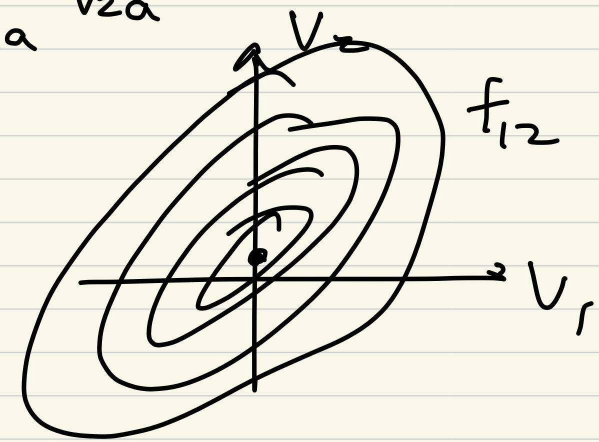
  · Joint PDF (JPDF) of U_1 and U_2

$$f_{12}(V_1, V_2) \equiv \frac{\partial^2}{\partial V_1 \partial V_2} F_{12}(V_1, V_2)$$

$$P\{V_{1a} \le U_1 < V_{1b}, V_{2a} \le U_2 < V_{2b}\} = \int_{V_{1a}}^{V_{1b}} \int_{V_{2a}}^{V_{2b}} f_{12}(V_1, V_2)\, dV_2\, dV_1$$

$$\int_{-\infty}^{\infty} f_{12}(V_1, V_2)\, dV_1 = f_2(V_2)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{12}(V_1, V_2)\, dV_1\, dV_2 = 1$$



- Mean of $Q(U_1, U_2)$

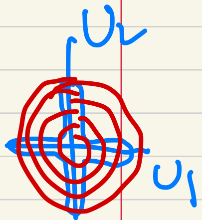
$$\langle Q(U_1, U_2) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(V_1, V_2)\, f_{12}(V_1, V_2)\, dV_1\, dV_2$$

• Covariance of u_1 and u_2

$$\mathrm{cov}(u_1, u_2) = \langle u_1 u_2 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (V_1 - \langle U_1 \rangle)(V_2 - \langle U_2 \rangle)\, f_{12}(V_1, V_2)\, dV_1\, dV_2$$

correlation coefficient

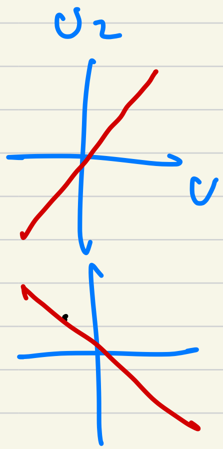
$$\rho_{12} \equiv \langle u_1 u_2 \rangle / (\langle u_1^2 \rangle \langle u_2^2 \rangle)^{\frac{1}{2}}$$



$\rho_{12} = 0$: u_1 and u_2 are uncorrelated

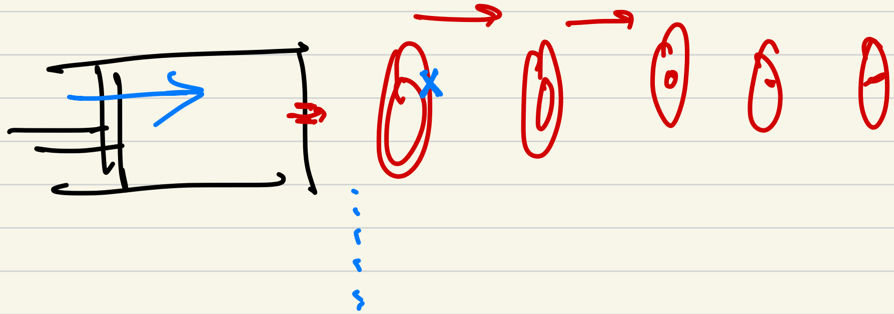
$\rho_{12} = 1$: " " are perfectly correlated

$\rho_{12} = -1$: " " " " negatively "



• Ensemble average (over N repetitions)

