CHAPTER 2. SECOND-ORDER LINEAR ODES

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서유택

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

2.1 Homogeneous Linear ODEs of Second Order

❖ **Linear ODEs of second order**: $y'' + p(x)y' + q(x)y = r(x)$ (the standard form)

- Homogeneous (**제차**): $r(x) = 0$
- Nonhomogeneous (**비제차**): $r(x) \neq 0$

☑ **Ex. A nonhomogeneous linear ODE (**비제차 상미분 방정식**):**

$$y'' + 25y = e^{-x}\cos x$$

A homogeneous linear ODE: $xy'' + y' + xy = 0$ in standard form $y'' + \frac{1}{x}y' + y = 0$

A nonlinear ODE: $y''y + (y')^2 = 0$

2.1 Homogeneous Linear ODEs of Second Order

❖ Homogeneous Linear ODEs: Superposition Principle (중첩원리)

❖ **Theorem 1 Fundamental Theorem for the Homogeneous Linear ODE**

For a homogeneous linear ODE,

- any linear combination of two solutions on an open interval I is
- again solution of the equation on I.

In particular, for such an equation, sums and constant multiples of solutions are again solutions.

❖ This highly important theorem holds for **homogeneous linear ODEs only**

but does not hold for nonhomogeneous linear or nonlinear ODEs.

2.1 Homogeneous Linear ODEs of Second Order

❖ Homogeneous Linear ODEs: Superposition Principle

☑ **Ex. 2 A nonhomogeneous linear ODE** $y''+y=1$

The functions $y=1+\cos x$ and $y=1+\sin x$ are solutions. Neither is $2(1+\cos x)$ or $5(1+\sin x)$

☑ **Ex. 3 A nonlinear ODE** $y''y-xy'=0$

The functions $y=1$ and $y=x^2$ are solutions. But their sum is not a solution. Neither is $-x^2$, so you cannot even multiply by -1.

2.1 Homogeneous Linear ODEs of Second Order

❖ Initial Value Problem. Basis. General Solution.

- Initial Value Problems (**초기값 문제**)

: A differential equation consists of the homogeneous linear ODE and two initial conditions.

- Initial Conditions : $y(x_0) = K_0, \quad y'(x_0) = K_1$
- This results in a *particular solution* of ODE.

☑ Ex. 4 Solve the initial value problem

$$y'' + y = 0, \qquad y(0) = 3.0, \qquad y'(0) = -0.5$$

Step 1 General solution (일반해)

$$y = c_1 \cos x + c_2 \sin x \quad \left(\because \lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm i\right)$$

Step 2 Particular solution (특수해)

$$y(0) = c_1 = 3.0, \quad y'(0) = c_2 = -0.5 \quad \left(\because \ y' = -c_1 \sin x + c_2 \cos x\right) \quad \Rightarrow \quad \therefore \ y = 3.0\cos x - 0.5\sin x$$

2.1 Homogeneous Linear ODEs of Second Order

❖ Definition General Solution, Basis, Particular Solution

A **general solution** of an ODE on an open interval I is

- a solution $y = c_1y_1+c_2y_2$ in which y_1 and y_2 are solutions of the equation on I that are not proportional and c_1, c_2 are arbitrary constants.

There y_1, y_2 are called a **basis (기저)** (or a **fundamental system**) of solutions of the equation on I.

A **particular solution** of the equation on I is obtained if we assign specific values to c_1 and c_2 in $y = c_1y_1+c_2y_2$.

* open interval: $a < x < b$ (NOT $a \le x \le b$), $-\infty < x < b$, $a < x < \infty$, $-\infty < x < \infty$

2.1 Homogeneous Linear ODEs of Second Order

- Two functions y_1 and y_2 are called **linearly independent** on I where they are defined if $k_1 y_1(x) + k_2 y_2(x) = 0$ everywhere on I implies $\boldsymbol{k_1 = 0}$ and $\boldsymbol{k_2 = 0}$.
- y_1 and y_2 are called **linearly dependent** on I if $k_1 y_1(x) + k_2 y_2(x) = 0$ also holds for some constants k_1, k_2 not both zero.

If $k_1 \neq 0$ or $k_2 \neq 0$, we can divide and see that y_1 and y_2 are proportional,

$$y_1 = -\frac{k_2}{k_1} y_2 \quad \text{or} \quad y_2 = -\frac{k_1}{k_2} y_1$$

❖ **Definition Basis (Reformulated)**

A **basis** of solutions of the equation on an open interval I is a pair of linearly independent solutions of the equation on I.

2.1 Homogeneous Linear ODEs of Second Order

❖ **Find a Basis if One Solution Is Known. Reduction of Order [차수축소법]**

(Extended Method, 확장 방법)

Apply **reduction of order** to the homogeneous linear ODE $y''+p(x)y'+q(x)y=0$.

$y=y_2=uy_1$ (Substitute) $\left(y'=y_2'=u'y_1+uy_1',\ y''=y_2''=u''y_1+2u'y_1'+uy_1''\right)$

$$\Rightarrow u''y_1+u'(2y_1'+py_1)+u(y_1''+py_1'+qy_1)=0$$

$$\Rightarrow u''+u'\frac{2y_1'+py_1}{y_1}=0 \qquad (\because\ y_1''+py_1'+qy_1=0)$$

$U=u',\quad U'=u''$ (Substitute) $\Rightarrow U'+\left(2\frac{y_1'}{y_1}+p\right)U=0 \Rightarrow \frac{dU}{dx}=-\left(2\frac{y_1'}{y_1}+p\right)U$

(Separation of variables and integration)

$$\Rightarrow \frac{dU}{U}=-\left(2\frac{y_1'}{y_1}+p\right)dx \ \ \&\ \ \ln|U|=-2\ln|y_1|-\int p\,dx$$

$$\Rightarrow \therefore\ U=\frac{1}{y_1^{\,2}}e^{-\int p\,dx},\qquad y_2=uy_1=y_1\int U\,dx$$

2.1 Homogeneous Linear ODEs of Second Order

☑ **Ex. 7 Find a basis of solution of the ODE** $\left(x^2 - x\right)y'' - xy' + y = 0$

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

One solution: $y_1 = x$

Apply reduction of order: $p = -\dfrac{x}{x^2 - x} = -\dfrac{1}{x-1}$

$$\boxed{U = \frac{1}{y_1^{\,2}} e^{-\int p dx}}$$

$$\Rightarrow \quad U = \frac{1}{y_1^{\,2}} e^{-\int p dx} = \frac{1}{x^2} e^{\int \frac{1}{x-1} dx} = \frac{1}{x^2} e^{\ln(x-1)} = \frac{x-1}{x^2}$$

$$\boxed{y_2 = u y_1 = y_1 \int U dx}$$

$$\Rightarrow \quad y_2 = y_1 \int U dx = x\left(\ln|x| + \frac{1}{x}\right) = x \ln|x| + 1$$

Q : Start from the original assumption.

$$y_2 = u y_1 = ux$$

$$\left(y' = y_2' = u' y_1 + u y_1',\ y'' = y_2'' = u'' y_1 + 2u' y_1' + u y_1''\right)$$

$$y'' + p(x)y' + q(x)y = 0$$

2.2 Homogeneous Linear ODEs with Constant Coefficients

- Second-order homogeneous linear ODEs with constant coefficients: $y'' + ay' + by = 0$
- We try $y = e^{\lambda x}$.
- Characteristic equation (Auxiliary Equation, **특성방정식**): $\lambda^2 + a\lambda + b = 0$
- Three kinds of the general solution of the equation

▪ **Case I** Two real roots $\lambda_1, \ \lambda_2$ if $a^2 - 4b > 0 \quad \Rightarrow \quad y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

▪ **Case II** A real double root $\lambda = -a/2$ if $a^2 - 4b = 0 \quad \Rightarrow \quad y = (c_1 + c_2 x) e^{-ax/2}$

▪ **Case III** Complex conjugate roots $\lambda = -a/2 \pm i\omega$

if $a^2 - 4b < 0 \quad \Rightarrow \quad y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$

- Euler formula: $e^{it} = \cos t + i\sin t$

2.2 Homogeneous Linear ODEs with Constant Coefficients

❖ **Case II** A real double root $\lambda = -a/2$ if $a^2 - 4b = 0$ $\Rightarrow$ $y = (c_1 + c_2 x)e^{-ax/2}$

Prove it by using the method of reduction of order!

$$y_1 = e^{-(a/2)x}$$

setting $y_2 = uy_1 \quad \Rightarrow y_2' = u'y_1 + uy_1'$ $\quad \boxed{y'' + ay' + by = 0}$

$$(u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + buy_1 = 0$$

$$u''y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0$$

here, $2y_1' = -ae^{-ax/2} = -ay_1$

$$u''y_1 = 0 \Rightarrow u'' = 0 \Rightarrow u = c_1 x + c_2$$

we can simply choose $c_1 = 1, c_2 = 0 \Rightarrow u = x \qquad y_2 = uy_1 = xy_1 = xe^{-(a/2)x}$

$$\boxed{y = c_1 y_1 + c_2 y_2 = (c_1 + c_2 x)e^{-ax/2}}$$

2.2 Homogeneous Linear ODEs with Constant Coefficients

☑ **Ex. 2 Solver the initial value problem** $y''+y'-2y=0, \quad y(0)=4, \quad y'(0)=-5$

Step 1 General solution

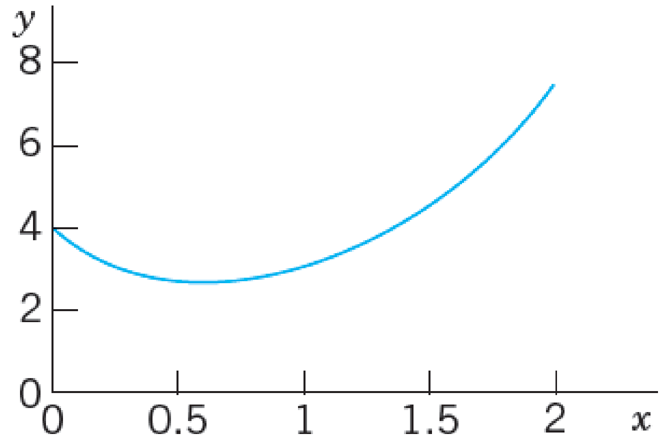
$$\lambda^2+\lambda-2=0 \text{ (Characteristic equation)} \Rightarrow \quad \lambda=1 \text{ or } -2 \quad \Rightarrow \quad \therefore\ y=c_1e^x+c_2e^{-2x}$$

Step 2 Particular solution

$$y'=c_1e^x-2c_2e^{-2x}$$

$$\Rightarrow \quad y(0)=c_1+c_2=4, \quad y'(0)=c_1-2c_2=-5 \quad \Rightarrow \quad c_1=1, \quad c_2=3$$

$$\Rightarrow \quad \therefore\ y=e^x+3e^{-2x}$$



☑ **Ex. 4 Solver the initial value problem** $y''+y'+0.25y=0, \quad y(0)=3.0, \quad y'(0)=-3.5$

Step 1 General solution

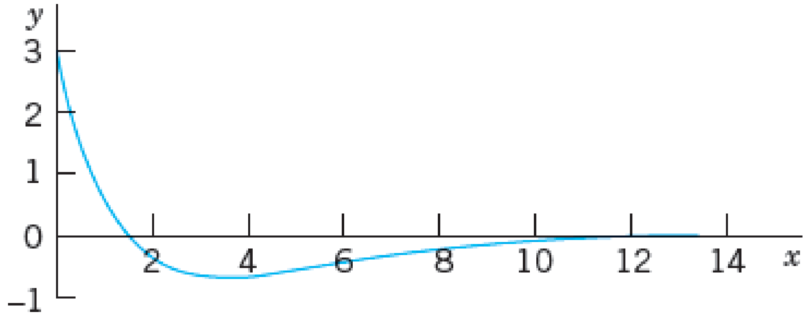
$$\lambda^2+\lambda+0.25=0 \text{ (Characteristic equation)} \Rightarrow \lambda=-0.5 \Rightarrow \therefore y=(c_1+c_2x)e^{-0.5x}$$

