Chapter 3: Truss elements and solutions for different strain measures

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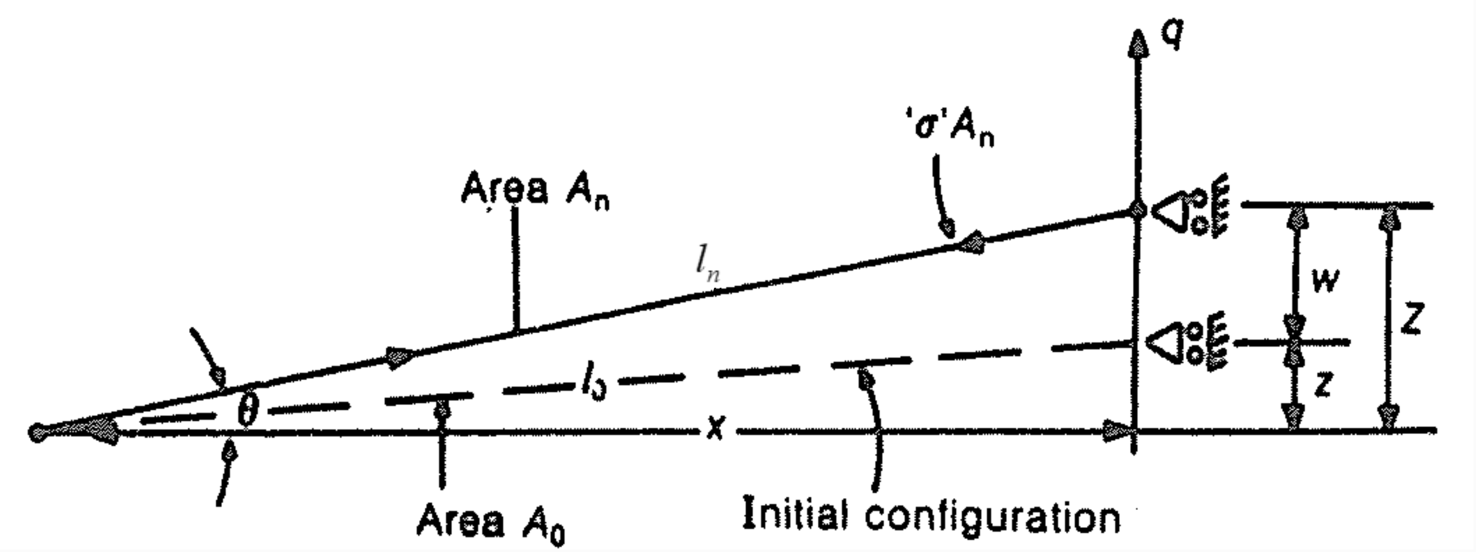
Contents of this chapter include:

- Formulations based on a **shallow truss** discussed in the previous chapter: area and length change of bar are negligible
- alternative **strain measures** in the truss element
- corresponding stress measures
- concepts of **total Lagrangian** and **updated Lagrangian** formulations, **corotational** formulation, equivalent constitutive laws
- finite element formulations of each strain measure
- general 3-D truss element formulation
- Fortran subroutines for the different strain measures



[Fig 3.1 Simple problem with one degree of freedom]

- Four different strain measures:
  - **rotated engineering strain**
  - **Green's strain**
  - **rotated log-strain**
  - **Almansi strain**

- For each strain measure, the virtual work relationship learned in ch1.2.3 and 2.1 is employed
- Virtual work can be formulated as:

$$V_n = \int \sigma \delta \varepsilon_v dV_n - q\delta w_v$$ [eq. 3.1]

or

$$V_0 = \int \sigma \delta \varepsilon_v dV_0 - q\delta w_v$$ [eq. 3.2]

- internal virtual work
- external virtual work

- eq. 3.1 is related to the **final configuration** with volume V_n ,

- eq. 3.2 is related to the **initial configuration** with volume V_0 .

- Strain based on a final configuration is subjected to eq. 3.1,
  while, strain based on initial configuration is subjected to eq. 3.2.

- Corresponding internal force, stress and tangent stiffness matrix will be formulated from the internal virtual work.

Strain measures

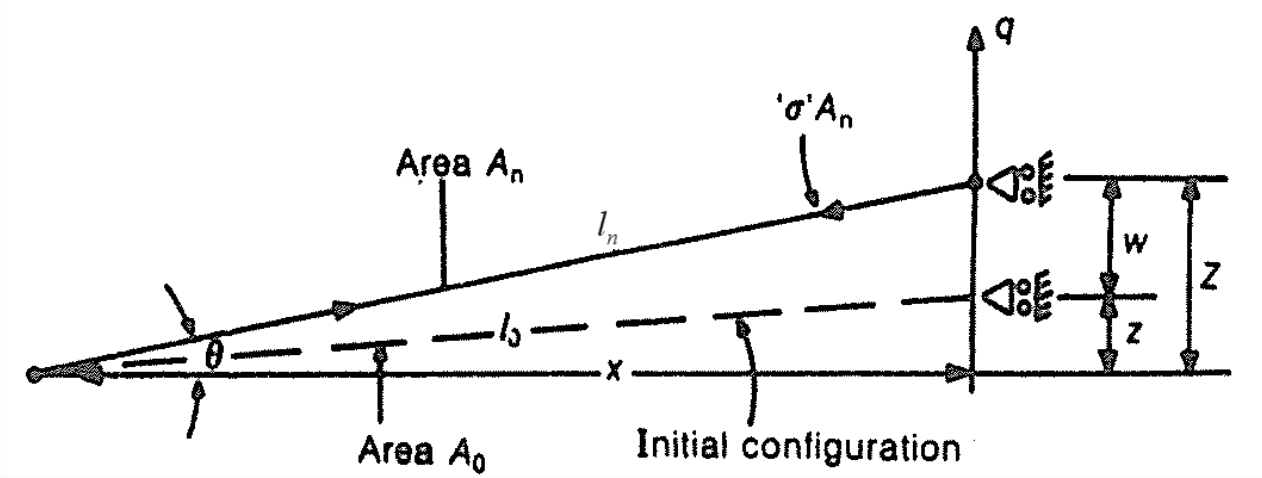
3.1.1 Rotated engineering strain

- The strain measure : $\varepsilon_E = \dfrac{l_n - l_0}{l_0}$ [eq. 3.3]

$$l_n = \left((z+w)^2 + x^2\right)^{1/2} = \left(Z^2 + x^2\right)^{1/2} = \left(Z^2 - z^2 + l_0^2\right)^{1/2}$$ [eq. 3.4]

