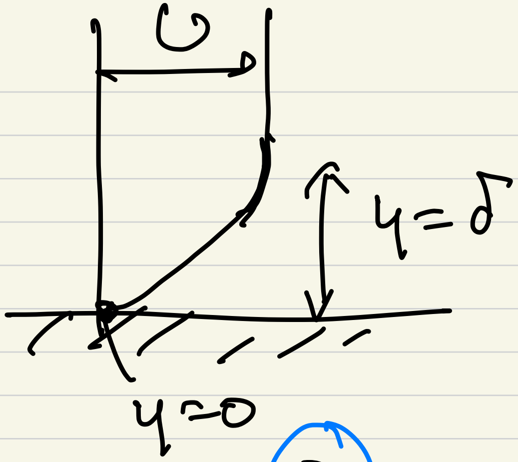
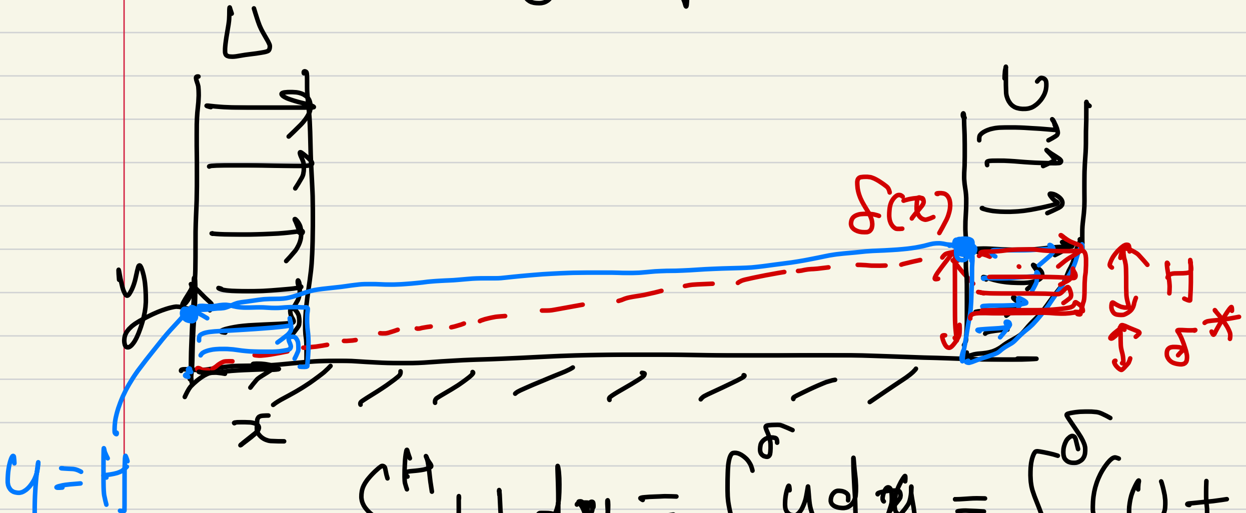


$$\left.\begin{aligned} u &= 0 \ @ \ y=0 \\ u &= U \ @ \ y=\delta \\ \frac{\partial u}{\partial y} &= 0 \ @ \ y=\delta \end{aligned}\right\} \rightarrow u(x,y) = U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$$

$$\Rightarrow \frac{\delta}{x} = \frac{5.5}{\sqrt{Re_x}}, \quad \frac{\theta}{x} = \frac{0.733}{\sqrt{Re_x}}, \quad \frac{\theta}{\delta} = \frac{2}{15}, \quad c_f = \frac{0.73}{\sqrt{Re_x}}$$

$$\delta \sim \sqrt{x} \qquad \theta \sim \sqrt{x} \qquad c_f \sim x^{-\frac{1}{2}}$$