Step 2 Particular solution

$$y'=c_2e^{-0.5x}-0.5(c_1+c_2x)e^{-0.5x}$$

$$\Rightarrow y(0)=c_1=3.0, \quad y'(0)=c_2-0.5c_1=-3.5 \Rightarrow c_1=3, \quad c_2=-2$$

$$\Rightarrow \therefore y=(3-2x)e^{-0.5x}$$



2.2 Homogeneous Linear ODEs with Constant Coefficients

☑ **Ex. 5 Solve the initial value problem** $y''+0.4y'+9.04y=0,\quad y(0)=0,\quad y'(0)=3$

Step 1 General solution

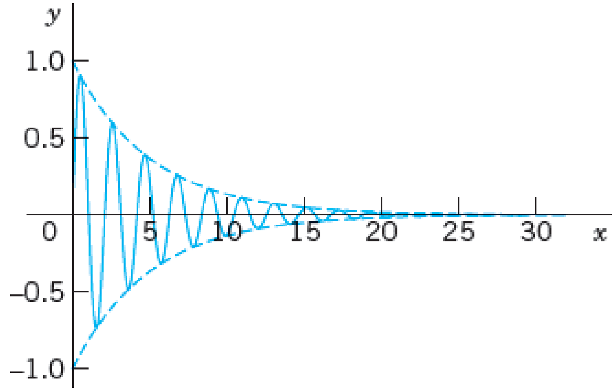
$$\lambda^2+0.4\lambda+9.04=0 \quad \textbf{(Characteristic equation)} \;\Rightarrow\; \lambda=-0.2\pm 3i \;\Rightarrow\; \therefore\; y=e^{-0.2x}\left(A\cos 3x+B\sin 3x\right)$$

Step 2 Particular solution

$$y'=-0.2e^{-0.2x}\left(A\cos 3x+B\sin 3x\right)+e^{-0.2x}\left(-3A\sin 3x+3B\cos 3x\right)$$

$$\Rightarrow\quad y(0)=A=0,\quad y'(0)=-0.2A+3B=3 \quad\Rightarrow\quad A=0,\quad B=1$$

$$\Rightarrow\quad \therefore\; y=e^{-0.2x}\sin 3x$$



Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega,$ $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos \omega x$ $e^{-ax/2}\sin \omega x$	$y = e^{-ax/2}(A\cos \omega x + B\sin \omega x)$

2.2 Homogeneous Linear ODEs with Constant Coefficients

Q: Solve the following initial value problem.

☑ **Ex.** $y''+4y'+(\pi^2+4)y=0, \quad y\left(1/2\right)=1, \quad y'\left(1/2\right)=-2$

2.3 Differential Operators

- Operator **(연산자)**: A transformation that transforms a function into another function.
- Operational Calculus **(연산자법)**: The technique and application of operators.
- Differential Operator **(미분 연산자)** D

  : An operator which transforms a (differentiable) function into its derivative.

$$Dy = y' = \frac{dy}{dx}$$

- Identity Operator **(항등 연산자)**: $Iy = y$
- Second-order differential operator **(2계 미분 연산자)**

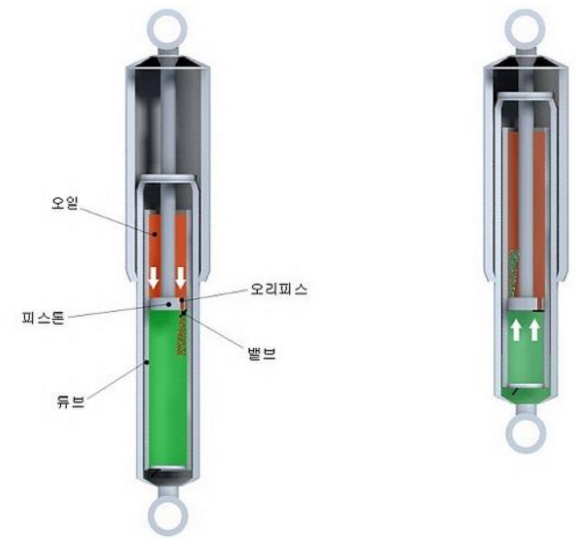
$$L = P(D) = D^2 + aD + bI \quad \Rightarrow \quad Ly = P(D)y = y'' + ay' + by$$

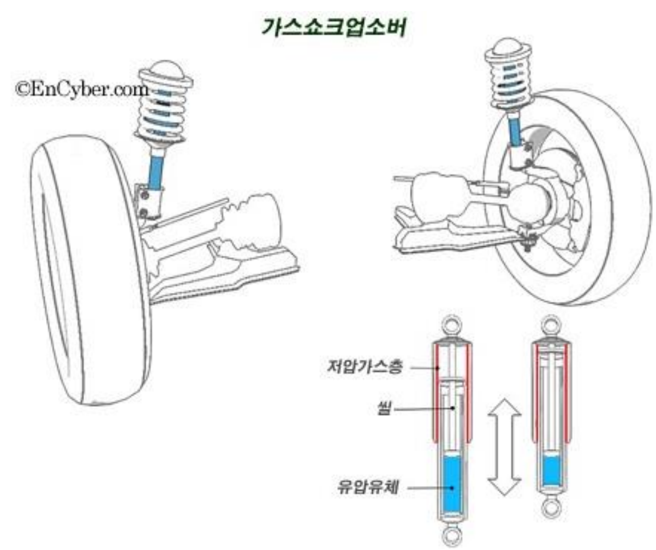
* P(D): Operator polynomial (연산자 다항식)
* L: Linear operator (2계 미분 연산자, 선형 연산자)

2.4 Modeling of Free Oscillations of Mass-Spring System

❖ Shock Absorber (흔히 “쇼바”)

- 자동차 서스펜션을 구성하는 주요 요소
- 원리: 스프링의 수축을 조절해, 노면 차이로 인해 충격을 받은 스프링이 위아래로 반복해서 되튐 운동을 하는 것을 막아 줌
- 스프링이 원상태로 천천히 돌아갈 수 있도록 하는 것
- 스프링의 신축 작용 즉, 차체가 위 아래로 흔들리거나 진동하는 것을 약화시켜줌
- 스프링의 진동을 억제하는 힘을 '감쇠력 (damping force)‘
- '딱딱한 shock absorber‘ vs. '부드러운 shock absorber‘ ?
- 오일과 가스 shock absorber



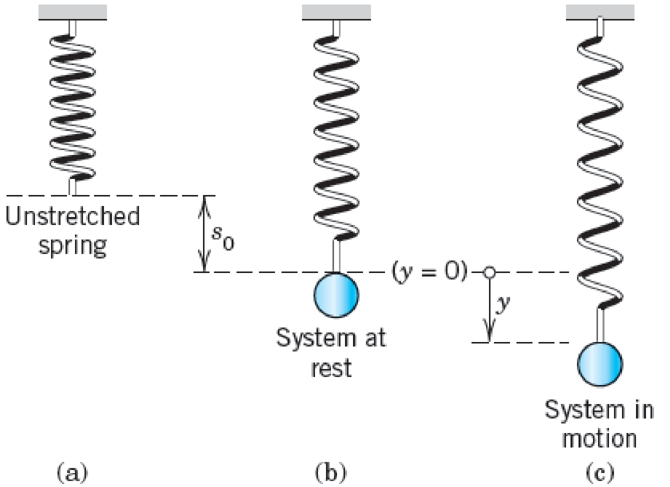


[네이버 지식백과] 쇼크업소버 [shock absorber] (두산백과, 두산백과)

2.4 Modeling of Free Oscillations of Mass-Spring System

- We consider a basic mechanical system, a mass on an elastic spring, which moves up and down.

Setting Up the Model



< Mechanical mass-spring system >

Physical Information

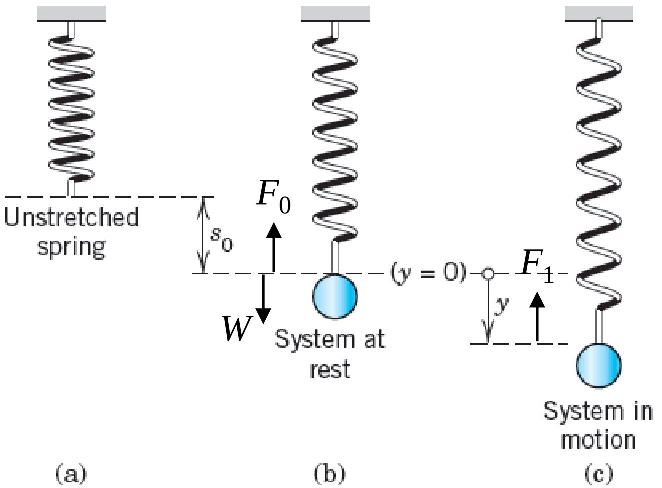
- Newton's second law: Mass x Acceleration = Force
- Hook's law

  : The restoring force is directly, inversely proportional to the distance.
- We choose the downward direction as the positive direction.

2.4 Modeling of Free Oscillations of Mass-Spring System

- We consider a basic mechanical system, a mass on an elastic spring, which moves up and down.

❖ Setting Up the Model



< Mechanical mass-spring system >

❖ Modeling

- System in static equilibrium

$$F_0 = -ks_0 \quad (k: \text{Spring constant})$$

Weight of body $W = mg$

$$F_0 + W = -ks_0 + mg = 0$$

- System in motion

Restoring force $F_1 = -ky$ (Hook's law)

$$my'' = F_1 \quad \text{(Newton's second law)}$$

$$my'' + ky = 0$$

(At this time, F_0 and W cancel each other.)

2.4 Modeling of Free Oscillations of Mass-Spring System

❖ Undamped System: ODE and Solution

- ODE: $my'' + ky = 0 \qquad \Rightarrow \lambda^2 + \dfrac{k}{m} = 0$

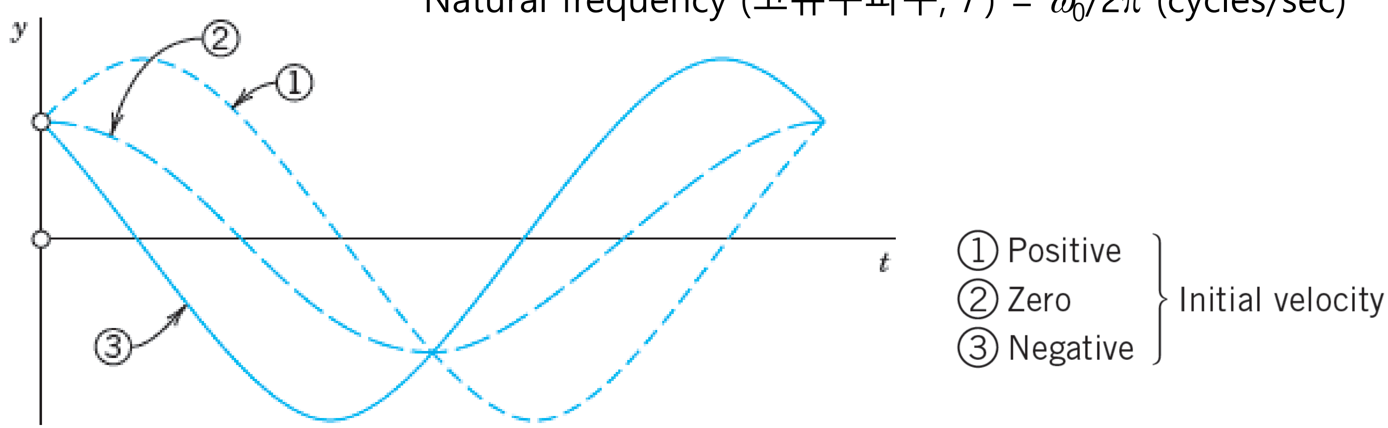
- Harmonic oscillation (**조화진동**):

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \delta), \quad \omega_0{}^2 = \frac{k}{m}$$

where, $C = \sqrt{A^2 + B^2}, \ \tan\delta = B/A$

$$A\cos x + B\sin x = \sqrt{A^2 + B^2}\cos(x - \delta)$$

Period (주기, T) = $2\pi/\omega_0$ (sec)
Natural frequency (고유주파수, f) = $\omega_0/2\pi$ (cycles/sec)