- Virtual strain increment:

$$\delta\varepsilon_E = \frac{\delta l_n}{l_0} = \frac{(z+w)}{l_n l_0}\delta w_v = \frac{Z\delta w_v}{l_n l_0}$$ [eq. 3.5]



$$\delta\varepsilon_E = \frac{\delta l_n}{l_0} = \frac{(z+w)}{l_n l_0}\delta w_v = \frac{Z\delta w_v}{l_n l_0}$$ [eq. 3.5]

Or,

$$\Rightarrow \quad b_E = \frac{Z}{l_n l_0}$$

$$\Downarrow \quad q_E = \int b_E^T \sigma_E dV_0$$

- $\sigma_E = E\varepsilon_E$ corresponds to engineering stress
- Internal virtual work and internal force:

$$\int \sigma_E \delta\varepsilon_E dV_0 = \left(\frac{\sigma_E Z\delta w_v}{l_n l_0}\right) A_0 l_0 = q_E^T \delta w_v \quad \Rightarrow \quad q_E = \frac{\sigma_E A_0 Z}{l_n} = \frac{\sigma_E A_0 (z+w)}{l_n}$$ [eq. 3.6]

$$\Rightarrow \quad q_E = \frac{EA_0 (z+w)\left(\left((z+w)^2 + x^2\right)^{1/2} - l_0\right)}{l_0\left((z+w)^2 + x^2\right)^{1/2}}$$ [eq. 3.7]

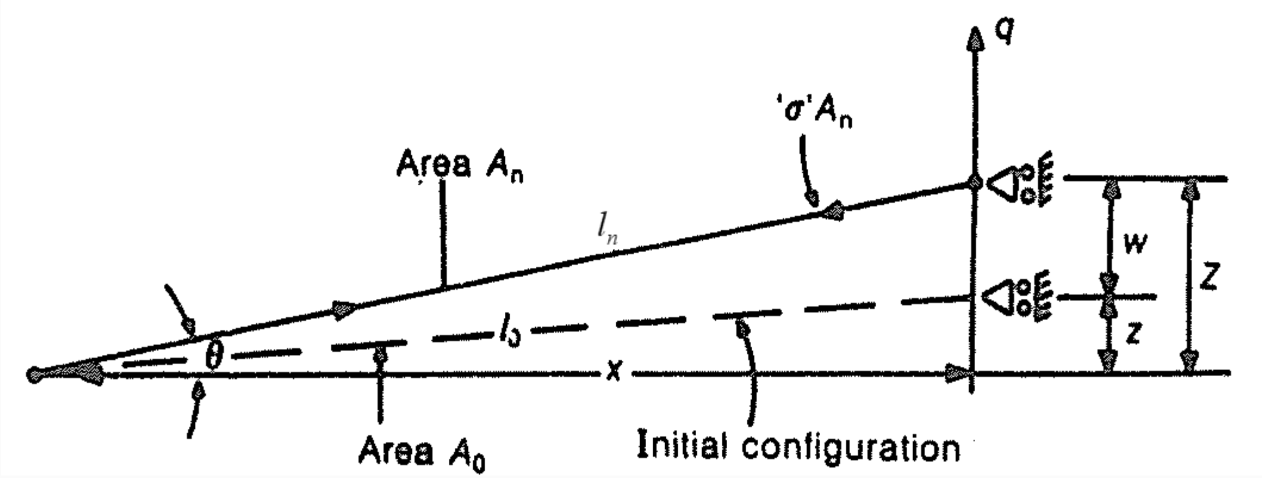
3.1.2 Green strain

- The strain measure : $$\varepsilon_G = \frac{1}{2}\left(\left(\frac{l_n}{l_0}\right)^2 - 1\right) = \frac{l_n^2 - l_0^2}{2l_0^2}$$ [eq. 3.8]

- General formulation will be discussed in ch 4.4.

- Virtual strain increment: $$l_n = \left((z+w)^2 + x^2\right)^{1/2} = \left(Z^2 + x^2\right)^{1/2} = \left(Z^2 - z^2 + l_0^2\right)^{1/2}$$

$$\delta\varepsilon_G = \frac{l_n}{l_0^2}\delta l_n = \frac{(z+w)}{l_0^2}\delta w_v = \frac{Z}{l_0^2}\delta w_v$$ [eq. 3.11]



$$\delta\varepsilon_G = \frac{l_n}{l_0^2}\delta l_n = \frac{(z+w)}{l_0^2}\delta w_v = \frac{Z}{l_0^2}\delta w_v$$ [eq. 3.11]

$$\Rightarrow\quad b_G = \frac{Z}{l_0^2}$$

$$q_G = \int b_G \sigma_G dV_0$$

$$\Downarrow$$

- $\sigma_G = E\varepsilon_G$ corresponds to 2nd Piola-Kirchhoff stress.
- Internal virtual work and internal force:

$$\int \sigma_G \delta\varepsilon_G dV_0 = \sigma_G \left(\frac{Z\delta w_v}{l_0^2}\right) A_0 l_0 = q_G^T \delta w_v \Rightarrow \quad q_G = \frac{\sigma_G A_0 Z}{l_0} = \frac{\sigma_G A_0 (z+w)}{l_0}$$ [eq. 3.12]

$$\Rightarrow\quad q_G = \frac{EA_0(z+w)(2zw+w^2)}{2l_0^3} = \frac{EA_0 Z(2zw+w^2)}{2l_0^3}$$ [eq. 3.13]

3.1.3 Rotated log-strain (without volume change)

- The strain measure : $\delta\varepsilon = \frac{\delta l}{l}$ ➡ $\varepsilon_L = \int_{l_0}^{l_n} \delta\varepsilon = \ln\left(\frac{l_n}{l_0}\right)$ [eq. 3.14, 3.15]

- Virtual strain increment:

$$l_n = \left(\left(z+w\right)^2 + x^2\right)^{1/2} = \left(Z^2 + x^2\right)^{1/2} = \left(Z^2 - z^2 + l_0^2\right)^{1/2}$$

$$\delta\varepsilon_L = \frac{\delta l_n}{l_n} = \frac{Z}{l_n^2}\delta w \quad ➡ \quad b_L = \frac{Z}{l_n^2}$$

- $\sigma_L = E\varepsilon_L$ corresponds to true (Cauchy) stress.

