Chapter 3. Higher Order Linear ODEs

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서유택

※ 본 강의 자료는 이규열, 장범선, 노명일 교수님께서 만드신 자료를 바탕으로 일부 편집한 것입니다.

3.1 Homogeneous Linear ODEs

- ODE of *n*th order (*n*계 상미분 방정식): $F\left(x, y, y', \cdots, y^{(n)}\right) = 0$
- Linear ODE of *n*th order: $y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = r(x)$

  (Standard Form)

  - Homogeneous: $r(x) = 0$
  - Nonhomogeneous: $r(x) \neq 0$

- **Homogeneous Linear ODE: Superposition Principle, General Solution**

Theorem 1 Fundamental Theorem for the Homogeneous Linear ODE

For a homogeneous linear ODE, sums and constant multiples of solutions on some open interval I are again solutions on I.

3.1 Homogeneous Linear ODEs

❖ **Definition General Solution (일반해), Basis (기저), Particular Solution (특수해)**

A **general solution** of an ODE on an open interval I is a solution on I of the form arbitrary

$$y = c_1 y_1(x) + \cdots + c_n y_n(x) \qquad (c_1, \ldots, c_n)$$

where $y_1, \ldots, y_n$ is a **basis** (or a **fundamental system**) of solutions of the equation on I; that is, these solutions are linearly independent on I.

A **particular solution** of the equation on I is obtained if we assign specific values to the n constants $y_1, \ldots, y_n$ in $c_1, \ldots, c_n$.

3.1 Homogeneous Linear ODEs

❖ **Definition Linear Independence and Dependence (1차독립과 1차종속)**

n functions $y_1(x), \ldots, y_n(x)$ are called linearly independent on some interval I where they are defined if the equation

$$k_1 y_1(x) + \cdots + k_n y_n(x) = 0 \quad (c_1, \ldots, c_n) \text{ on } I$$

implies that all $k_1, \ldots, k_n$ are zero.

These functions are called linearly dependent on I if this equation also holds on I for some $k_1, \ldots, k_n$ not all zero.

☑ **Ex. 1 Linear Dependence**

Show $y_1 = x^3$, $y_2 = -2x$, $y_3 = 3x^3$ are linearly dependent on any interval.

3.1 Homogeneous Linear ODEs

❖ **Initial Value Problem (초기값문제). Existence and Uniqueness**

- Initial value problem

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0 \quad \text{and} \quad y(x_0) = K_0,\ y'(x_0) = K_1,\ \cdots,\ y^{(n-1)}(x_0) = K_{n-1}$$

❖ **Theorem 2 Existence and Uniqueness Theorem for Initial Value Problems**

If the coefficients $p_0(x), \ldots, p_{n-1}(x)$ of the equation are continuous on some open interval I and x_0 is in I, then the initial value problem has a unique solution $y(x)$ on I.

❖ **Initial Value Problem for a Third-Order Euler-Cauchy Equation**

☑ **Ex. 2 Solve the following initial value problem on any open interval I on the positive x-axis containing x = 1.**

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0, \quad y(1) = 2, \quad y'(1) = 1, \quad y''(1) = -4$$

Step 1 General solution of the homogeneous ODE

Substitution of $y = x^m \quad \Rightarrow \quad m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0 \quad \Rightarrow \quad m = 1,\ 2,\ 3$

Corresponding general solution of the homogeneous ODE:

$$y_h = c_1 x + c_2 x^2 + c_3 x^3$$

Step 2 Particular Solution

$$\begin{aligned} y(1) &= c_1 + c_2 + c_3 = 2 \\ y'(1) &= c_1 + 2c_2 + 3c_3 = 1 \\ y''(1) &= 2c_2 + 6c_3 = -4 \end{aligned} \quad \Rightarrow c_1 = 2, c_2 = 1, c_3 = -1$$

Answer $y_h = 2x + x^2 - x^3$

3.1 Homogeneous Linear ODEs

❖ Linear Independence of Solutions. Wronskian

$$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

❖ Theorem 3 Linear Dependence and Independence of Solutions

Let the ODE have continuous coefficients $p_0(x), \ldots, p_{n-1}(x)$ on an open interval I.