(배제두께)
δ^* : displacement thickness

$$\int_0^H U\,dy = \int_0^\delta u\,dy = \int_0^\delta (U+u-U)\,dy = \underbrace{\int_0^\delta U\,dy}_{\shortparallel} - \int_0^\delta (U-u)\,dy$$

$$\int_0^H U\,dy = \cancel{U}H$$

$$U\delta = \cancel{U}(H+\delta^*)$$

$\rightarrow \; U\delta^* = \int_0^\delta (U-u)\,dy$ $\qquad \theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right)dy$:

displacement thickness

$\rightarrow \; \boxed{\delta^* = \int_0^\delta \left(1-\frac{u}{U}\right)dy} = \frac{1}{3}\delta \qquad \delta > \delta^* > \theta$

$$\frac{\delta^*}{x} = \frac{\delta^*}{\delta}\frac{\delta}{x} = \frac{1}{3}\cdot\frac{5.5}{\sqrt{Re_x}} = \frac{1.83}{\sqrt{Re_x}} \rightarrow \delta^* \sim \sqrt{x} \quad 6\% \text{ error}$$

cf. $\frac{\delta^*}{x} = \frac{1.721}{\sqrt{Re_x}}$ Blasius

$Re_x = \frac{Ux}{\nu}$, $Re_\delta = \frac{U\delta}{\nu}$, $Re_{\delta^*} = \frac{U\delta^*}{\nu}$, $Re_\theta = \frac{U\theta}{\nu}$

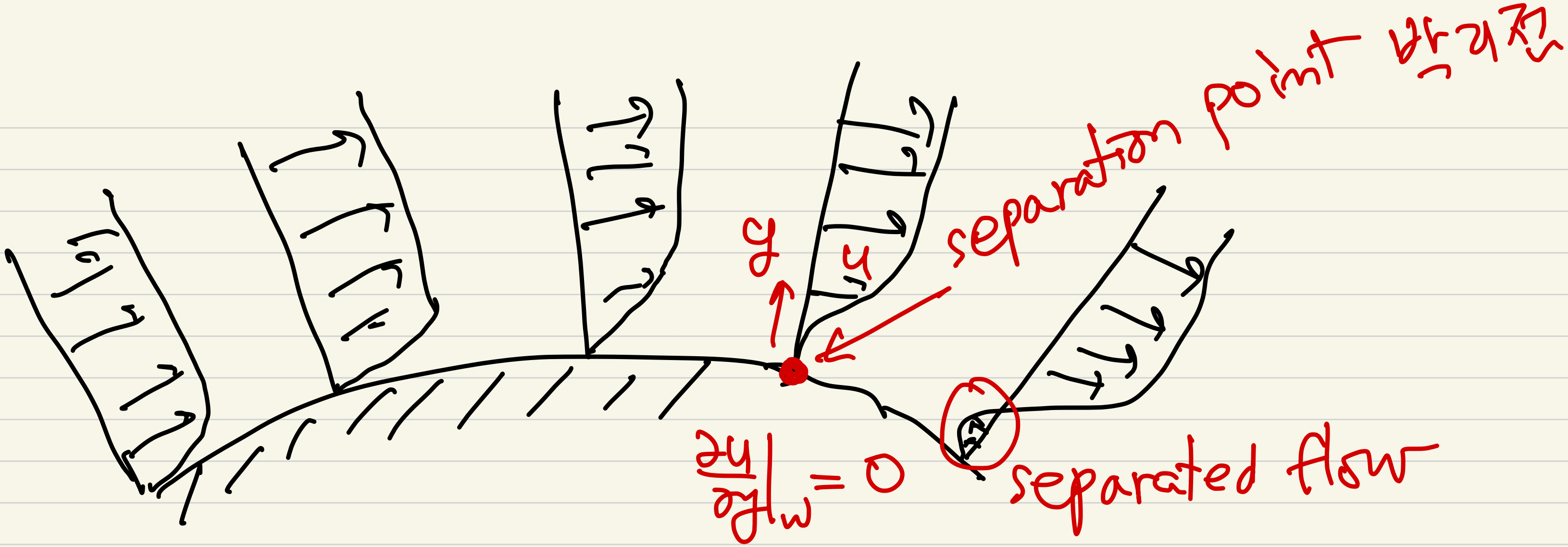
$Re_\tau = \frac{u^*\delta}{\nu}$

$H \equiv \frac{\delta^*}{\theta} = 2.5$: shape factor $\quad$ cf. $H = 2.59$ Blasius

laminar flow

turbulent flow $\quad H = 1.3$

$\rightarrow$ indicates whether or not boundary layer separation is about to occur.