$$q_L = \int b_L \sigma_L dV_n$$

- Internal virtual work and internal force:

$$\int \sigma_L \delta\varepsilon_L dV_n = \sigma_L \frac{Z}{l_n^2}\delta w A_n l_n \quad ➡ \quad q_L = \frac{\sigma_L A_n l_n Z}{l_n^2} = \frac{\sigma_L A_0 l_0 \left(z+w\right)}{l_n^2}$$ [eq. 3.18]

$$A_n l_n = A_0 l_0$$

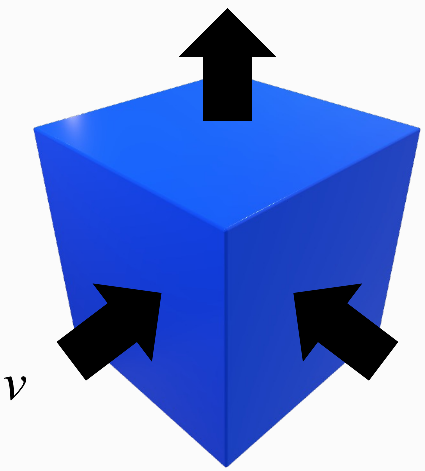
$$➡ \quad q_L = \frac{EA_0\left(z+w\right)l_0}{2\left(\left(z+w\right)^2 + x^2\right)} \ln\left(\frac{\left(z+w\right)^2 + x^2}{l_0^2}\right)$$ [eq. 3.19]

3.1.4 Rotated log-strain formulation allowing for volume change

$$q_L = \frac{\sigma_L A_n l_n Z}{l_n^2}$$

- Current area

$$A + dA = A(1 - v d\varepsilon)^2 \simeq A(1 - 2v d\varepsilon)$$ [eq. 3.20]



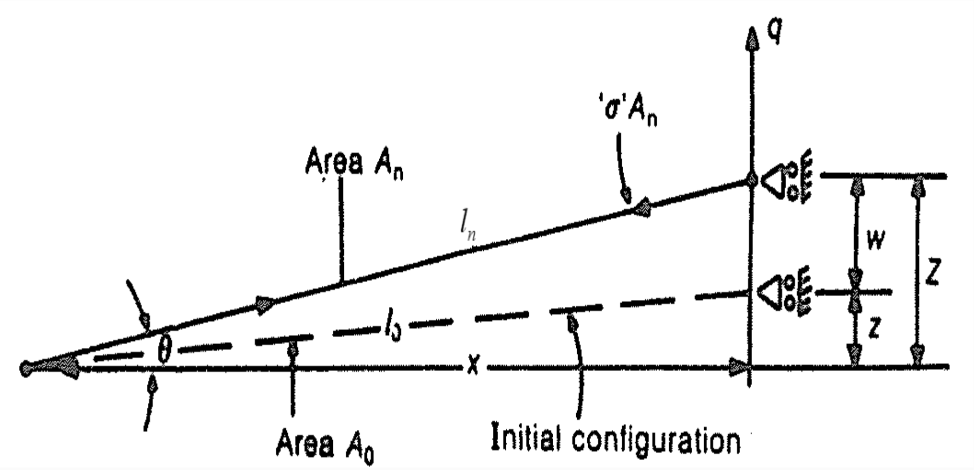
$$\int_{A_0}^{A_n} \frac{dA}{A} = -2v \int_{l_0}^{l_n} \delta\varepsilon = -2v \int_{l_0}^{l_n} \frac{dl}{l}$$ [eq. 3.21]

$$\frac{A_n}{A_0} = \left(\frac{l_0}{l_n}\right)^{2v}$$ [eq. 3.23]

➡ $$q_L = \frac{\sigma_L A_0 l_0^{2v} Z}{l_n^{1+2v}}$$ [eq. 3.25]

➡ $$q_L = \frac{A_0 (z+w) l_0^{2v}}{2\left((z+w)^2 + x^2\right)^{(1+2v)/2}} \ln\left(\frac{(z+w)^2 + x^2}{l_0^2}\right)$$ [eq. 3.26]

3.1.5 Comparisons



- Summary

$$q_E = \frac{\sigma_E A_0 Z}{l_n} = \frac{\sigma_E A_0 (z+w)}{l_n}$$

$$q_G = \frac{\sigma_G A_0 Z}{l_0} = \frac{\sigma_G A_0 (z+w)}{l_0}$$

$$q_L = \frac{\sigma_L A_n l_n Z}{l_n^2} = \frac{\sigma_L A_0 l_0 (z+w)}{l_n^2}$$

3.1.5 Comparisons

- From equilibrium, a general form can be introduced

$$q = \frac{A_n \, '\sigma' \, Z}{l_n}$$

[eq. 3.27]

$'\sigma'$: *true stress*

- Referenced to "current" area and length

$$q_L = \frac{\sigma_L A_n Z}{l_n}$$ [eq. 3.18]

$$\Rightarrow \quad \sigma_L = '\sigma'$$ [eq. 3.28]