< Harmonic oscillation >

2.4 Modeling of Free Oscillations of Mass-Spring System

❖ Damped System: ODE and Solutions

- Damping force (감쇄력): inversely proportional to the velocity

$$F_2 = -cy' \ (c : \text{damping constant})$$

$$F_1 = -ky \ (k : \text{spring constant})$$

$$my'' = F_1 + F_2$$

$$\boxed{my'' + cy' + ky = 0}$$

where, $c, k > 0$

- Characteristic equation is

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \Rightarrow \lambda_1 = -\alpha + \beta,\ \lambda_2 = -\alpha - \beta, \ \text{where} \ \alpha = \frac{c}{2m} \ \text{and} \ \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$

- Three types of motion

Case 1 **(Overdamping)** $c^2 > 4mk$ Distinct real roots λ_1 , λ_2

Case 2 **(Critical damping)** $c^2 = 4mk$ A real double root

Case 3 **(Underdamping)** $c^2 < 4mk$ Complex conjugate roots

2.4 Modeling of Free Oscillations of Mass-Spring System

$$\boxed{\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0}$$

$$\boxed{\lambda_1 = -\alpha + \beta,\ \lambda_2 = -\alpha - \beta,\quad \alpha = \frac{c}{2m},\ \ \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}}$$

- Discussion of the Three Cases

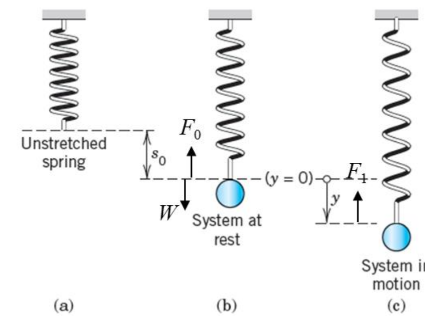
Case 1 Overdamping $\left(c^2 > 4mk\right)$

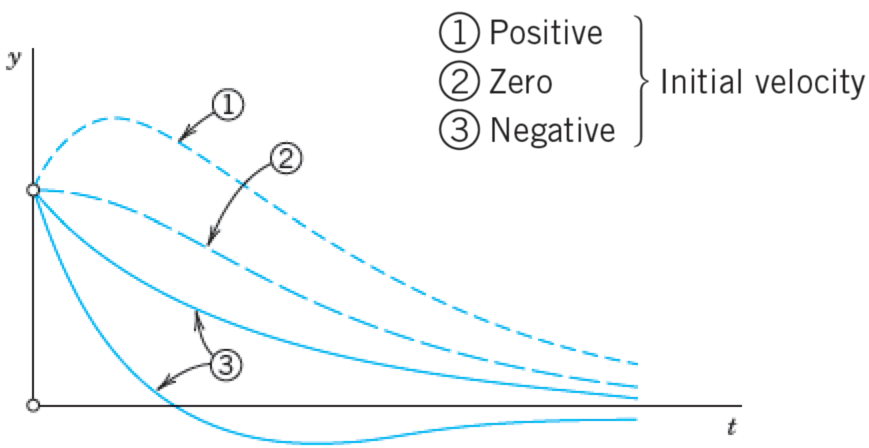
$$: \ y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t} \quad \text{where} \quad \alpha = \frac{c}{2m}, \quad \beta = \frac{\sqrt{c^2 - 4mk}}{2m}$$

$$\beta^2 = \left(\frac{1}{2m}\right)^2 (c^2 - 4mk) = \alpha^2 - \frac{k}{m} < \alpha^2 \quad \Rightarrow \alpha - \beta > 0, \quad \alpha + \beta > 0 \ \Rightarrow \ y(t) \to 0$$

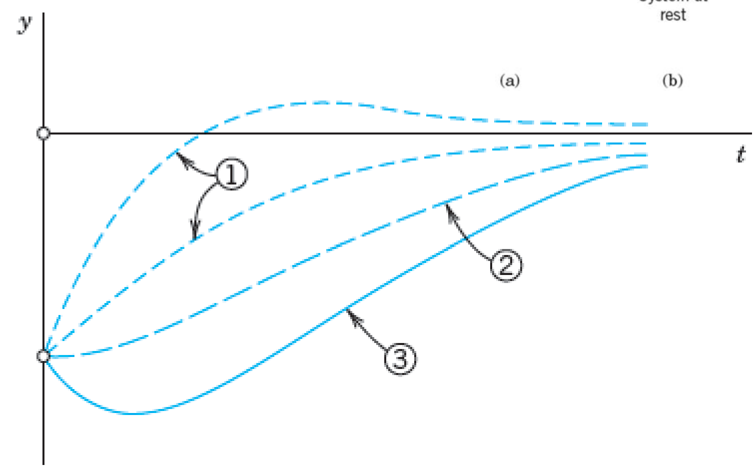
Damping takes out energy so quickly

that the body does not oscillate.





< Positive initial displacement (tension) >



< Negative initial displacement (compression) >

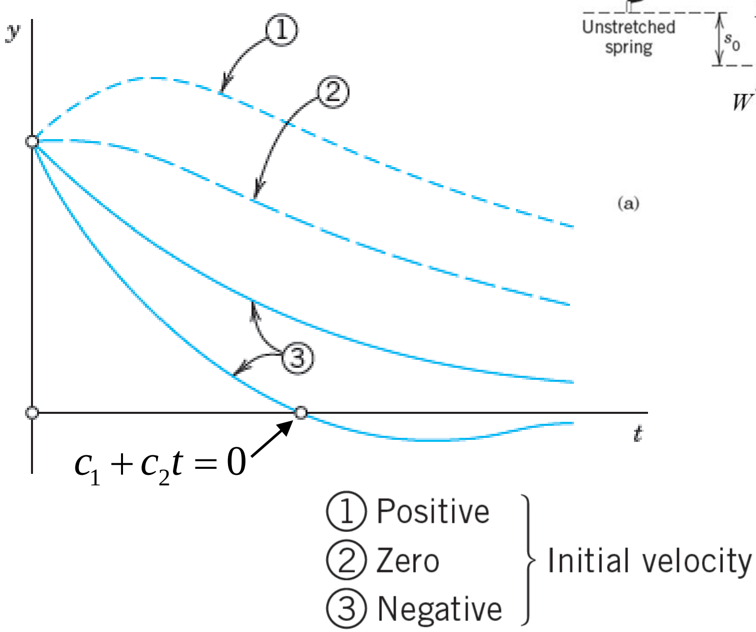
2.4 Modeling of Free Oscillations of Mass-Spring System

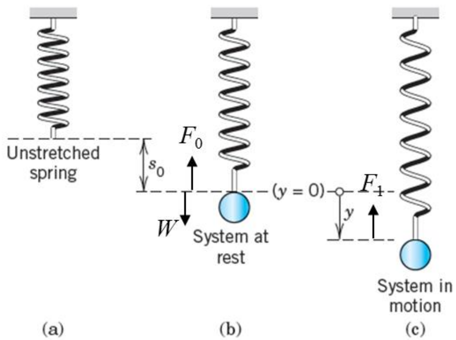
$$\lambda_1 = -\alpha + \beta,\ \lambda_2 = -\alpha - \beta,$$
$$\alpha = \frac{c}{2m},\quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$

Case 2 Critical damping $\left(c^2 = 4mk\right)$ $\quad \beta = 0$

$$: \ y(t) = (c_1 + c_2 t)\underbrace{e^{-\alpha t}}_{>0}, \qquad \alpha = \frac{c}{2m}$$

Damping takes out energy so quickly that the body does not oscillate.





$$y(0) = c_1 > 0$$

$$y'(t) = c_2 - \alpha(c_1 + c_2 t)e^{-\alpha t} \ \Rightarrow y'(0) = c_2 - \alpha c_1$$

Case ① Positive initial velocity

$$y'(0) > 0 \ \ c_2 - \alpha c_1 > 0,\ \ c_2 > \alpha c_1 > 0 \ \ \Rightarrow y(t) \neq 0$$

Case ② Zero initial velocity

$$y'(0) = 0 \ \ c_2 - \alpha c_1 = 0,\ \ c_2 = \alpha c_1 > 0 \ \ \Rightarrow y(t) \neq 0$$

Case ③ Negative initial velocity

$$y'(0) < 0 \ \ c_2 - \alpha c_1 < 0,\ \ c_2 < \alpha c_1,$$

$$\underline{c_2 < 0} \text{ or } c_2 > 0 \ \ \Rightarrow y(t) = 0 \text{ or } y(t) \neq 0$$

$$c_1 + c_2 t = 0$$

$$\boxed{\lambda_1 = -\alpha + \beta,\ \lambda_2 = -\alpha - \beta, \quad \alpha = \frac{c}{2m},\ \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}}$$

Case 3 Underdamping $\left(c^2 < 4mk\right)$

$$\beta = i\omega^* \text{ where } \omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \quad (>0)$$

$$\lambda_1 = -\alpha + i\omega^*, \quad \lambda_2 = -\alpha - i\omega^*,$$

$$: \ y(t) = e^{-\alpha t}\left(A\cos\omega^* t + B\sin\omega^* t\right) = Ce^{-\alpha t}\cos\left(\omega^* t - \delta\right)$$

where, $C = \sqrt{A^2 + B^2}, \ \tan\delta = B/A$

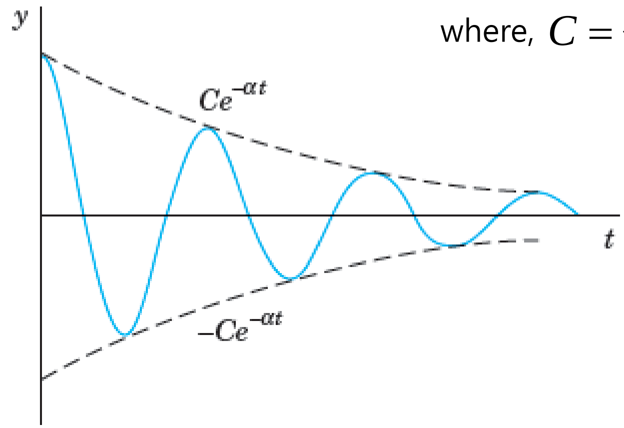


Fig. 39. Damped oscillation in Case III [see (10)]

2.5 Euler-Cauchy Equations

❖ **Euler-Cauchy Equations**: $x^2 y'' + axy' + by = 0$

$$y = x^m,\ \ y' = mx^{m-1},\ \ y'' = m(m-1)x^{m-2}$$

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0 \quad \Rightarrow m(m-1)x^m + amx^m + bx^m = 0$$

❖ **Auxiliary Equation (보조 방정식)**: $m^2 + \left(a-1\right)m + b = 0$

$$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b},\ m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}$$

❖ **Three kinds of the general solution of the equation**

- **Case 1 Two real roots** $m_1,\ m_2 \quad \Rightarrow \quad y = c_1 x^{m_1} + c_2 x^{m_2}$

2.5 Euler-Cauchy Equations

- Euler-Cauchy Equations : $x^2 y'' + axy' + by = 0$

$$m_1 = \frac{1}{2}(1-a) + \sqrt{\frac{1}{4}(1-a)^2 - b}, \quad m_2 = \frac{1}{2}(1-a) - \sqrt{\frac{1}{4}(1-a)^2 - b}$$

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

$$\boxed{U = \frac{1}{y_1^2} e^{-\int p dx}}$$

$$\boxed{y_2 = uy_1 = y_1 \int U dx}$$

- **Case 2 A real double root**

$$m_1 = \frac{1}{2}(1-a) \quad \text{only if} \quad b = \frac{1}{4}(1-a)^2 \quad \Rightarrow y_1 = x^{(1-a)/2}$$

$$\boxed{x^2 y'' + axy' + by = 0} \Rightarrow \quad x^2 y'' + axy' + \frac{1}{4}(1-a)^2 y = 0 \quad \text{or} \quad \boxed{y'' + \frac{a}{x} y' + \frac{(1-a)^2}{4x^2} y = 0}$$