$H = 3.5$ laminar

$\quad\;\; 2.4$ turbulent

7.3 Boundary layer equations

Navier-Stokes eqs. (steady 2-D incomp. flow)

$$
\begin{cases}
\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 & \text{— ①} \\[2ex]
\rho\left(u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial p}{\partial x} + \mu\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) & \text{— ②} \\[2ex]
\rho\left(u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y}\right) = -\dfrac{\partial p}{\partial y} + \mu\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right) & \text{— ③}
\end{cases}
$$

No analytic sol. for external flow

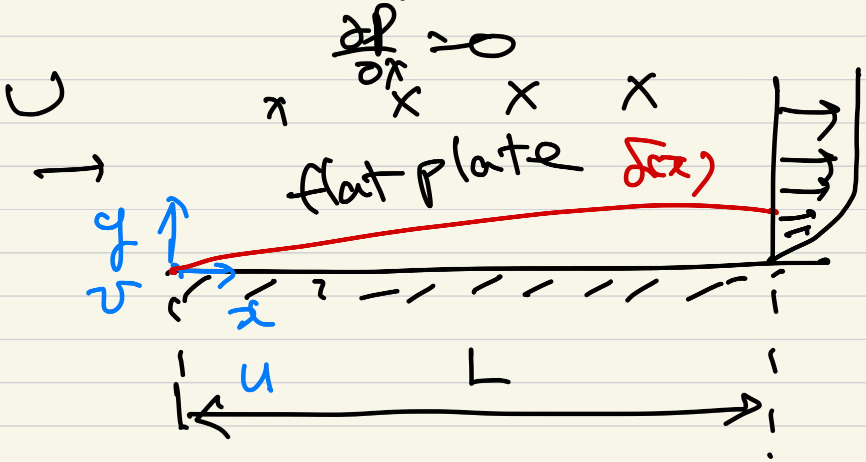
i) numerical sol. – CFD (Computational fluid dynamics)

ii) experiment – wind tunnel + velocimetries
water "

iii) $\boxed{\text{boundary layer theory}}$ – Ludwig Prandtl (1904)

경계층 이론

- Boundary layer approximation : Prandtl (1904)



$\delta \ll L$

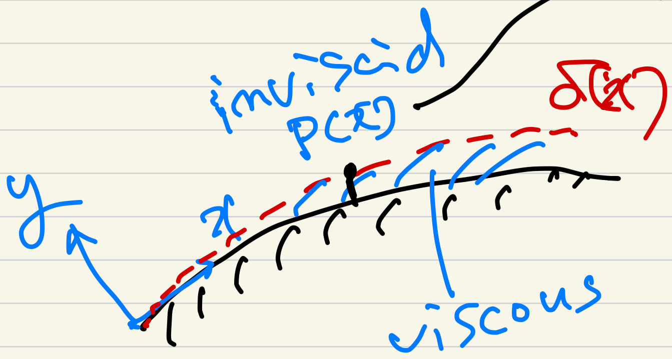
thin boundary layer

$$\boxed{\frac{\partial}{\partial x}(\cdot) \ll \frac{\partial}{\partial y}(\cdot), \quad \frac{\partial^2}{\partial x^2}(\cdot) \ll \frac{\partial^2}{\partial y^2}(\cdot), \quad v \ll u}$$

② + ③ $\Rightarrow$ $\frac{\partial p}{\partial y} \doteq 0 \rightarrow p = p(x)$ only

$\frac{\partial p}{\partial x} \neq 0$ $\left(\frac{\partial p}{\partial x} = 0 \text{ for flat plate}\right)$

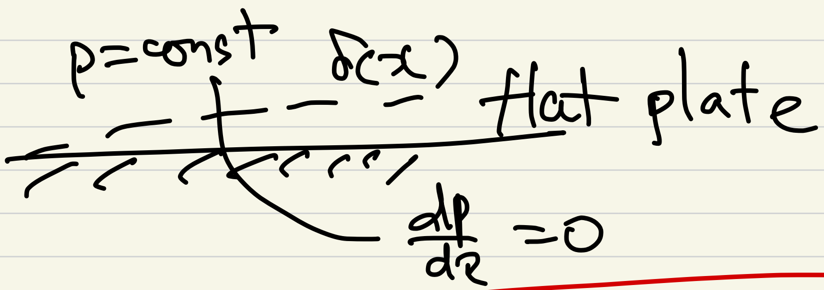
pressure varies only along the boundary layer.