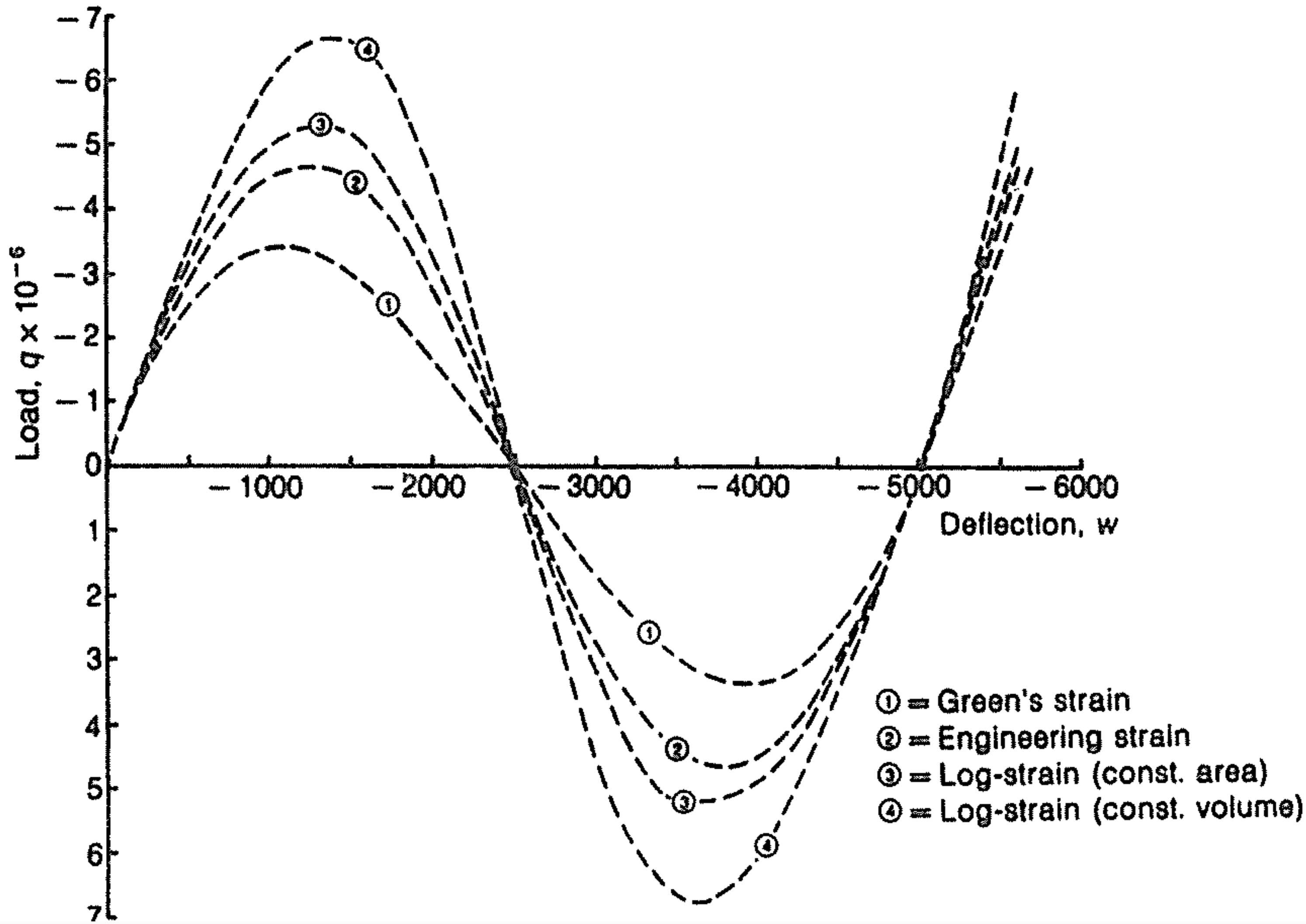
$$q_E = \frac{\sigma_E A_0 Z}{l_n}$$ [eq. 3.6]

$$\Rightarrow \quad \sigma_E = '\sigma' \left(\frac{A_n}{A_0} \right)$$ [eq. 3.29]

$$q_G = \frac{\sigma_G A_0 Z}{l_0}$$ [eq. 3.12]

$$\Rightarrow \quad \sigma_G = '\sigma' \left(\frac{A_n l_0}{A_0 l_n} \right) = \frac{l_0}{l_n} \sigma_E$$ [eq. 3.30]

$$when \quad x = 2500, \quad A_0 = 100, \quad E = 5 \times 10^5, \quad z = 2500$$



[Fig 3.2 Load/deflection relationships for deep truss]

Conjugate stresses

- Engineering strain,

$$\delta\varepsilon_E = \delta\left(\frac{u}{l_0}\right) = \frac{\delta u}{l_0} = b\delta u \quad \Rightarrow \quad b = \frac{1}{l_0} \quad \Rightarrow \quad q_E = \int b^T \sigma_E dV_0 = \frac{\sigma_E}{l_0} A_0 l_0 = \sigma_E A_0$$

[eq. 3.32]

$$\Rightarrow \quad \sigma_E = \frac{q_E}{A_0}$$

: engineering stress

- Green strain,

$$\delta\varepsilon_G = \delta\left(\frac{l_n^2 - l_0^2}{2l_0^2}\right) = \frac{l_n \delta u}{l_0^2} = b\delta u \quad \Rightarrow \quad b = \frac{l_n}{l_0^2} \quad \Rightarrow \quad q_G = \int b\ \sigma_G dV_0 = \frac{l_n}{l_0} \sigma_G A_0$$

[eq. 3.34]

$q_E = q_G$

$$\Rightarrow \quad \sigma_G = \left(\frac{l_0}{l_n}\right)\sigma_E \left(= \left(\frac{A_n l_n}{A_0 l_0}\right)\left(\frac{l_0}{l_n}\right)\left(\frac{q}{A_n}\right)\left(\frac{l_0}{l_n}\right) = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T} \right)$$

: 2nd order Piola-Kirchhoff stress

- Log-strain,

$$\delta\varepsilon_L = \frac{\delta u}{l_n} = b\delta u \Rightarrow b = \frac{1}{l_n} \Rightarrow q_L = \int b\,\sigma_L dV_n = \frac{\sigma_L}{l_n} A_n l_n = \sigma_L A_n = \sigma_L \left(\frac{l_0}{l_n}\right)^{2v} A_0$$

[eq. 3.36]

$$\Rightarrow \sigma_L = \frac{q_L}{A_n} = '\sigma'$$

: true (Cauchy) stress

Almansi strain

- The strain measure : $$\varepsilon_A = \frac{1}{2}\left(1-\left(\frac{l_n}{l_0}\right)^{-2}\right) = \frac{l_n^2 - l_0^2}{2l_n^2}$$ [eq. 3.41]

- Virtual strain increment:

$$\delta\varepsilon_A = \left(\frac{l_n}{l_0}\right)^{-3}\frac{\delta u}{l_0} = \frac{l_0^2}{l_n^3}\delta u$$ [eq. 3.42]

$$= b\delta u$$

$$\Rightarrow \quad b_A = \frac{l_0^2}{l_n^3}$$

$$\Downarrow \quad q_A = \int b_A \sigma_A dV_n$$

- Internal virtual work and internal force:

$$\int \sigma_A \delta\varepsilon_A dV_n = \sigma_A\left(\frac{l_0^2}{l_n^3}\delta u\right)A_n l_n = q_A^T \delta u$$