Method of reduction of order, $y_2 = uy_1$

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

$$u = \int U dx \quad \text{where} \quad U = \frac{1}{y^2} \exp\left(-\int p dx\right)$$

$$p = \frac{a}{x} \Rightarrow U = \frac{1}{y_1^2} \exp\left(-\int \frac{a}{x} dx\right) = \frac{1}{y_1^2} \exp(-a \ln x) = \frac{1}{y_1^2} \exp\left(\ln x^{-a}\right) = \frac{x^{-a}}{x^{(1-a)}} = \frac{1}{x}$$

$$u = \int U dx = \int \frac{1}{x} dx = \ln x$$

$$y_2 = uy_1 = x^{(1-a)/2} \ln x \quad \Rightarrow \quad y = (c_1 + c_2 \ln x) x^m, \quad m = \frac{1}{2}(1-a)$$

2.5 Euler-Cauchy Equations

❖ Euler-Cauchy Equations : $x^2y''+axy'+by=0$

$$m_1=\frac{1}{2}(1-a)+\sqrt{\frac{1}{4}(1-a)^2-b}\,,\quad m_2=\frac{1}{2}(1-a)-\sqrt{\frac{1}{4}(1-a)^2-b}$$

- **Case 3 Complex conjugate roots**

$$m_1=\mu+i\nu, m_2=\mu-i\nu,\quad \text{where}\quad \mu=\frac{1}{2}(1-a),\ \nu=\sqrt{b-\frac{1}{4}(1-a)^2}$$

trick of writing $x=e^{\ln x}$

❖ Euler formula: $e^{it}=\cos t+i\sin t$

$$y_1=x^{m_1}=x^{\mu+i\nu}=x^{\mu}(e^{\ln x})^{i\nu}=x^{\mu}e^{(\nu\ln x)i}=x^{\mu}(\cos(\nu\ln x)+i\sin(\nu\ln x))$$

$$y_2=x^{m_2}=x^{\mu-i\nu}=x^{\mu}(e^{\ln x})^{-i\nu}=x^{\mu}e^{-(\nu\ln x)i}=x^{\mu}(\cos(\nu\ln x)-i\sin(\nu\ln x))$$

$$(y_1+y_2)/2=x^{\mu}\cos(\nu\ln x)$$

$$(y_1-y_2)/2=x^{\mu}\sin(\nu\ln x)$$

⇨ These are also solutions of Euler-Cauchy equation and linearly independent.

$$\boxed{m=\mu\ \pm i\upsilon \quad\Rightarrow\quad y=x^{\mu}\left[A\cos\left(\nu\ln x\right)+B\sin\left(\nu\ln x\right)\right]}$$

2.5 Euler-Cauchy Equations

Q : Solve the followings.

☑ **Ex. 1** Solver the Euler-Cauchy equation $x^2 y'' + 1.5xy' - 0.5y = 0$

☑ **Ex. 2** Solver the Euler-Cauchy equation $x^2 y'' - 5xy' + 9y = 0$

☑ **Ex. 3** Solver the Euler-Cauchy equation $x^2 y'' + 0.6xy' + 16.04y = 0$

2.6 Existence and Uniqueness of Solutions. Wronskian

❖ **Theorem 1 Existence and Uniqueness Theorem for Initial Value Problem**

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is in I,

then the initial value problem consisting of $y''+p(x)y'+q(x)y=0$ and $y(0)=K_0$, $y'(0)=K_1$

has a unique solution $y(x)$ on the interval I.

2.6 Existence and Uniqueness of Solutions. Wronskian

❖ Linear Independence of Solutions

$y_1,\ y_2$ are linearly independent on I if equations

$$k_1 y_1(x) + k_2 y_2(x) = 0 \quad \text{on } I \quad \text{implies} \quad k_1 = 0,\ k_2 = 0$$

$y_1,\ y_2$ are linearly dependent on I if equations

$$y_1 = k y_2 \quad \text{or} \quad y_2 = k y_1$$

❖ Theorem 2 Linear Dependence and Independence of Solutions

Let the ODE $y''+p(x)y'+q(x)y=0$ have continuous coefficients $p(x)$ and $q(x)$ on an open interval I. **(a)** Then two solutions y_1, y_2 of the equation on I are linearly dependent on I if and only if their "**Wronskian**"

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

is 0 at some x_0 in I.

PROOF) (a) If y_1, y_2 be linear dependent on I (**$y_1 = ky_2$**) ➡ $W=0$ at an x_0 on I

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = k y_2 y_2' - y_2 k y_2' = 0$$

2.6 Existence and Uniqueness of Solutions. Wronskian

❖ **Theorem 2 Linear Dependence and Independence of Solutions**

Let the ODE $y''+p(x)y'+q(x)y=0$ have continuous coefficients $p(x)$ and $q(x)$ on an open interval I. **(a)** Then two solutions y_1, y_2 of the equation on I are linearly dependent on I if and only if their "**Wronskian**"

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

is 0 at some x_0 in I. **(b)** Furthermore, if $W = 0$ at an $x=x_0$ in I, then $W \equiv 0$ on I;

PROOF) Inverse of (a) if $W = 0$ at an $x=x_0$ in I ➡ y_1, y_2 linearly dependent

$$Let\ k_1 y_1(x) + k_2 y_2(x) = 0 \text{ for unknown } k_1, k_2.$$

$$\Rightarrow k_1 y_1'(x) + k_2 y_2'(x) = 0$$

$x=x_0$ ⇨

$$k_1 y_1(x_0) + k_2 y_2(x_0) = 0$$

$$k_1 y_1'(x_0) + k_2 y_2'(x_0) = 0$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⇨ if $W(y_1, y_2) = \det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} = \underline{y_1 y_2' - y_2 y_1'} = 0$

⇨ *Non zero solutions exist.* k_1, k_2 *are not both 0.*

$$k_1 = y_2', k_2 = -y_1' \text{ or } k_1 = -y_2, k_2 = y_1$$

2.6 Existence and Uniqueness of Solutions. Wronskian

$$k_1 y_1(x_0) + k_2 y_2(x_0) = 0$$
$$k_1 y_1'(x_0) + k_2 y_2'(x_0) = 0$$

⇨ Same formula ⇨ $y_1(x_0) = -\frac{k_2}{k_1} y_2(x_0)$

PROOF) (b) if $W = 0$ at an $x=x_0$ in I ➡ $W \equiv 0$ on I

$y = k_1 y_1(x) + k_2 y_2(x)$ is also solution of $\boxed{y'' + p(x)y' + q(x)y = 0}$

$$y(x_0) = k_1 y_1(x_0) + k_2 y_2(x_0) = 0$$
$$y'(x_0) = k_1 y_1'(x_0) + k_2 y_2'(x_0) = 0$$

“앞 페이지에서 두 식의 정의로 부터”

⇨ $\boxed{\begin{aligned} y(x_0) &= 0 \\ y'(x_0) &= 0 \end{aligned}}$ Initial conditions

Another solution satisfying the same initial condition is $y^* \equiv 0$ (constant 0).

Theorem 1 Uniqueness theorem ➡ $y \equiv y^*$.

$$k_1 y_1 + k_2 y_2 \equiv 0 \quad on \ I$$

Ex) $y_1 = \sin\omega x$, $y_2 = 2\sin\omega x$
$y_1 = x$, $y_2 = 3x$

Now k_1, k_2 are not both zero ➡ linear dependence of y_1, y_2.

2.6 Existence and Uniqueness of Solutions. Wronskian

❖ **Theorem 2 Linear Dependence and Independence of Solutions**

Let the ODE $y''+p(x)y'+q(x)y=0$ have continuous coefficients $p(x)$ and $q(x)$ on an open interval I. **(a)**Then two solutions y_1, y_2 of the equation on I are linearly dependent on I if and only if their "**Wronskian**"

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

is 0 at some x_0 in I. Furthermore, **(b)** if $W = 0$ at an $x=x_0$ in I, then $W \equiv 0$ on I; hence **(C)** if there is an x_1 in I at which W is not 0, then y_1, y_2 are linearly independent on I.

PROOF)(c)

From (b) y_1, y_2 linearly dependent ➡ $W = 0$ at an $x=x_0$ in I ➡ $W \equiv 0$ on I

NOT($W \equiv 0$ on I) ➡ $W(x_1) \neq 0$ at an x_1 on I ➡ y_1, y_2 linearly independent

2.6 Existence and Uniqueness of Solutions. Wronskian

Q: Show linear independence using the Wronskian.

☑ Ex. 1 $e^{-x}\cos \omega x,\ e^{-x}\sin \omega x$

☑ Ex. 2 $e^{-4x}\ e^{-1.5x}$

2.6 Existence and Uniqueness of Solutions. Wronskian

❖ Theorem 3 Existence of a General Solution

If $p(x)$ and $q(x)$ are continuous on an open interval I, then $y''+p(x)y'+q(x)y=0$ has a general solution on I.

❖ Theorem 4 A General Solution Includes All Solutions

If the ODE $y''+p(x)y'+q(x)y=0$ has continuous coefficients $p(x)$ and $q(x)$ on some open interval I, then every solution $y = Y(x)$ of the equation on I is of the form

$$Y(x) = C_1 y_1(x) + C_2 y_2(x)$$

where y_1, y_2 is any basis of solutions of the equation on I and C_1, C_2 are suitable constants.

Hence the equation does not have **singular solutions** (that is, solutions not obtainable from a general solution).

2.6 Existence and Uniqueness of Solutions. Wronskian

Proof) Let $y=Y(x)$ be any solution of $y''+p(x)y+q(x)y=0$ on *I*.
if we prove $Y(x) = C_1y_1(x) + C_2y_2(x)$ ➡ **no singular solutions**

" we know $Y(x)$ is a general solution of $y''+p(x)y+q(x)y=0$, but we don't know whether there is any other solution or not"

The ODE has a general solution

$y(x)= c_1y_1(x) + c_2y_2(x)$ on *I*.

We have to find suitable values of c_1, c_2 such that $y(x) = Y(x)$ on *I*.
For any x_0

$$\begin{aligned} c_1y_1(x_0)+c_2y_2(x_0)&=Y(x_0) \times y_2'(x_0) \\ c_1y_1'(x_0)+c_2y_2'(x_0)&=Y'(x_0) \times -y_2(x_0) \end{aligned} \Rightarrow \begin{aligned} c_1y_1y_2'+c_2y_2y_2'&=Yy_2' \\ -c_1y_2y_1'-c_2y_2y_2'&=-Y'y_2 \end{aligned}$$

$$\Rightarrow c_1y_1y_2'-c_1y_2y_1'=c_1W(y_1,y_2)=Yy_2'-y_2Y' \Rightarrow c_1=\frac{Yy_2'-y_2Y'}{W(y_1,y_2)}=C_1$$

Similarly $c_2y_1y_2'-c_2y_2y_1'=c_2W(y_1,y_2)=y_1Y'-Yy_1' \Rightarrow c_2=\frac{y_1Y'-Yy_1'}{W(y_1,y_2)}=C_2$

2.6 Existence and Uniqueness of Solutions. Wronskian

Particular solution

$y^*(x) = C_1 y_1(x) + C_2 y_2(x)$

Therefore $y^*(x_0) = Y(x_0)$ and $y'^*(x_0) = Y'(x_0)$
That is,

$$y^*(x_0) = C_1 y_1(x_0) + C_2 y_2(x_0) = Y(x_0)$$
$$y'^*(x_0) = C_1 y_1'(x_0) + C_2 y_2'(x_0) = Y'(x_0)$$

From the uniqueness in Theorem 1, $y^* \equiv Y$ *must be equal everywhere on* I.

2.7 Nonhomogeneous ODEs

❖ **Nonhomogeneous linear ODEs:** $y'' + p(x)y' + q(x)y = r(x), \quad r(x) \neq 0$

❖ **Definition General Solution, Particular Solution**

A general solution of the nonhomogeneous ODE $y''+p(x)y'+q(x)y=r(x)$ on an open interval I is a solution of the form

$$y(x) = y_h(x) + y_p(x)$$

here, $\boldsymbol{y_h = c_1 y_1 + c_2 y_2}$ is a general solution of the homogeneous ODE $y''+p(x)y'+q(x)y=0$ on I and $\boldsymbol{y_p}$ is any solution of $y''+p(x)y'+q(x)y=r(x)$ on I containing no arbitrary constants.