$U(x)$ from Bernoulli eq.

$$\frac{p}{\rho} + \frac{U^2}{2} = \text{const}.$$

$$\rightarrow \frac{1}{\rho}\frac{dp}{dx} + U\frac{dU}{dx} = \text{const}$$



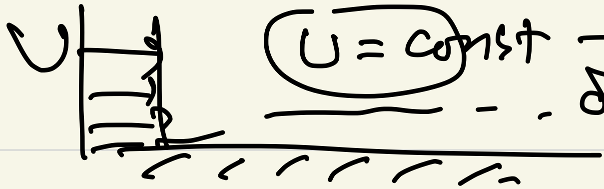
$$\rightarrow \boxed{\frac{dp}{dx} = -\rho U \frac{dU}{dx}}$$

$$\boxed{\begin{aligned} &① \rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &② \rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \end{aligned}}$$

Boundary layer eq.
경계층 방정식

boundary cond. $\begin{cases} u = v = 0 & @\ y = 0 \\ u = U(x) & @\ y = \delta \end{cases}$

7.4 Flat-plate boundary layer



$\boxed{U = const} \longrightarrow \frac{dp}{dx} = 0$

↳ simple but the most important

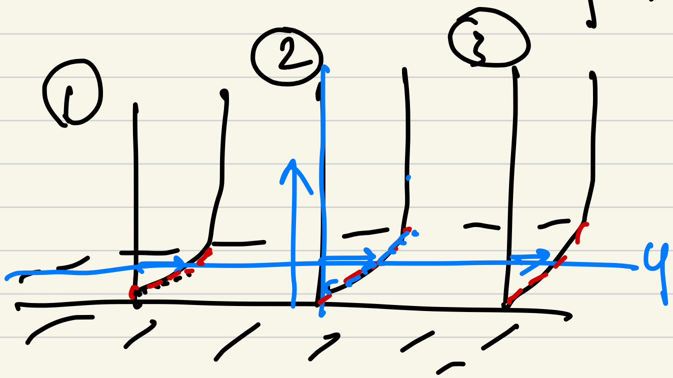
laminar flow

$$U = const \rightarrow \frac{dU}{dx} = 0 \rightarrow \frac{dP}{dx} = 0$$

Blasius (1908) using coord. transformation showed that $\frac{u}{U}$ is a ft. only of a single dimensionless variable $\eta = y\sqrt{\frac{U}{\nu x}}$. $\Rightarrow$ $\boxed{\frac{u}{U} = f'(\eta)}$ $\frac{u}{U} = F(\eta)$

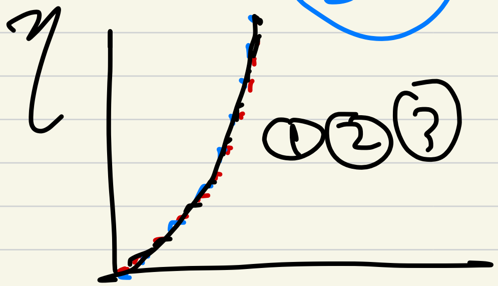
$$L \sqrt{\frac{m/s}{m^2/s \cdot m}} = \frac{m}{m^3} = \frac{1}{m^2} = \frac{1}{L}$$

$\int F$



as $x\uparrow$, $u\downarrow$

as $y\uparrow$, $u\uparrow$



$(x, y) \to \eta$

$\to \frac{u}{U} = f'(\eta)$, $\eta = y\sqrt{\frac{U}{\nu x}}$, $u = f'U$

bdry layer eqs.