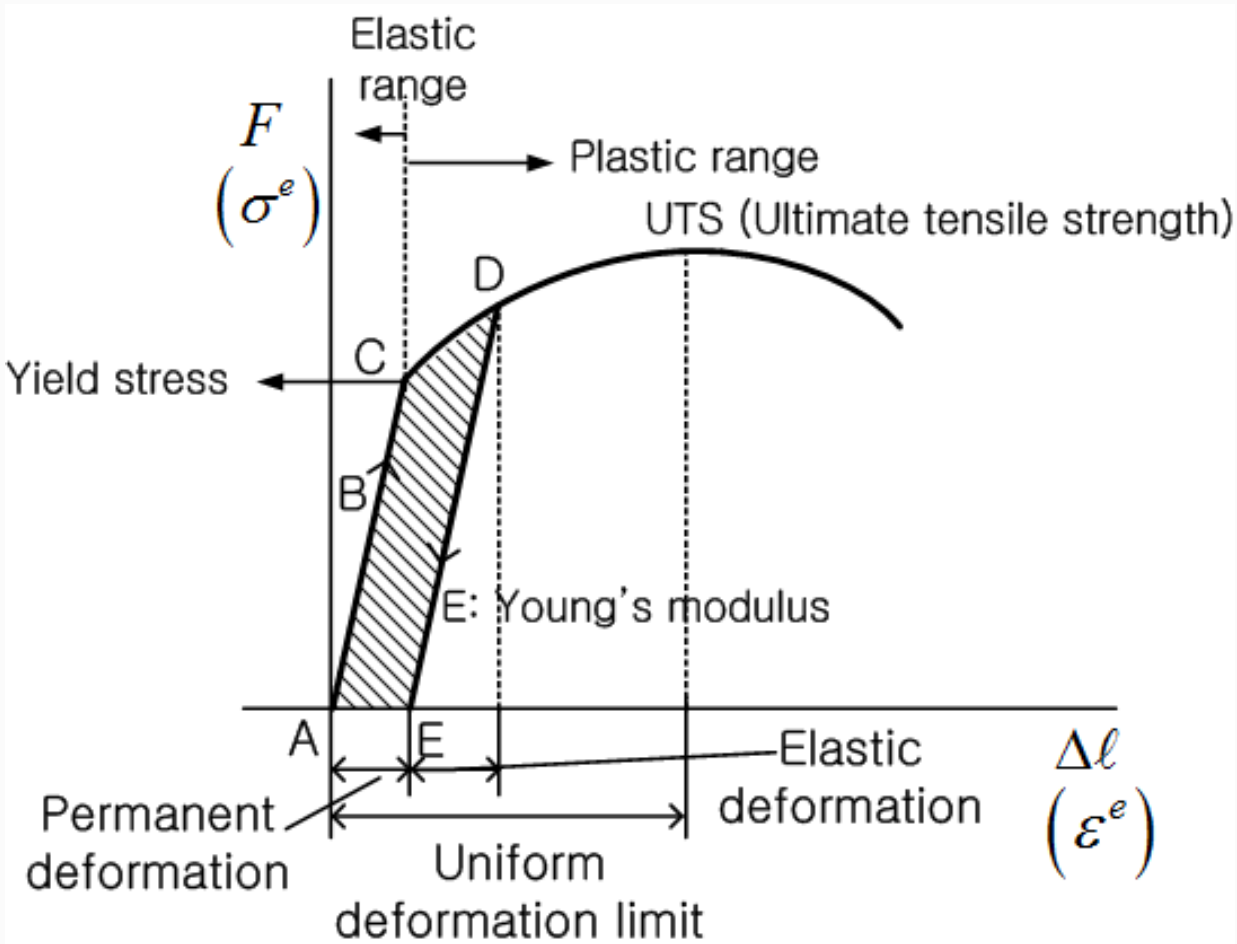
$$\Rightarrow \quad q_A = \sigma_A A_n \frac{l_0^2}{l_n^2}$$

$q_L = q_A$

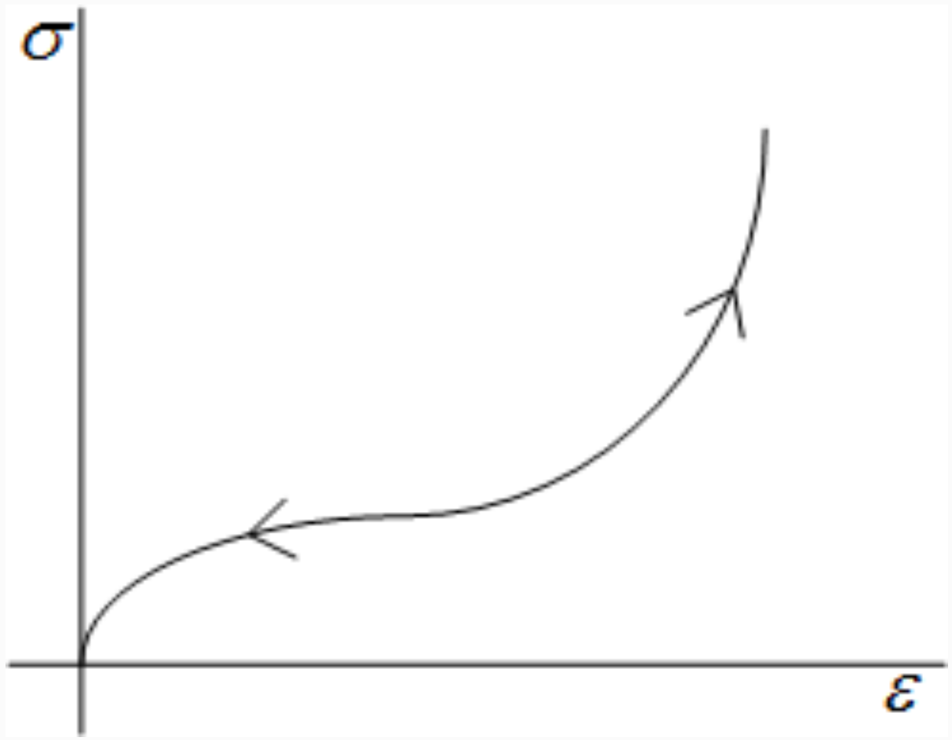
$$\Rightarrow \quad \sigma_A = {}'\sigma'\frac{l_n^2}{l_0^2}$$ [eq. 3.44]

True & engineering measures in “engineering”

Simple tension test
Eng. S-S curve of metals



Simple tension test
Nonlinear elastic behavior of rubber

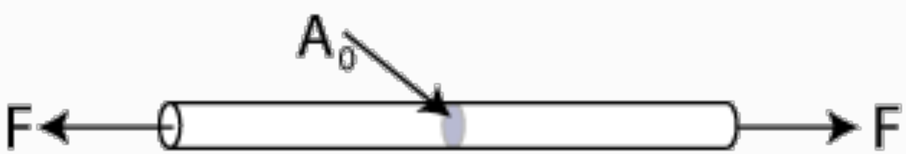


Stress in tension: 1D

- True stress $\sigma^t = \dfrac{F}{A}$

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Engineering strain $\sigma^e = \dfrac{F}{A_o}$



$$\text{Stress, } \sigma = \frac{\text{Force}}{\text{Cross-Sectional Area}} = \frac{F}{A_0}$$