Then n solutions $y_1, \ldots, y_n$ of the equation on I are linearly dependent on I if and only if their Wronskian is zero for some $x = x_0$ in I.

Furthermore, if W is zero for $x = x_0$, then W is identically zero on I.

Hence if there is an x_1 in I at which W is not zero, then $y_1, \ldots, y_n$ are linearly independent on I,

so that they form a basis of solutions of the equation on I.

3.1 Homogeneous Linear ODEs

❖ **Initial Value Problem for a Third-Order Euler-Cauchy Equation**

☑ Ex. Show that the given functions are solutions and form a basis on any interval.

Q:?

$1, x, x^2, x^3$

3.1 Homogeneous Linear ODEs

❖ A General Solution Includes All Solutions

❖ Theorem 4 Existence of a General Solution

If the coefficients $p_0(x), \ldots, p_{n-1}(x)$ of the equation are continuous on some open interval I, then the equation has a general solution on I.

❖ Theorem 5 General Solution Includes All Solutions

If the ODE has continuous coefficients $p_0(x), \ldots, p_{n-1}(x)$ on some open interval I, then every solution $y = Y(x)$ on I is of the form

$$Y(x) = C_1 y_1(x) + \cdots + C_n y_n(x)$$

where $y_1, \ldots, y_n$ is a basis of solutions and $C_1, \ldots, C_n$ are suitable constants.

3.2 Homogeneous Linear ODEs with Constant Coefficients

- ***n*th-order homogeneous linear ODEs with constant coefficients:**

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

- **We try** $y = e^{\lambda x}$**.**

- **Characteristic equation (특성방정식):** $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$

- **General Solutions**

  - Distinct Real Roots: all the n roots $\lambda_1, \ldots, \lambda_n$ are real and different.

    ➡ $y_1 = e^{\lambda_1 x}, \quad \cdots, \quad y_n = e^{\lambda_n x}$ constitute a basis.

  - Simple Complex Roots: $\lambda = \gamma \pm i\omega$ are simple roots

    ➡ two corresponding linearly independent solutions are

$$y_1 = e^{\gamma x}\cos\omega x, \quad y_2 = e^{\gamma x}\sin\omega x.$$

3.2 Homogeneous Linear ODEs with Constant Coefficients

❖ General Solutions

- Multiple Real Roots: λ is a real root of order m.

  ➡ m corresponding linearly independent solutions are

$$e^{\lambda x},\ xe^{\lambda x},\ x^2e^{\lambda x},\ \cdots,\ x^{m\text{-}1}e^{\lambda x}.$$

- Multiple Complex Roots: $\lambda = \gamma \pm i\omega$ are double roots.

  ➡ corresponding linearly independent solutions are

$$e^{\gamma x}\cos \omega x,\ e^{\gamma x}\sin \omega x,\ xe^{\gamma x}\cos \omega x,\ xe^{\gamma x}\sin \omega x.$$

❖ Simple Complex Roots. Initial Value Problem

☑ **Ex 2. Solve the initial value problem.**

$$y''' - y'' + 100y' - 100y = 0 \qquad y(0) = 4 \quad y'(0) = 11 \quad y''(0) = -299$$

$$\lambda^3 - \lambda^2 + 100\lambda - 100 = 0 \qquad \lambda = 1, \pm 10i$$

$$y = c_1 e^x + A\cos 10x + B\sin 10x$$

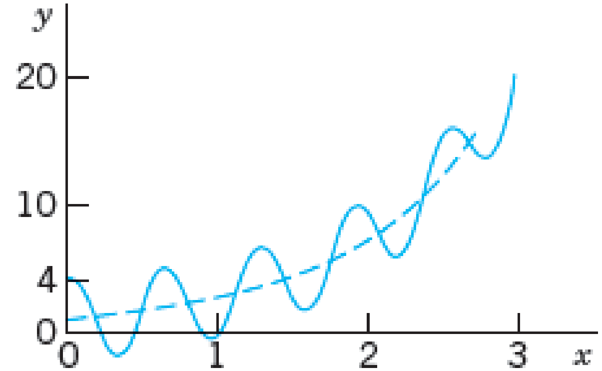
$$y' = c_1 e^x - 10A\sin 10x + 10B\cos 10x$$