$$\langle U \rangle_N \equiv \frac{1}{N} \sum_{n=1}^{N} U^{(n)}$$



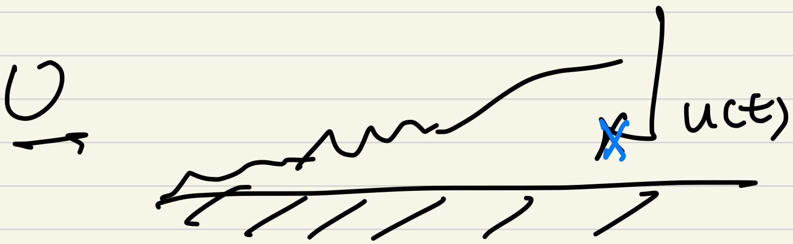
① Random process

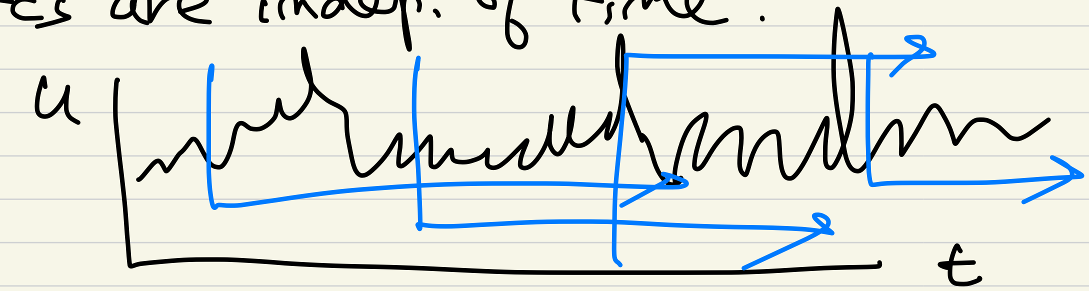
A process is statistically stationary

→ All multi-time statistics are invariant under a shift in time, i.e., for all T,

$$f(V_1, t_1+T; V_2, t_2+T; \cdots; V_N, t_N+T)$$
$$= f(V_1, t_1; V_2, t_2; \cdots; V_N, t_N)$$

A turbulent flow can reach a statistically steady state in which the statistics are indep. of time.





- Auto covariance $R(s) \equiv \langle u(t)\, u(t+s) \rangle$ where $u = U - \langle U \rangle$ fluctuation

Auto correlation ft $\rho(s) \equiv \dfrac{\langle u(t)\, u(t+s) \rangle}{\langle u(t)^2 \rangle}$

$\rho(s) = \langle u(t)\, u(t+s) \rangle$

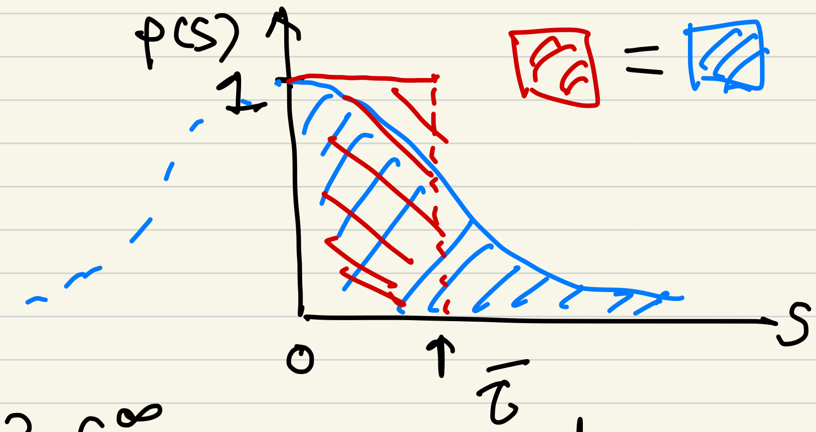
$\rho(-s) = \langle u(t)\, u(t-s) \rangle$

$\rho(0) = 1 \qquad |\rho(s)| \le 1$

$\boxed{\rho(s) = \rho(-s)}$: ρ is an even fc.

If $u(t)$ is periodic w/ period T, $\rho(s) = \rho(s+T)$

In most turb. flow, $\rho(s) \to 0$ as $s \to \infty$.



- Integral timescale

$$\bar{\tau} = \int_0^\infty \rho(s)\, ds$$

- Frequency spectrum $E(\omega)$

$$\begin{cases} E(\omega) \equiv \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{\infty} R(s) e^{-i\omega s} ds = \dfrac{2}{\pi} \displaystyle\int_0^\infty R(s) \cos(\omega s)\, ds \\ R(s) = \dfrac{1}{2} \displaystyle\int_{-\infty}^{\infty} E(\omega) e^{i\omega s} d\omega = \displaystyle\int_0^\infty E(\omega) \cos(\omega s)\, d\omega \end{cases}$$

Fourier transform $\quad g(\omega) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dT$

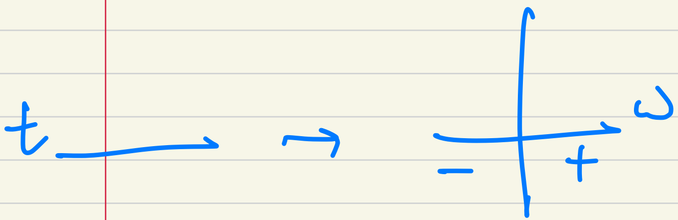
$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{+i\omega t} d\omega$$

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(\omega)\, e^{i\omega t}\, d\omega \qquad \hat{u} : \text{Fourier coeff.}$$

$$u(t+s) = \int_{-\infty}^{\infty} \hat{u}(\omega)\, e^{i\omega(t+s)}\, d\omega$$