Strain in tension: 1D

- True strain

$$d\varepsilon^t = \frac{dl}{l}$$

$$\varepsilon^t = \int_{l_o}^{l} \frac{1}{l} dl = \ln \frac{l}{l_o}$$

- Engineering strain

$$d\varepsilon^e = \frac{dl}{l_o}$$

$$\varepsilon^e = \int_{l_o}^{l} \frac{dl}{l_o} = \frac{l - l_0}{l_0}$$

Stress in tension: Eng. vs. True

- Tension case

$$\varepsilon^t = \ln\frac{l}{l_o} = \ln\left(\frac{l-l_o}{l_o}+1\right) = \ln\left(\varepsilon^e+1\right)$$

$$\sigma^t = \frac{F}{A} = \frac{F}{A_o}\frac{A_o}{A} = \frac{F}{A_o}\frac{l}{l_o} = \sigma^e\left(\frac{l-l_o}{l_o}+1\right) = \sigma^e\left(\varepsilon^e+1\right)$$

$(A_o l_o = Al : \text{volume constant})$

Stress in tension: Eng. vs. True

- Relation for small deformation

$$\varepsilon^t = \ln\frac{l}{l_o} = \ln\left(\frac{l-l_o}{l_o}+1\right) = \ln\left(\varepsilon^e+1\right)$$

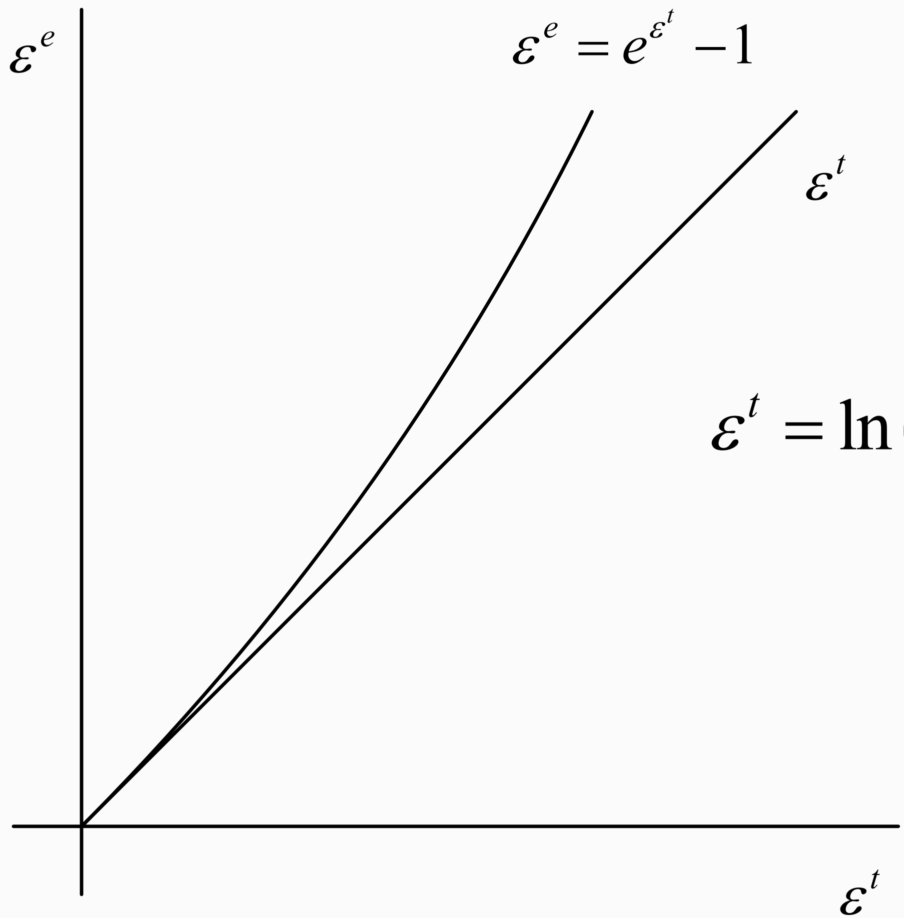
With Taylor series,

$$\varepsilon^t = \ln\left(1+\varepsilon^e\right) = \ln 1 + \left.\frac{1}{1+\varepsilon^e}\right|_{\varepsilon^e=0}\varepsilon^e - \frac{1}{2!}\left.\frac{1}{\left(1+\varepsilon^e\right)^2}\right|_{\varepsilon^e=0}\left(\varepsilon^e\right)^2 + \cdots = \varepsilon^e - \frac{1}{2}\left(\varepsilon^e\right)^2 + \ldots$$