$$y'' = c_1 e^x - 100A\cos 10x - 100B\sin 10x$$

$$c_1 + A = 4 \longrightarrow 101A = 303, \Rightarrow A = 3$$

$$c_1 - 10B = 11 \longrightarrow B = 1$$

$$c_1 - 100A = -299 \longrightarrow c_1 = 1$$



$$\therefore y = e^x + 3\cos 10x + \sin 10x$$

3.2 Homogeneous Linear ODEs with Constant Coefficients

❖ Real Double and Triple Roots

☑ **Ex. Solve the ODE.**

Q:?

$$y^{iv} + 2y'' + y = 0$$

3.3 Nonhomogeneous Linear ODEs

❖ Nonhomogeneous linear ODEs of nth order [n계 비제차 선형 상미분방정식]

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x), \quad r(x) \neq 0$$

❖ General solution: $y(x) = y_h(x) + y_p(x)$

- $y_h = c_1y_1(x) + \cdots + c_ny_n(x)$ is a general solution of $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$
- y_p is any solution of $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)$ on I containing no arbitrary constants.

❖ Initial value problem

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x) \text{ and } y(x_0) = K_0,\ y'(x_0) = K_1,\ \cdots,\ y^{(n-1)}(x_0) = K_{n-1}$$

3.3 Nonhomogeneous Linear ODEs

❖ Method of Undetermined Coefficients (미정계수법)

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + \cdots + p_1(x)\,y' + p_0(x)\,y = r(x), \quad r(x) \neq 0$$

❖ Choice Rules for the Method of Undetermined Coefficients

a. **Basic Rule.** If $r(x)$ is one of the functions in the first column in Table 2.1, choose $\boldsymbol{y_p}$ in the same line and determine its undetermined coefficients by substituting y_p and its derivatives into $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x)$.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

3.3 Nonhomogeneous Linear ODEs

❖ Method of Undetermined Coefficients (미정계수법)

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x), \quad r(x) \neq 0$$

❖ Choice Rules for the Method of Undetermined Coefficients

b. Modification Rule. If a term in your choice for y_p is a solution of the homogeneous ODE corresponding to $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = r(x)$ then multiply this term by x^k, where k is the smallest positive integer such that this term times x^k is not a solution of the homogeneous ODE.

Table 2.1 Method of Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$\left.\right\}K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$\left.\right\}e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

3.3 Nonhomogeneous Linear ODEs

❖ Method of Undetermined Coefficients (미정계수법)

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x), \quad r(x) \neq 0$$

❖ **Choice Rules for the Method of Undetermined Coefficients**

c. **Sum Rule.** If $r(x)$ is a sum of functions in the first column of Table 2.1, choose for y_p the sum of the functions in the corresponding lines of the second column.

Table 2.1 **Method of Undetermined Coefficients**

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

3.3 Nonhomogeneous Linear ODEs

❖ **Simple Complex Roots. Initial Value Problem**

☑ Ex 1. Solve the initial value problem.

$$y''' + 3y + 3y' + y = 30e^{-x} \qquad y(0) = 3 \quad y'(0) = -3 \quad y''(0) = -47$$

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0.$$

$$y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$$

$$= (c_1 + c_2 x + c_3 x^2) e^{-x}.$$

$y_p = Ce^{-x}$ ⇨ $-C + 3C - 3C + C = 30$, (not satisfied)

$y_p = Cx^3 e^{-x}$. (from modification rule)

$$y_p' = C(3x^2 - x^3)e^{-x},$$

$$y_p'' = C(6x - 6x^2 + x^3)e^{-x},$$

$$y_p''' = C(6 - 18x + 9x^2 - x^3)e^{-x}.$$

3.3 Nonhomogeneous Linear ODEs

❖ **Simple Complex Roots. Initial Value Problem**

☑ Ex 3. Solve the initial value problem.