A particular solution of $y''+p(x)y'+q(x)y=r(x)$ on I is a solution obtained from $y(x) = y_h(x) + y_p(x)$ by assigning specific values to the arbitrary constants c_1 and c_2 in y_h.

2.7 Nonhomogeneous ODEs

❖ **Theorem 1**

Relations of Solution of $y''+p(x)y'+q(x)y=r(x)$ **to those of** $y''+p(x)y'+q(x)y=0$

- y : a solution of $y''+p(x)y'+q(x)y=r(x)$ on some open interval I
- $\tilde{y}$: a solution of $y''+p(x)y'+q(x)y=0$

(a) $y+\tilde{y}$: a solution of $y''+p(x)y'+q(x)y=r(x)$ on I.

In particular, $y(x) = y_h(x) + y_p(x)$ is a solution of $y''+p(x)y'+q(x)y=r(x)$ on I.

PROOF) (a) Let $L[y]$ denotes the left side of $y''+p(x)y'+q(x)y=r(x)$

$$L[y+\tilde{y}] = L[y] + L[\tilde{y}] = r(x) + 0 = r(x)$$

2.7 Nonhomogeneous ODEs

❖ **Theorem 1**

Relations of Solution of $y''+p(x)y'+q(x)y=r(x)$ **to those of** $y''+p(x)y'+q(x)y=0$

- y, y^* : two solutions of $y''+p(x)y'+q(x)y=r(x)$ on some open interval I

(b) the difference of two solutions ($y-y^*$) of $y''+p(x)y'+q(x)y=r(x)$ on I

⇨ a solution of $y''+p(x)y'+q(x)y=0$ on I.

PROOF) (b) $L[y-y^*]=L[y]-L[y^*]=r-r=0$

2.7 Nonhomogeneous ODEs

❖ **Theorem 2 A General Solution of a Nonhomogeneous ODE Includes All Solutions**

If the coefficients $p(x)$, $q(x)$, and the function $r(x)$ in $y''+p(x)y'+q(x)y=r(x)$ are continuous on some open interval I,

then every solution of $y''+p(x)y'+q(x)y=r(x)$ on I is obtained by assigning suitable values to the arbitrary constants c_1 and c_2 in a general solution $y(x) = y_h(x) + y_p(x)$ of $y''+p(x)y'+q(x)y=r(x)$ on I.

2.7 Nonhomogeneous ODEs

PROOF) Let y^* any solution of $y''+p(x)y'+q(x)y=r(x)$ on I

y_p is particular solution of $y''+p(x)y'+q(x)y=r(x)$

$Y=y^*-y_p$: a solution of $y''+p(x)y'+q(x)y=0$ ⇦ *Theorem 1(b)*

At x_0 *we have* $Y(x_0)=y^*(x_0)-y_p(x_0)$, $Y'(x_0)=y^{*\prime}(x_0)-y'_p(x_0)$

Theorem 4 in Sec. 2.6 ⇨ There exists a unique particular solution (Y) of $y''+p(x)y'+q(x)y=0$ obtained by assigning suitable values to c_1 and c_2 in $\boldsymbol{y_h=c_1y_1+c_2y_2}$.

⇨ From this and $y^* = Y\,(=y_h)+y_p$, the statement follows.

(y_h: general solution of $y''+p(x)y'+q(x)y=0$)

2.7 Nonhomogeneous ODEs

❖ Method of Undetermined Coefficients (미정계수법)

❖ Choice Rules for the Method of Undetermined Coefficients

a. Basic Rule (기본규칙). If $r(x)$ in $y''+p(x)y'+q(x)y=r(x)$ is one of the functions in the first column in Table 2.1, choose $\boldsymbol{y_p}$ in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into $y''+ay'+by=r(x)$.

b. Modification Rule (변형규칙). If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to $y''+ay'+by=r(x)$, multiply your choice of by x (or by x^2 if this solution corresponding to a double root of the characteristic equation of the homogeneous ODE). y_p

c. Sum Rule (합규칙). If $r(x)$ is a sum of functions in the first column of Table 2.1, choose for y_p the sum of the functions in the corresponding lines of the second column.

2.7 Nonhomogeneous ODEs

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$\left.\right\} K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$\left.\right\} e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

2.7 Nonhomogeneous ODEs

☑ **Ex. 1 Solve the initial value problem** $y''+y=0.001x^2,\quad y(0)=0,\quad y'(0)=1.5$

Step 1 General solution of the homogeneous ODE. $y_h = A\cos x + B\sin x$

Step 2 Solution y_p of the nonhomogeneous ODE. **("Basic Rule" 이용)**

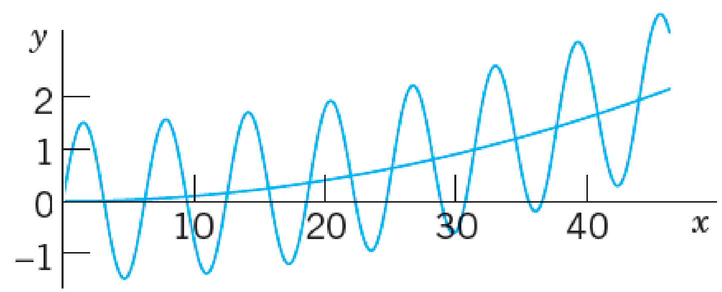
$$r(x)=0.001x^2 \quad\Rightarrow\quad y_p = K_2x^2+K_1x+K_0 \quad\Rightarrow\quad K_2=0.001,\quad K_1=0,\quad K_0=-0.002$$

$$\Rightarrow\quad y_p=0.001x^2-0.002 \quad\Rightarrow\quad \therefore\ y=A\cos x+B\sin x+0.001x^2-0.002$$

Step 3 Solution of the initial value problem.

$$y(0)=A-0.002=0,\quad y'(0)=B=1.5 \quad\Rightarrow\quad \therefore y=0.002\cos x+1.5\sin x+0.001x^2-0.002$$

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n\ (n=0,1,\cdots)$	$K_nx^n+K_{n-1}x^{n-1}+\cdots+K_1x+K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x+M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x+M\sin\omega x)$



2.7 Nonhomogeneous ODEs

☑ **Ex. 2 Solve the initial value problem** $y'' + 3y' + 2.25y = -10e^{-1.5x}, \quad y(0) = 1, \quad y'(0) = 0$

Step 1 General solution of the homogeneous ODE. $y_h = (c_1 + c_2 x)e^{-1.5x}$

Step 2 Solution y_p of the nonhomogeneous ODE. **("Modification Rule" 이용)**

$$r(x) = -10e^{-1.5x} \quad \Rightarrow \quad y_p = Cx^2 e^{-1.5x} \quad \Rightarrow \quad C = -5$$

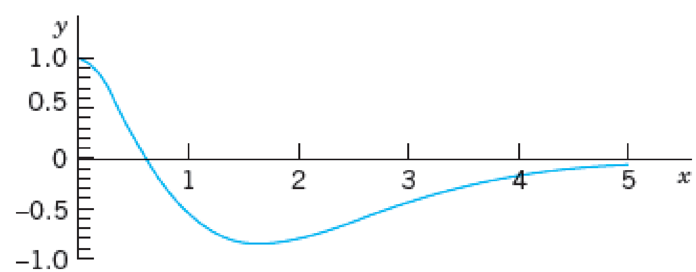
Solution of the homogeneous ODE and corresponding to a double root ➡ Multiply by x^2

$$\Rightarrow \quad y_p = -5x^2 e^{-1.5x} \quad \Rightarrow \quad \therefore\ y = (c_1 + c_2 x)e^{-1.5x} - 5x^2 e^{-1.5x}$$

Step 3 Solution of the initial value problem.

$$y(0) = c_1 = 1, \quad y'(0) = c_2 - 1.5c_1 = 0 \quad \Rightarrow \quad \therefore\ y = (1 + 1.5x)e^{-1.5x} - 5x^2 e^{-1.5x}$$

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	



2.7 Nonhomogeneous ODEs

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$

☑ **Ex. 3 Solve the initial value problem**

$$y'' + 2y' + 0.75y = 2\cos x - 0.25\sin x + 0.09x, \quad y(0) = 2.78, \quad y'(0) = -0.43$$

Step 1 General solution of the homogeneous ODE. $y_h = c_1 e^{-x/2} + c_2 e^{-3x/2}$

Step 2 Solution y_p of the nonhomogeneous ODE. **("Sum Rule" 이용)**

$$r_1(x) = 2\cos x - 0.25\sin x \quad \Rightarrow \quad y_{p1} = K\cos x + M\sin x \quad \Rightarrow \quad K = 0, \quad M = 1$$

$$r_2(x) = 0.09x \quad \Rightarrow \quad y_{p2} = K_1 x + K_0 \quad \Rightarrow \quad K_1 = 0.12, \quad K_0 = -0.32$$

$$\Rightarrow \quad y_{p1} = \sin x, \quad y_{p2} = 0.12x - 0.32$$

$$\Rightarrow \quad \therefore\ y = c_1 e^{-x/2} + c_2 e^{-3x/2} + \sin x + 0.12x - 0.32$$

Step 3 Solution of the initial value problem.

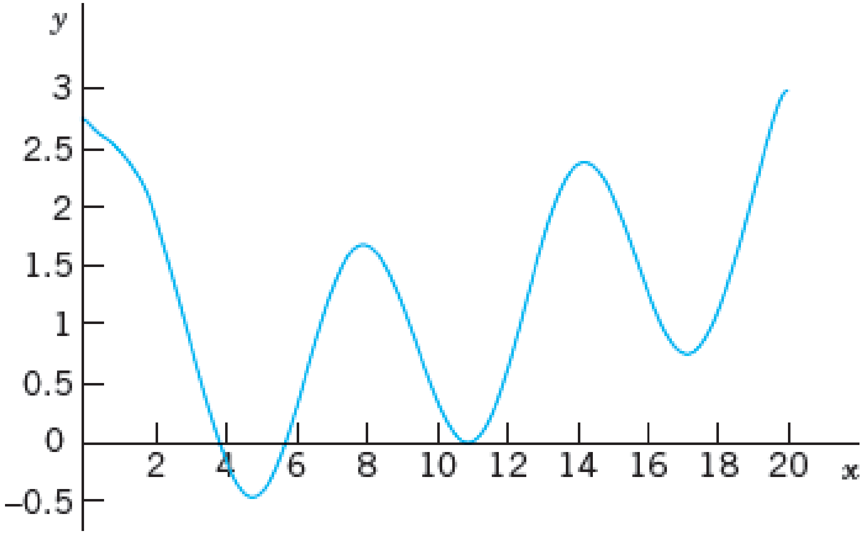
$$y(0) = c_1 + c_2 - 0.32 = 2.78, \quad y'(0) = -\frac{1}{2}c_1 - \frac{3}{2}c_2 + 1 + 0.12 = -0.4$$

$$\Rightarrow \quad c_1 = 3.1, \ \ c_2 = 0$$

$$\Rightarrow \quad \therefore\ y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32$$

2.7 Nonhomogeneous ODEs

$$y = 3.1e^{-x/2} + \sin x + 0.12x - 0.32$$



2.2 Homogeneous Linear ODEs with Constant Coefficients

Q: Solve the following problem.

☑ **Ex.** $y''+5y'+6y=2e^{-x}$

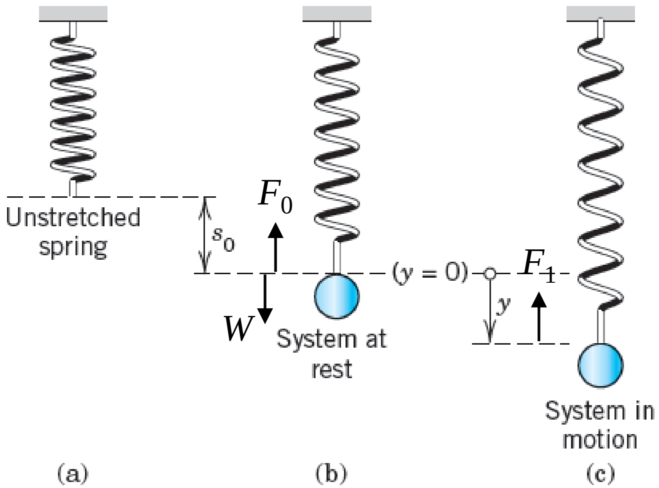
Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n=0,1,\cdots)$	$K_nx^n + K_{n-1}x^{n-1} + \cdots + K_1x + K_0$
$k\cos\omega x$	$K\cos\omega x + M\sin\omega x$
$k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{\alpha x}\cos\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$
$ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x + M\sin\omega x)$

(Review) 2.4 Modeling of Free Oscillations of Mass-Spring System

- We consider a basic mechanical system, a mass on an elastic spring, which moves up and down.