for small deformation,

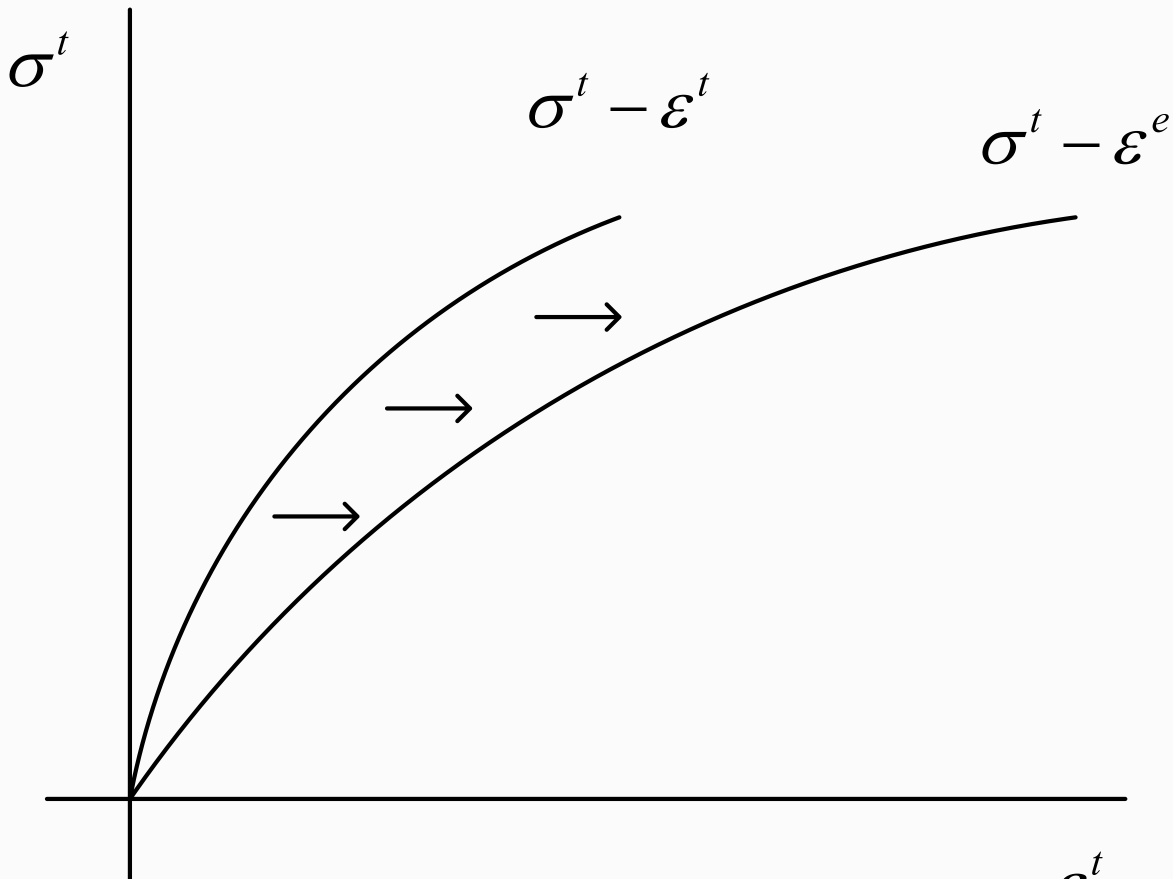
$$\varepsilon^t \approx \varepsilon^e$$

Stress in tension: Eng. vs. True



$$\varepsilon^t = \ln\left(\varepsilon^e + 1\right) \rightarrow \varepsilon^e = e^{\varepsilon^t} - 1$$

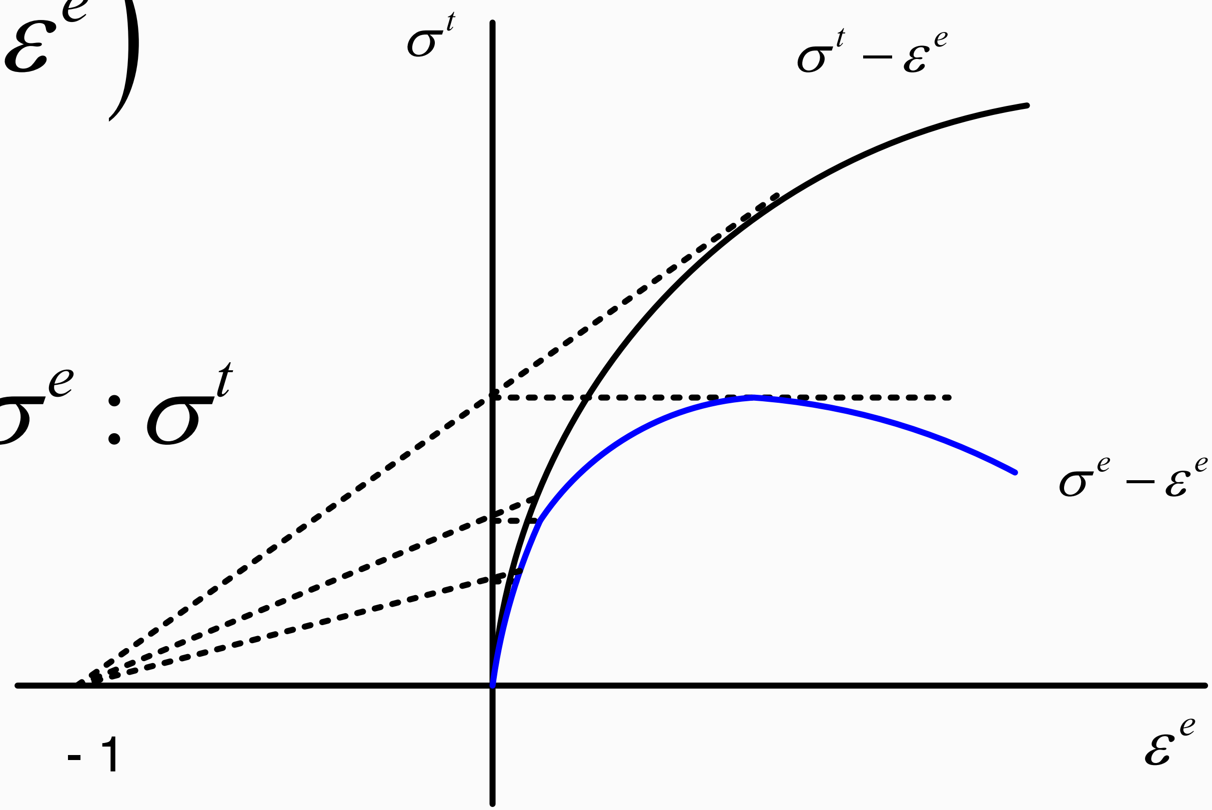
Stress in tension: Eng. vs. True



Stress in tension: Eng. vs. True

$$\sigma^t = \sigma^e\left(1+\varepsilon^e\right)$$

$$1:\left(1+\varepsilon^e\right) = \sigma^e : \sigma^t$$



Stress in compression: Eng. vs. True

- Compressive case

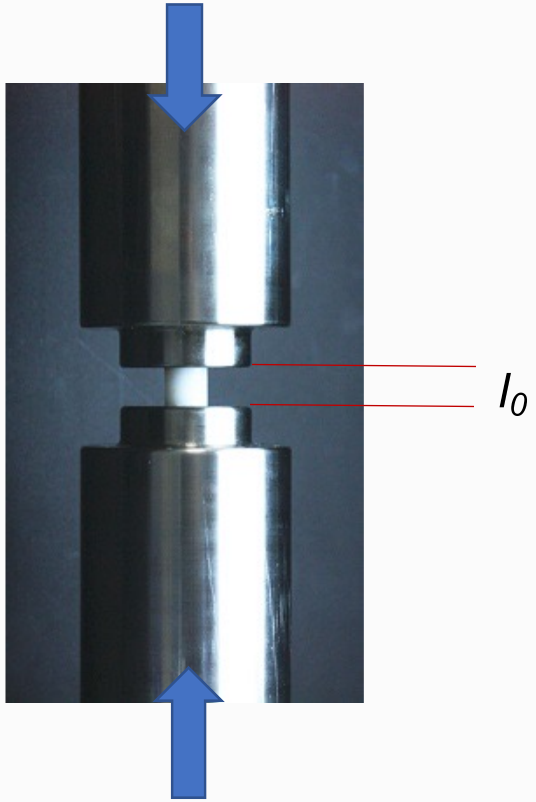
$$d\varepsilon^t = -\frac{dl}{l} \quad (\geq 0)$$