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \end{cases}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = f''U\cdot\left(-\frac{1}{2}\right)y\sqrt{\frac{U}{\nu}}\,x^{-\frac{3}{2}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = f''U\cdot\sqrt{\frac{U}{\nu x}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial \eta}{\partial y} = f'''U\sqrt{\frac{U}{\nu x}}\sqrt{\frac{U}{\nu x}}$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \to v = -\frac{\partial}{\partial x}\int u\,dy = -\frac{\partial}{\partial x}\int f'U\cdot\sqrt{\frac{\nu x}{U}}\,d\eta$$

$$= \sim$$

$$= -f'\left(-\frac{1}{2}\right)\sqrt{\frac{U}{\nu}}\,y\,x^{-\frac{3}{2}}U\sqrt{\frac{\nu}{U}}\,x^{\frac{1}{2}} - fU\sqrt{\frac{\nu}{U}}\cdot\frac{1}{2}x^{-\frac{1}{2}}$$

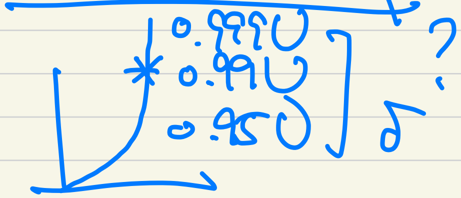
$$\Rightarrow \boxed{f''' + \frac{1}{2}ff'' = 0}$$

Blasius eq. (exact eq.)

@ $y=0$. $u = v = 0 \to$ @ $\eta = 0$, $f'(0) = 0$, $f(0) = 0$

@ $y = \delta$ (∞), $u = U \to$ @ $\eta \to \infty$, $f'(\infty) = 1$

$u = U$ @ $y = \delta$?



$\eta = y[U/(\nu x)]^{1/2}$ f'

$y[U/(\nu x)]^{1/2}$	u/U	$y[U/(\nu x)]^{1/2}$	u/U
0.0	0.0	2.8	0.81152
0.2	0.06641	3.0	0.84605
0.4	0.13277	3.2	0.87609
0.6	0.19894	3.4	0.90177
0.8	0.26471	3.6	0.92333
1.0	0.32979	3.8	0.94112
1.2	0.39378	4.0	0.95552
1.4	0.45627	4.2	0.96696
1.6	0.51676	4.4	0.97587
1.8	0.57477	4.6	0.98269
2.0	0.62977	4.8	0.98779
2.2	0.68132	$\boxed{5.0}$	$\boxed{0.99155}$
2.4	0.72899	∞	1.00000
2.6	0.77246		

numerical sol.

δ

$\Downarrow$

$$\eta = y\sqrt{\frac{U}{\nu x}} = 5.0$$

@ $\frac{u}{U} = 0.99$

$\Downarrow$

$$\boxed{\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}}}$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = f''(0)\cdot U\sqrt{\frac{U}{\nu x}} \rightarrow \tau_w = \frac{0.332\sqrt{\rho\mu U^3}}{\sqrt{x}}$$

$$c_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{\mu \left.\frac{\partial u}{\partial y}\right|_{y=0}}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re_x}}$$

$\rightarrow c_f \sim x^{-\frac{1}{2}}$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

$\downarrow$

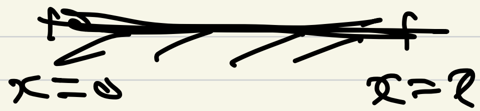
$$\frac{\delta^*}{x} = \frac{1.721}{\sqrt{Re_x}}$$

$$\int_0^\delta u\, dy = \int_0^{\delta\sqrt{\frac{U}{\nu x}}} f' U \sqrt{\frac{\nu x}{U}}\, d\eta$$

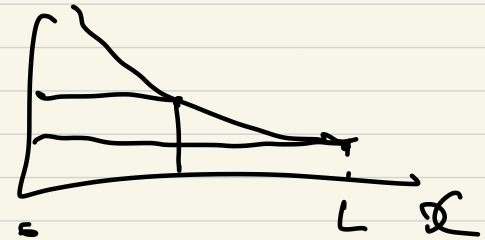
$$= f(\eta(\delta)) \cdot \cancel{xxx}$$

$$\eta(\delta) = 5.0$$

$$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$



Cf



$$D(x) = b\int_0^x \tau_w\, dx = 0.664\, \rho^{\frac{1}{2}} \mu^{\frac{1}{2}} U^{\frac{3}{2}} x^{\frac{1}{2}}$$

Drag coefficient : $\underline{C_D} = \dfrac{D}{\frac{1}{2}\rho U^2 b \cdot L} = \dfrac{1.328}{\sqrt{Re_L}} = \underline{2C_f(L)}$

Shape factor : $H = \dfrac{\delta^*}{\theta} = \dfrac{1.721}{0.664} = 2.59$