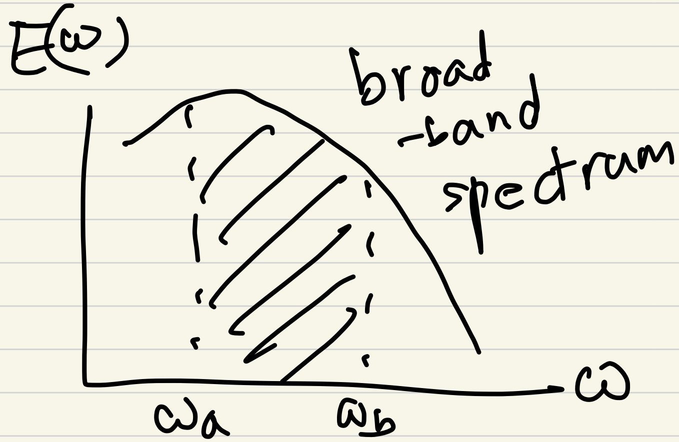
$$R(s) = \langle u(t)\, u(t+s) \rangle = \left\langle \int_{-\infty}^{\infty} \hat{u}(\omega)\, e^{i\omega t}\, d\omega \cdot \int_{-\infty}^{\infty} \hat{u}(\omega')\, e^{i\omega'(t+s)}\, d\omega' \right\rangle$$

orthogonality $\omega' = -\omega$



$$= \int_{-\infty}^{\infty} \hat{u}(\omega)\, \underbrace{\hat{u}(-\omega)}_{= \hat{u}^*(\omega)}\, e^{-i\omega s}\, d\omega$$

$$= \int_{-\infty}^{\infty} \underbrace{\boxed{\hat{u}(\omega)\, \hat{u}^*(\omega)}}_{= \frac{1}{2}E(\omega)}\, e^{-i\omega s}\, d\omega \qquad E(\omega) = 2\hat{u}\hat{u}^*$$



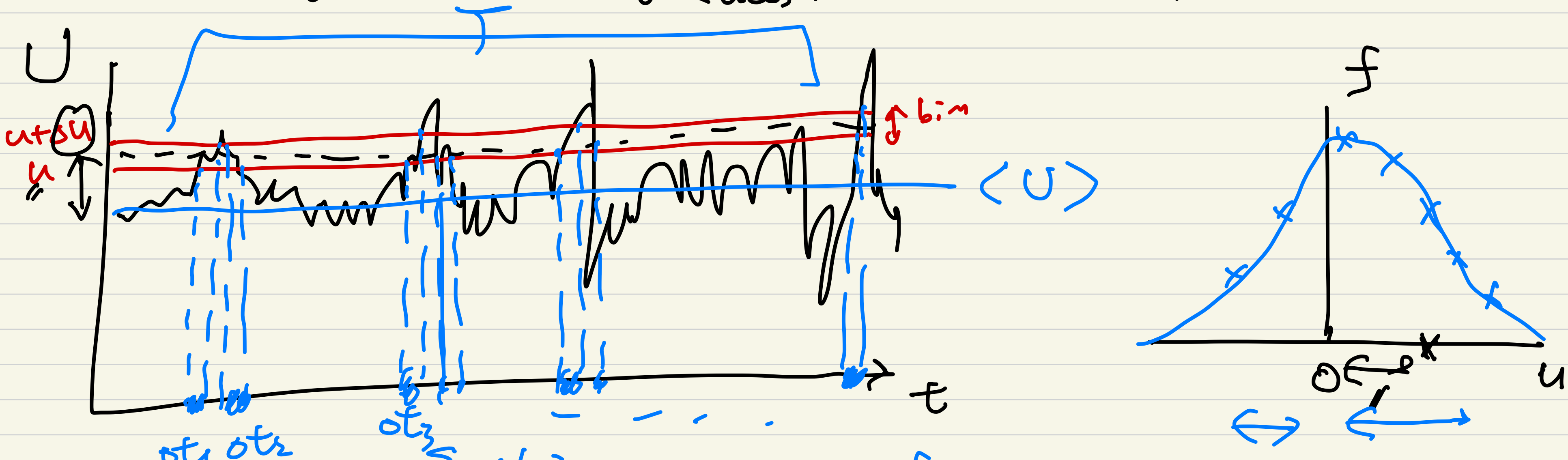
$\int_{\omega_a}^{\omega_b} E(\omega)\, d\omega$: contribution to $\langle u(t)^2 \rangle$ in $\omega_a \le \omega \le \omega_b$.

$u(t) \xrightarrow{FT} \hat{u} \to \hat{u}\hat{u}^*(\omega)$

$$\boxed{R(0) = \langle u(t)^2 \rangle = \int_0^\infty E(\omega)\, d\omega}$$

$$E(0) = \frac{2}{\pi}\int_0^\infty R(s)\, ds$$

$$\bar{\tau} = \int_0^\infty \rho(s)\, ds = \int_0^\infty \frac{R(s)}{\langle u(t)^2 \rangle}\, ds = \frac{\pi/2 \cdot E(0)}{\langle u^2 \rangle} = \frac{\pi E(0)}{2\langle u^2 \rangle}$$



$$f = \frac{\sum \delta t_i}{T}$$ for a specific u.

- covariance of velocity : $\langle u_i(x,t)\, u_j(x,t) \rangle$ @ $s \neq 0$

↑ Reynolds stresses

$u_i \neq u_j$: cross correlation

$u_i = u_j$: auto correlation