❖ **Setting Up the Model**



< Mechanical mass-spring system >

❖ Modeling

- System in static equilibrium

$$F_0 = -ks_0 \quad (k : \text{Spring constant})$$
$$\text{Weight of body} \quad W = mg$$
$$F_0 + W = -ks_0 + mg = 0$$

- System in motion

$$\text{Restoring force} \quad F_1 = -ky \quad \text{(Hook's law)}$$
$$my'' = F_1 \quad \text{(Newton's second law)}$$
$$my'' + ky = 0$$

(At this time, F_0 and W cancel each other.)

❖ Undamped System: ODE and Solution

- ODE: $my'' + ky = 0 \qquad \Rightarrow \lambda^2 + \dfrac{k}{m} = 0$

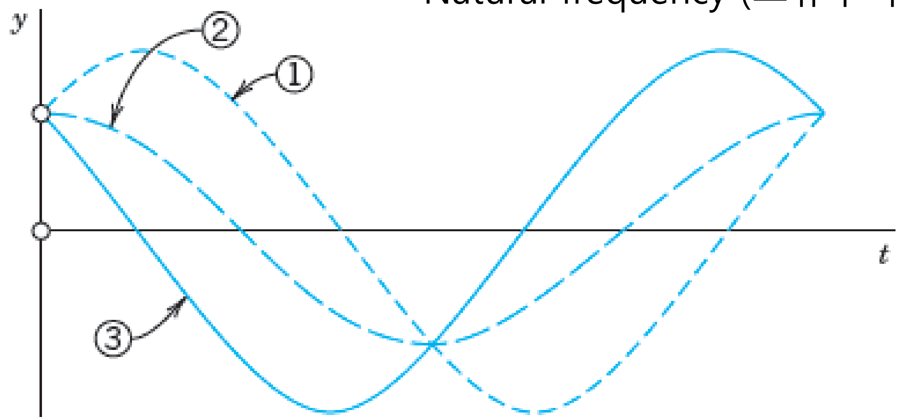
- Harmonic oscillation (**조화진동**):

$$y(t) = A\cos\omega_0 t + B\sin\omega_0 t = C\cos(\omega_0 t - \delta), \qquad \omega_0{}^2 = \frac{k}{m}$$

where, $C = \sqrt{A^2 + B^2}$, $\tan\delta = B/A$

$$A\cos x + B\sin x = \sqrt{A^2 + B^2}\cos(x - \delta)$$

Period (주기, T) = $2\pi/\omega_0$ (sec)
Natural frequency (고유주파수, f) = $\omega_0/2\pi$ (cycles/sec)



① Positive
② Zero
③ Negative

} Initial velocity

< Harmonic oscillation >

(Review) 2.4 Modeling of Free Oscillations of Mass-Spring System

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda_1 = -\alpha + \beta,\ \lambda_2 = -\alpha - \beta,$$
$$\alpha = \frac{c}{2m},\quad \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$

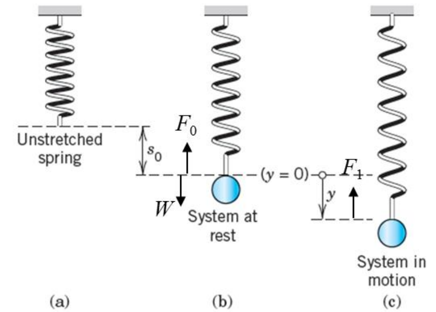
- **Overdamping** $\left(c^2 > 4mk\right)$

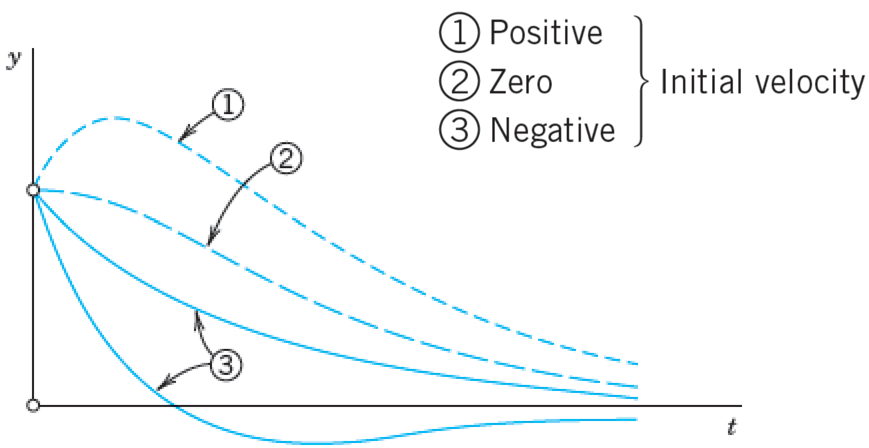
$$: \ y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t} \quad \text{where} \quad \alpha = \frac{c}{2m}, \quad \beta = \frac{\sqrt{c^2 - 4mk}}{2m}$$

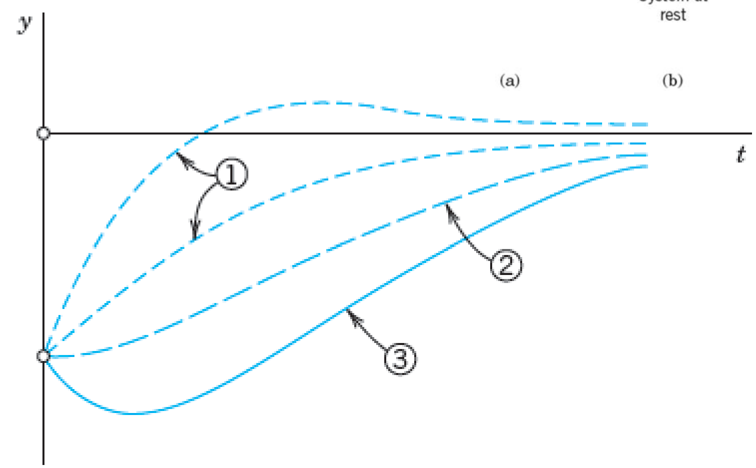
$$\beta^2 = \left(\frac{1}{2m}\right)^2 (c^2 - 4mk) = \alpha^2 - \frac{k}{m} < \alpha^2 \quad \Rightarrow \alpha - \beta > 0, \quad \alpha + \beta > 0 \quad \Rightarrow \quad y(t) \to 0$$

Damping takes out energy so quickly

that the body does not oscillate.







< Positive initial displacement (tension) > **< Negative initial displacement (compression) >**

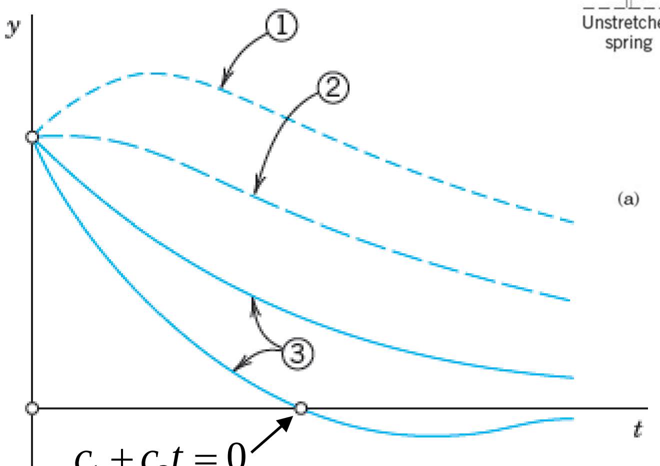
(Review) 2.4 Modeling of Free Oscillations of Mass-Spring System

$$\lambda_1 = -\alpha + \beta,\ \lambda_2 = -\alpha - \beta,$$
$$\alpha = \frac{c}{2m},\ \ \beta = \frac{1}{2m}\sqrt{c^2 - 4mk}$$

❖ **Critical damping** $\left(c^2 = 4mk\right)$ $\quad \beta = 0$

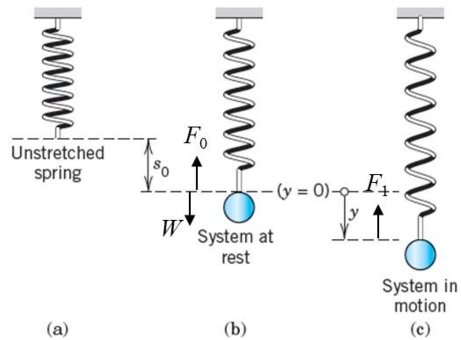
$$:\ y(t) = \left(c_1 + c_2 t\right)\underbrace{e^{-\alpha t}}_{>0}, \qquad \alpha = \frac{c}{2m}$$

Damping takes out energy so quickly that the body does not oscillate.



$c_1 + c_2 t = 0$

① Positive
② Zero
③ Negative
} Initial velocity



$$y(0) = c_1 > 0$$
$$y'(t) = c_2 - \alpha(c_1 + c_2 t)e^{-\alpha t} \quad \Rightarrow y'(0) = c_2 - \alpha c_1$$

Case ① Positive initial velocity

$$y'(0) > 0 \ \ c_2 - \alpha c_1 > 0,\ \ c_2 > \alpha c_1 > 0 \quad \Rightarrow y(t) \neq 0$$

Case ② Zero initial velocity

$$y'(0) = 0 \ \ c_2 - \alpha c_1 = 0,\ \ c_2 = \alpha c_1 > 0 \quad \Rightarrow y(t) \neq 0$$

Case ③ Negative initial velocity

$$y'(0) < 0 \ \ c_2 - \alpha c_1 < 0,\ \ c_2 < \alpha c_1,$$
$$\underline{c_2 < 0} \text{ or } c_2 > 0 \quad \Rightarrow y(t) = 0 \text{ or } y(t) \neq 0$$

$$c_1 + c_2 t = 0$$

2.8 Modeling: Forced Oscillations. Resonance

- **Free Motion (자유진동)**: Motions in the absence of external forces caused solely by internal forces. $my''+cy'+ky=0$

- **Forced Motion (강제진동): Model by including an external force**

$$my''+cy'+ky=r(t)$$

  - $r(t)$: Input or Driving Force
  - $y(t)$: Output or Response

Table 2.1 **Method of Undetermined Coefficients**

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n\ (n=0,1,\cdots)$	$K_nx^n+K_{n-1}x^{n-1}+\cdots+K_1x+K_0$
$k\cos\omega x$ $k\sin\omega x$	$K\cos\omega x+M\sin\omega x$
$ke^{\alpha x}\cos\omega x$ $ke^{\alpha x}\sin\omega x$	$e^{\alpha x}(K\cos\omega x+M\sin\omega x)$

- **Motion with periodic external forces**
  - Nonhomogeneous ODE: $my''+cy'+ky=F_0\cos\omega t$
  - Use the method of undetermined coefficients

$$y_p(t)=a\cos\omega t+b\sin\omega t$$

$$y_p'(t)=-\omega a\sin\omega t+\omega b\cos\omega t$$

$$y_p''(t)=-\omega^2 a\sin\omega t-\omega^2 b\cos\omega t$$

2.8 Modeling: Forced Oscillations. Resonance

$$\boxed{[(k - m\omega^2)a + \omega cb]}\cos\omega t + \underbrace{\boxed{[-\omega ca + (k - m\omega^2)b]}}_{=0}\sin\omega t = \boxed{F_0}\cos\omega t$$

$$(k - m\omega^2)a + \omega cb = F_0$$

$$-\omega ca \quad + (k - m\omega^2)b = 0$$

$$a = F_0 \frac{k - m\omega^2}{(k - m\omega^2)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{(k - m\omega^2)^2 + \omega^2 c^2}$$

$$\text{if we set } \sqrt{\mathrm{k/m}} = \omega_0 \,(> 0)$$

$$a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}$$

$$y_p(t) = a\cos\omega t + b\sin\omega t$$

2.8 Modeling: Forced Oscillations. Resonance

Case 1 Undamped Forced Oscillations. Resonance

$$\boxed{a = F_0 \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}, \quad b = F_0 \frac{\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2 c^2}}$$

$$c = 0 \quad \Rightarrow \quad y_p = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t \quad \Rightarrow \quad y = \underbrace{C \cos\left(\omega_0 t - \delta\right)}_{= y_h} + \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)} \cos \omega t$$

where, $\omega \neq \omega_0$

where, $C = \sqrt{a^2 + b^2} = a, \ \tan \delta = b / a = 0$

$$\boxed{y_p = \frac{F_0}{k[1 - (\omega / \omega_0)^2]} \cos \omega t, \quad \omega_0^2 = k / m}$$

- This output is a superposition of two harmonic oscillations of the frequencies just mentioned.