$$\therefore \varepsilon^t = \ln\frac{l_o}{l} \quad \left(0 \leq \varepsilon^t < \infty\right)$$

$$\varepsilon^e = \frac{l_o - l}{l_o} = 1 - \frac{l}{l_o} \quad \left(0 \leq \varepsilon^e < 1\right)$$

$$\sigma^t = \frac{F}{A} = \frac{F}{A_o}\frac{A_o}{A} = \frac{F}{A_o}\frac{l}{l_o} = \sigma^e\left(1 - \varepsilon^e\right)$$

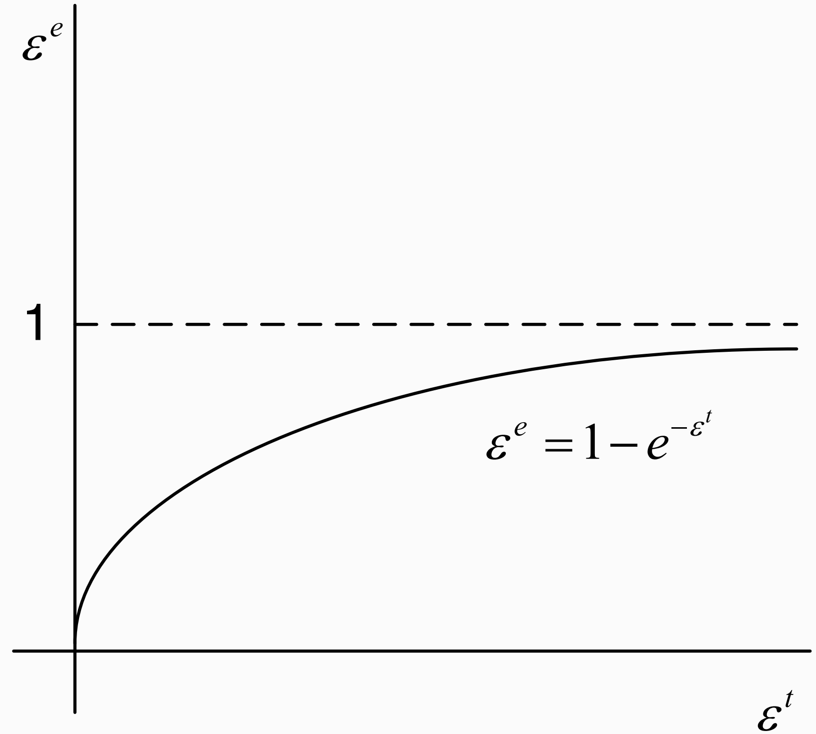
$(A_o l_o = Al$: volume constant)



Stress in compression: Eng. vs. True

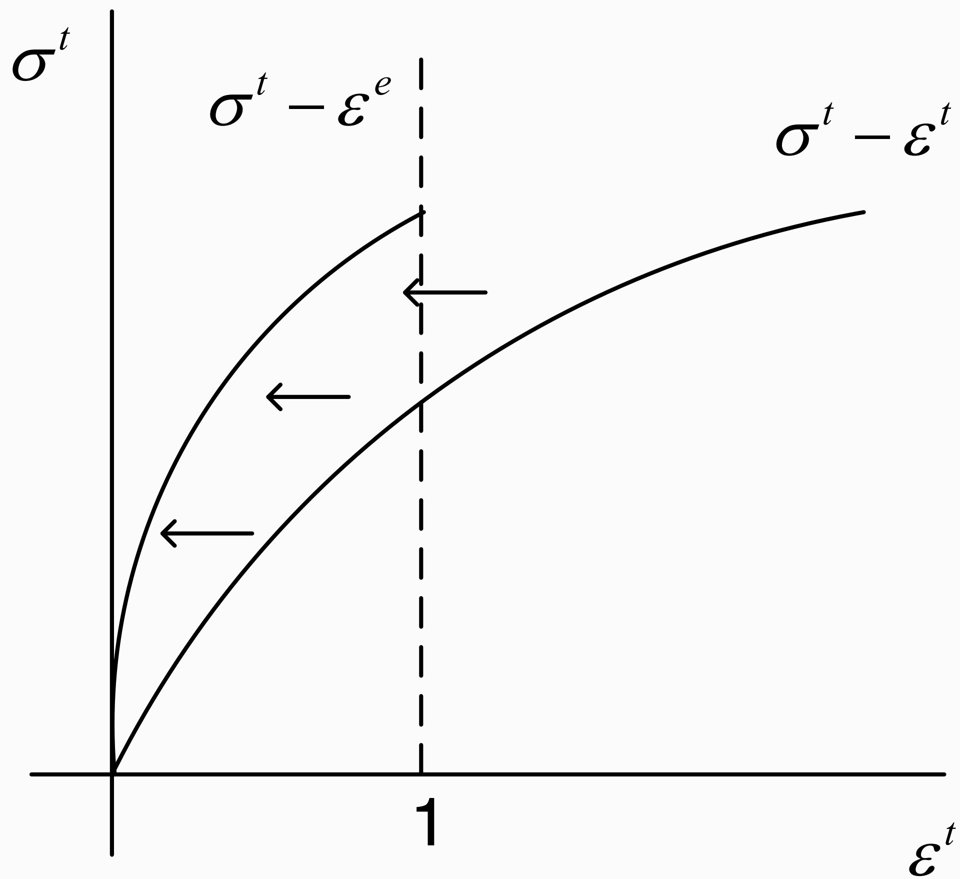
$$\frac{l}{l_o} = 1 - \varepsilon^e$$

$$\varepsilon^t = \ln \frac{l_o}{l} \Rightarrow e^{\varepsilon^t} = \frac{1}{1-\varepsilon^e} \Rightarrow \varepsilon^e = 1 - e^{-\varepsilon^t}$$



Stress in compression: Eng. vs. True

$$\varepsilon^t > \varepsilon^e$$




Thank you!