$$y''' + 3y'' + 3y' + y = 30e^{-x} \qquad y(0) = 3 \quad y'(0) = -3 \quad y''(0) = -47$$

$$C(6 - 18x + 9x^2 - x^3) + 3C(6x - 6x^2 + x^3) + 3C(3x^2 - x^3) + Cx^3 = 30. \Rightarrow y_p = 5x^3e^{-x}$$

$$y = y_h + y_p = (c_1 + c_2x + c_3x^2)e^{-x} + 5x^3e^{-x}, \qquad y(0) = c_1 = 3$$

$$y' = [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, \qquad y'(0) = -3 + c_2 = -3, \qquad c_2 = 0$$

$$y'' = [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, \qquad y''(0) = 3 + 2c_3 = -47, \qquad c_3 = -25.$$

$$\boxed{y = (3 - 25x^2)e^{-x} + 5x^3e^{-x}.}$$

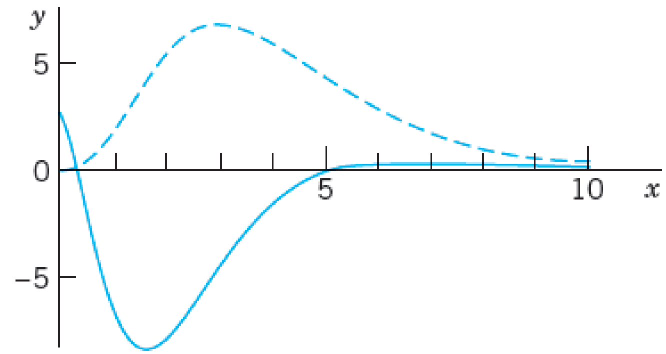


Fig. 74. y and y_p (dashed) in Example 1

3.3 Nonhomogeneous Linear ODEs

- Method of Undetermined Coefficients (미정계수법)
- Method of Variation of Parameters (매개변수변환법)

$$y_p(x) = y_1(x)\int \frac{W_1(x)}{W(x)} r(x)dx + \cdots + y_n(x)\int \frac{W_n(x)}{W(x)} r(x)dx$$

W_j is obtained from W by replacing the jth column of W by the column $[0\ 0\ \cdots\ 0\ 1]^T$

When $n = 2$, this becomes identical with

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1$$

$$y_p(x) = -y_1\int \frac{y_2 r}{W} dx + y_2\int \frac{y_1 r}{W} dx$$

If it starts with $f(x)y'''$, divide first by $f(x)$.

3.3 Nonhomogeneous Linear ODEs

☑ Ex. 2 Solve the nonhomogeneous Euler-Cauchy equation.

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x$$

Step 1 General solution of the homogeneous ODE

Substitution of $y = x^m$ $\Rightarrow$ $m(m-1)(m-2) - 3m(m-1) + 6m - 6 = 0$ $\Rightarrow$ $m = 1,\ 2,\ 3$

Corresponding general solution of the homogeneous ODE:

$$y_h = c_1 x + c_2 x^2 + c_3 x^3$$

$$\Rightarrow y_1 = x,\ y_2 = x^2,\ y_3 = x^3$$

Step 2 Determinants

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4, \quad W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3, \quad W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

3.3 Nonhomogeneous Linear ODEs

☑ Ex. 2 Solve the nonhomogeneous Euler-Cauchy equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x$$

$$y_1 = x,\ y_2 = x^2,\ y_3 = x^3$$

$$W = 2x^3, \quad W_1 = x^4, \quad W_2 = -2x^3, \quad W_3 = x^2$$

Step 3 Integration

Here, since it starts with $x^3 y'''$, divide first by x^3.

Then, $r(x) = x^3 \ln x / x^4 = x \ln x$.

$$y_p = x\int \frac{x}{2} x \ln x\,dx - x^2 \int x \ln x\,dx + x^3 \int \frac{1}{2x} x \ln x\,dx = \frac{1}{6} x^4 \left(\ln x - \frac{11}{6} \right)$$

$$\therefore \quad y = c_1 x + c_2 x^2 + c_3 x^3 + \frac{1}{6} x^4 \left(\ln x - \frac{11}{6} \right)$$

$$y_p(x) = y_1(x) \int \frac{W_1(x)}{W(x)} r(x)\,dx + \cdots + y_n(x) \int \frac{W_n(x)}{W(x)} r(x)\,dx$$