- Natural frequency (자유비감쇠진동의 주파수): $\frac{\omega_0}{2\pi} \left[\text{cycles}/\text{sec} \right]$
- Frequency of the driving force (강제비감쇠진동의 주파수): $\frac{\omega}{2\pi} \left[\text{cycles}/\text{sec} \right]$

2.8 Modeling : Forced Oscillations. Resonance

- **Resonance:** Excitation of large oscillations by matching input and natural frequencies. $(\omega = \omega_0)$

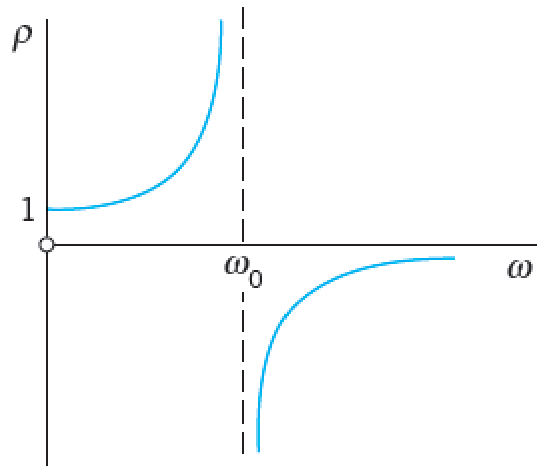
$$y_p = \frac{F_0}{k[1-(\omega/\omega_0)^2]}\cos\omega t$$

Maximum amplitude (a_0) of y_p (when $\cos\omega t = 1$)

$$a_0 = \frac{F_0}{k}\rho \quad where \quad \rho = \frac{1}{1-(\omega/\omega_0)^2}$$: resonance factor

The ratio of the amplitudes and the input $F_0 cos\omega t$

$$\frac{\rho}{k} = \frac{a_0}{F_0}$$



< Resonance factor ρ >

2.8 Modeling: Forced Oscillations. Resonance

- **Resonance**: $(\omega = \omega_0)$

We obtained $y_p = \dfrac{F_0}{m(\omega_0^{\,2} - \omega^2)}\cos\omega t \quad \Rightarrow \quad y = C\cos(\omega_0 t - \delta) + \dfrac{F_0}{m(\omega_0^{\,2} - \omega^2)}\cos\omega t$

where, $\omega \neq \omega_0$

Valid or not when resonance?

$\omega = \omega_0$

$\sqrt{k/m} = \omega = \omega_0$

$$my'' + \cancel{cy'} + ky = F_0 \cos\omega t \quad \Longrightarrow \quad y'' + \omega_0 y = \frac{F_0}{m}\cos\omega_0 t$$

How to solve?

(Review) 2.7 Nonhomogeneous ODEs

❖ Choice Rules for the Method of Undetermined Coefficients (미정계수법)

a. **Basic Rule (기본규칙).** If $r(x)$ in $y''+p(x)y'+q(x)y=r(x)$ is one of the functions in the first column in Table 2.1, choose y_p in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into $y''+ay'+by=r(x)$.

b. **Modification Rule (변형규칙).** If a term in your choice for y_p happens to be a solution of the homogeneous ODE corresponding to $y''+ay'+by=r(x)$, multiply your choice of by x (or by x^2 if this solution corresponding to a double root of the y_p).

characteristic equation of the homogeneous ODE).

c. **Sum Rule (합규칙).** If $r(x)$ is a sum of functions in the first column of Table 2.1, choose for y_p the sum of the functions in the corresponding lines of the second column.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

2.8 Modeling: Forced Oscillations. Resonance

- **Resonance**: $(\omega = \omega_0)$

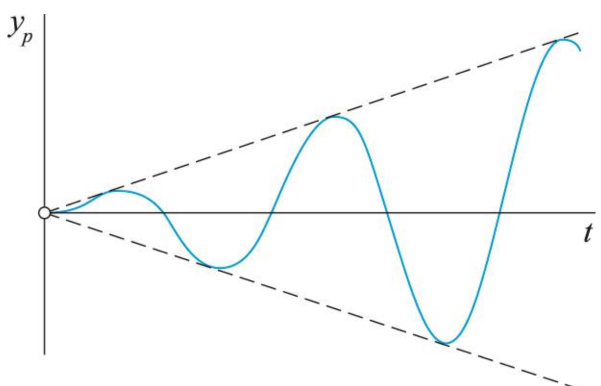
$$my'' + \cancel{cy'} + ky = F_0 \cos \omega t \quad \xrightarrow[\sqrt{k/m} = \omega = \omega_0]{\omega = \omega_0} \quad y'' + \omega_0 y = \frac{F_0}{m} \cos \omega_0 t$$

$$y_p = t(a \cos \omega_0 t + b \sin \omega_0 t)$$ (From the modification rule, we multiply y_p by t)

$$y_p = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	



< Particular solution in the case of resonance >

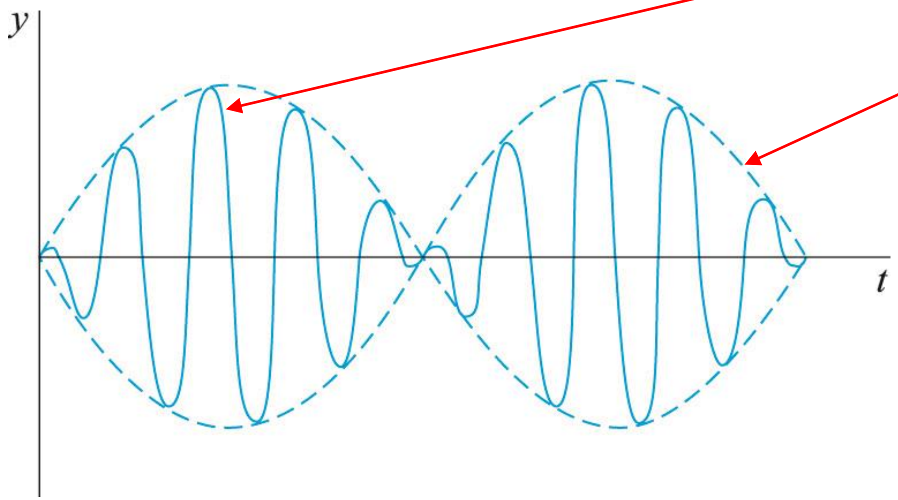
2.8 Modeling: Forced Oscillations. Resonance

- Beats (맥놀이): Forced undamped oscillation (강제비감쇠진동) when the difference of the input and natural frequencies (ω - ω_0) is small.

$$\boxed{y = C\cos(\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t}$$

⇨ Take a particular solution

$$y = \frac{F_0}{m\left(\omega_0^{\ 2} - \omega^2\right)}\left(\cos\omega t - \cos\omega_0 t\ \right) = \frac{2F_0}{m\left(\omega_0^{\ 2} - \omega^2\right)}\underline{\sin\left(\frac{\omega_0 + \omega}{2}t\right)}\,\underline{\sin\left(\frac{\omega_0 - \omega}{2}t\right)}$$



Resulting from the second sine factor.
➡ This is what musicians are listening to when they tune (조율) the instruments.

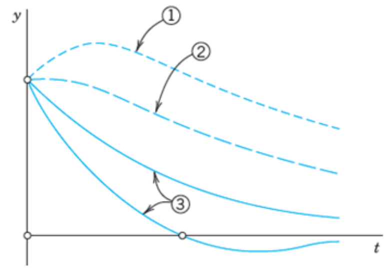
< Beasts >

$$\sin A \cdot \sin B = -\frac{1}{2}\left\{\cos(A+B) - \cos(A-B)\right\}$$

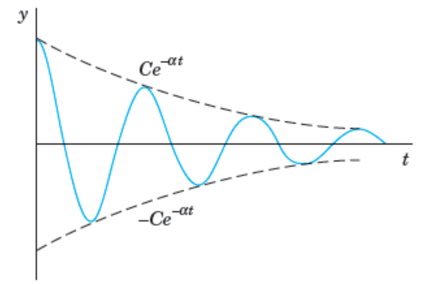
2.8 Modeling: Forced Oscillations. Resonance

Case 2 Damped Forced Oscillations

- $y_h \to 0$ *as t goes infinity.*



Overdamping



Underdamping

- Transient Solution: The general solution $y = y_p + y_h$ of the nonhomogeneous ODE
- Steady-State Solution: The particular solution y_p (because $y_h \to 0$)

❖ **Steady-State Solution**

After a sufficiently long time the output of a damped vibrating system under a purely sinusoidal driving force will practically be a harmonic oscillation whose frequency is that of the input.

2.8 Modeling: Forced Oscillations. Resonance

❖ **Amplitude of the Steady-State Solution. Practical Resonance**

$$\boxed{y_p = a\cos\omega t + b\sin\omega t = C^*\cos(\omega t - \eta) \qquad C^*:\ amplitude,\ \eta:\ pahse\ lag}$$

$$C^*(\omega) = \sqrt{a^2+b^2} = \frac{F_0}{\sqrt{\underbrace{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}_{=R}}} \quad \Leftarrow \quad \boxed{a = F_0\frac{m(\omega_0^2-\omega^2)}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2},\quad b = F_0\frac{\omega c}{m^2(\omega_0^2-\omega^2)^2+\omega^2c^2}}$$

$$\tan\eta(\omega) = \frac{b}{a} = \frac{\omega c}{m(\omega_0^2-\omega^2)}$$

where, η (phase lag): The lag of the output behind the input (also called phase angle)

- For what ω, $C^*(\omega)$ has maximum? what is the size?

$$\frac{dC^*}{d\omega} = F_0\left(-\frac{1}{2}R^{-3/2}\right)[\underbrace{2m^2(\omega_0^2-\omega^2)(-2\omega)+2\omega c^2}_{=0}] = 0 \qquad \boxed{C^*(\omega_{\max})}$$

$$c^2 = 2m^2(\omega_0^2-\omega^2) \quad (\omega_0^2 = k/m)$$

$$2m^2\omega^2 = 2m^2\omega_0^2 - c^2 = 2mk - c^2$$

- If $c^2 > 2mk$ ⇨ $\dfrac{dC^*}{d\omega} < 0$
- If $c^2 < 2mk$ ⇨ a real solution $\omega = \omega_{max}$

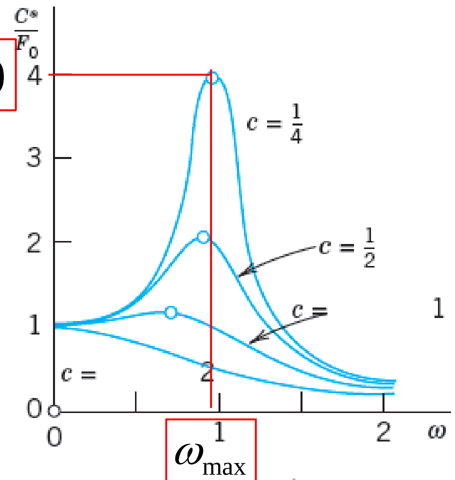


Fig. 57. Amplification C^*/F_0 as a function of ω for $m = 1, k = 1$, and various values of the damping constant c

2.8 Modeling: Forced Oscillations. Resonance

$$\boxed{my'' + cy' + ky = F_0 \cos \omega t}$$

$$\boxed{y_p = a\cos\omega t + b\sin\omega t = C^*\cos(\omega t - \eta) \qquad C^*:\ amplitude,\ \eta:\ pahse\ lag}$$

❖ **Amplitude of the Steady-State Solution. Practical Resonance**

$$2m^2\omega^2 = 2m^2\omega_0^2 - c^2 = 2mk - c^2 \Rightarrow \boxed{\omega_{\max}^2 = \omega_0^2 - \frac{c^2}{2m^2}}$$

$$C^*(\omega_{\max}) = \sqrt{a^2+b^2} = \frac{F_0}{\sqrt{\boxed{m^2(\omega_0^2 - \omega_{\max}^2)^2 + \omega_{\max}{}^2 c^2}}}$$

$$\Rightarrow \quad m^2(\omega_0^2 - \omega_{\max}^2)^2 + \omega_{\max}^2 c^2 = \frac{c^4}{4m^2} + (\omega_0^2 - \frac{c^4}{2m^2})c^2$$

$$= \frac{c^4 + 4m^2\omega_0^2 c^2 - 2c^4}{4m^2}$$

$$= \frac{c^2(4m^2\omega_0^2 - c^2)}{4m^2}$$

$$C^*(\omega_{\max}) = \frac{2mF_0}{c\sqrt{4m^2\omega_0{}^2 - c^2}}$$

$c \rightarrow 0$ then $C^* \rightarrow$ infinity

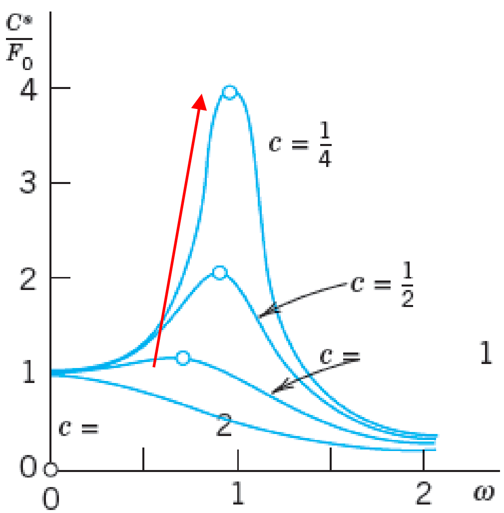


Fig. 57. Amplification C^*/F_0 as a function of ω for $m = 1, k = 1$, and various values of the damping constant c

2.8 Modeling: Forced Oscillations. Resonance

$$\boxed{my''+cy'+ky=F_0\cos\omega t}$$

$$\boxed{y_p=a\cos\omega t+b\sin\omega t=C^*\cos(\omega t-\eta)\qquad C^*:\ amplitude,\ \eta:\ pahse\ lag}$$

$$\tan\eta(\omega)=\frac{b}{a}=\frac{\omega c}{m(\omega_0^2-\omega^2)}$$

where, η (phase lag): The lag of the output behind the input (also called phase angle)

$$\eta(\omega)=\tan^{-1}\left(\frac{\omega c}{m(\omega_0^2-\omega^2)}\right)$$

$$if\ \ \omega<\omega_0\ \ \Rightarrow\eta<\frac{\pi}{2}$$

$$if\ \ \omega>\omega_0\ \ \Rightarrow\eta>\frac{\pi}{2}$$

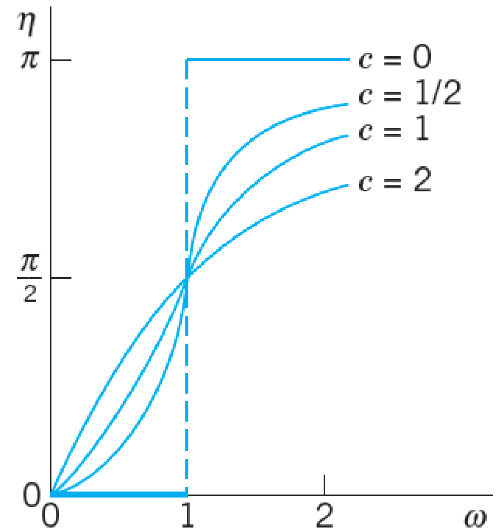
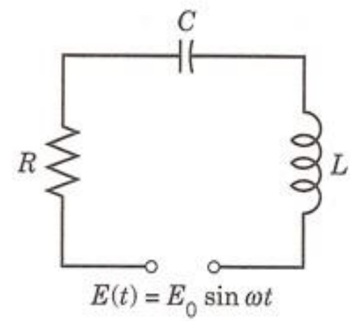


Fig. 58. Phase lag η as a function of ω for $m = 1, k = 1$, thus $\omega_0 = 1$, and various values of the damping constant c

2.9 Modeling: Electric Circuits - Skip



< *RLC*-circuit >

Name	Symbol	Notation		Unit	Voltage Drop
Ohm's resistor		R	Ohm's resistance	ohms (Ω)	RI
Inductor		L	Inductance	henrys (H)	$L\dfrac{dI}{dt}$
Capacitor		C	Capacitance	farads (F)	Q/C

< Elements in an *RLC*-circuit >

2.9 Modeling: Electric Circuits - Skip

- Kirchhoff's Voltage Law (KVL): The voltage (the electromotive force) impressed on a closed loop is equal to the sum of the voltage drops across the other elements of the loop.

- Voltage Drops

RI	(Ohm's law) Voltage drop for a resistor of resistance *R* ohms (W)
$LI' = L\dfrac{dI}{dt}$	Voltage drop for an inductor of inductance *L* henrys(*H*)
$\dfrac{Q}{C}$	Voltage drop for a capacitor of capacitance *C* farads (*F*)

- Model of an *RLC*-circuit with electromotive force: $L\dfrac{d^2I}{dt^2} + R\dfrac{dI}{dt} + \dfrac{1}{C}I = E'(t) = E_0\omega\cos\omega t$

$$I_p = a\cos\omega t + b\sin\omega t = I_0\sin(\omega t - \theta)$$

$$a = \frac{-E_0 S}{R^2 + S^2}, \quad b = \frac{-E_0 R}{R^2 + S^2}, \quad I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}, \quad \tan\theta = -\frac{a}{b} = \frac{S}{R}$$

2.9 Modeling: Electric Circuits - Skip

❖ Analogy of Electrical and Mechanical Quantities

- Entirely different physical or other systems may have the same mathematical model.
- Practical importance of this analogy
  1. Electric circuits are easy to assemble.
  2. Electric quantities can be measured much more quickly and accurately than mechanical ones.

Electrical System	Mechanical System
Inductance L	Mass m
Resistance R	Damping constant c
Reciprocal $1/C$ of capacitance	Spring modulus k
Derivative $E_0\omega\cos\omega t$ of electromotive force	Driving force $F_0\cos\omega t$
Current $I(t)$	Displacement $y(t)$

< Analogy of Electrical and Mechanical Quantities >

2.10 Solution by Variation of Parameters

- $y'' + p(x)\,y' + q(x)y = r(x)$

  $\boldsymbol{y} = \boldsymbol{y_h}$ (solution of $y''+p\,y'+qy=0$) + $\boldsymbol{y_p}$ (solution of $y''+py'+qy=r$)

❖ **Method of undetermined coefficient**

- If $r(x)$ is not complicated (ex. e^{rx}, $\cos\omega x$, $\sin\omega x$, $e^{\alpha x}\cos\omega x$, $e^{\alpha x}\sin\omega x$)

  $\rightarrow$ Method of undetermined coefficient

❖ **Method of Variation of Parameter for more general r(x) (매개변수 변환법)**

- $p(x)$, $q(x)$, $r(x)$ in $y'' + p(x)\,y' + q(x)y = r(x)$ are continuous on some open interval I.
- Solution formula: $y_p(x) = -y_1\int \frac{y_2 r}{W}dx + y_2\int \frac{y_1 r}{W}dx$ $\quad W = y_1 y_2' - y_1' y_2$

  $\boldsymbol{y_1, y_2}$: a basis of solution of the homogeneous ODE $y'' + p(x)\,y' + q(x)y = 0$

- If it starts with $f(x)y''$, divide first by $f(x)$.
- The integration in $y_p(x) = -y_1\int \frac{y_2 r}{W}dx + y_2\int \frac{y_1 r}{W}dx$ may often cause difficulties.

2.10 Solution by Variation of Parameters

☑ **Ex. 1 Solve the nonhomogeneous ODE** $y'' + y = \sec x$

A basis of solutions of the homogeneous ODE: $y_1 = \cos x, \quad y_2 = \sin x$

Wronskian: $W(y_1, y_2) = \cos x \cos x - \sin x(-\sin x) = 1$

Apply the method of variation of parameters: $\boxed{y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx}$

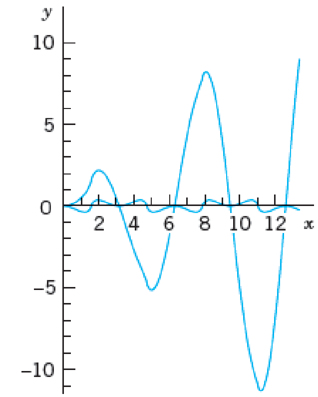
- Particular solution

$$y_p = -\cos x \int \sin x \sec x dx + \sin x \int \cos x \sec x dx = \cos x \ln|\cos x| + x \sin x$$

- General solution

$$y_h = c_1 y_1 + c_2 y_2 = c_1 \cos x + c_2 \sin x$$

$$y = y_h + y_p = (c_1 + \ln|\cos x|) \cos x + (c_2 + x) \sin x$$



The particular solution y_h

$\sec x = 1/\cos x, \ \ \text{cosec} x = 1/\sin x, \ \ \cot x = 1/\tan x$

2.10 Solution by Variation of Parameters

❖ Idea of the Method. Derivation

$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ is a general solution of $y'' + py' + qy = 0$

$\boldsymbol{y_p(x) = u(x)y_1(x) + v(x)y_2(x)}$ is assumed to be a particular solution of $y'' + p\,y' + qy = r$

Here, $\boldsymbol{u(x)}$ **and** $\boldsymbol{v(x)}$ should be determined.

$$y'_p = \cancel{u'y_1} + uy'_1 + \cancel{v'y_2} + vy'_2$$

a second condition: $u'y_1 + v'y_2 = 0$ (assumption)

⇨ $y'_p = uy'_1 + vy'_2$

$$y''_p = u'y'_1 + uy''_1 + v'y'_2 + vy''_2$$

$$u(y''_1 + \cancel{py'_1} + qy'_1) + v(y''_2 + \cancel{py'_2} + qy'_2) + u'y'_1 + v'y'_2 = r \quad \Leftarrow \quad \boxed{y'' + p\,y' + qy = 0}$$

⇨ $u'y'_1 + v'y'_2 = r$

$u'y_1 + v'y_2 = 0$ from a second condition

2.10 Solution by Variation of Parameters

❖ Idea of the Method. Derivation

$$u'y'_1 + v'y'_2 = r$$

$$u'y_1 + v'y_2 = 0$$ from a second condition

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \Rightarrow \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$= \frac{1}{y_1y_2' - y_1'y_2}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ r \end{bmatrix} = \frac{1}{y_1y_2' - y_1'y_2}\begin{bmatrix} -y_2r \\ y_1r \end{bmatrix}$$

$$u' = \frac{-y_2r}{y_1y_2' - y_1'y_2} = -\frac{y_2r}{W}$$

$$v' = \frac{y_1r}{y_1y_2' - y_1'y_2} = \frac{y_1r}{W}$$

$$\boxed{u = -\int \frac{y_2r}{W}dx, \quad v = \int \frac{y_1r}{W}dx}$$

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

$$\boxed{y_p(x) = -y_1\int \frac{y_2r}{W}dx + y_2\int \frac{y_1r}{W}dx}$$